Chapter 3

FUZZY LATTICE AND FUNCTION APPROXIMATION IN IMAGE PROCESSING

3.1 Introduction.

In this chapter we will focus on distributive lattices and the algebra of lattices and their applications in image processing. Using a set $X$ and a complete distributive lattice $L$ we define an $L$-fuzzy subset $A$ of $X$ as a function from $X \rightarrow L$ such that $A(x)$ is the degree of membership of element $x$ respectively. Then by introducing unit hypercube $I''$ such that $L$ can be reduced to a particular case $I$ where $I$ is complete distributive lattice. The order relation can be reduced to the interval $[0,1/2]$ or $[1/2,1]$ for under exposed images or over exposed images respectively, so that the approximation in image enhancement takes place at the fuzzy point $C(x,I)$ and hence it is
called the fuzzy crossover point. The three independent parameters can be found out by gradient descent algorithm. An experimental study shows that the color image can be enhanced using newly designed intensification operator. It was also observed that approximation in image enhancement takes place at the fuzzy point \( C(x,l) \). We prove that at the fuzzy cross over point \( C(x,l) \) the color image is enhanced. The study has revealed that fuzzy cross over point for under exposed images is \( C(x,l) = 0.31 \). At this point \( f(x) \) tends to \( g^*(x) \) such that the contrast of the image is increased. Dualizing this approximate value of \( C(x,l) \), brightness can be reduced for over exposed images. In section 3.2, two isomorphic functions namely valuations are considered. All valuations must obey the requirement that the mapping of lattice elements to real numbers must be consistent with the lattice structure. Unit hypercube and fuzzy entropy are defined. The notion of valuation is used to define the Shannon’s entropy. The gradient descent method defined in the previous chapter
is used to find the parameters in the reconstructed Gaussian membership function such that the objective function $J_c$ can be minimized. We also propose an algorithm which leads to the function approximation in the unit hyper cube structure. Use of modulus of continuity gives the approximation bounds. Thus the function approximation in image enhancement is obtained from a newly designed intensification operator

3.1.1 Definition (Fuzzy point) [37]

Let $L$ is a complete lattice and $A: X \rightarrow L$ be an $L$-fuzzy set. By a fuzzy point $(x, l)$ we mean that $S(x, l) \in A$ iff $l \leq A(x)$.

3.1.2 Remark

a. Let $A$ and $B$ be $L$ fuzzy subsets of $X$. Then $A \subset B$ iff $S(x, l) \in B$, $\forall S(x, l) \in A$.

b. $[0,1]$ is a complete distributitive lattice where least and greatest element are denoted by 0 and 1 respectively. Therefore, as a particular case, we can take $L = [0,1]$.

c. $F(X)$ can be considered as the set of all possible digital images over $X$. For $f, g \in F(X)$ we define
\[ f \leq g \iff f(x) \leq g(x), \forall x \in X. \]

With this ordering \( F(X) \) becomes a lattice where the “meet” and “join” are given by

\[
(f \land g)(x) = \min\{f(x), g(x)\}, \text{ and}
\]

\[
(f \lor g)(x) = \max\{f(x), g(x)\}, \forall f, g \in F(X), x \in X.
\]

Hence the set of all digital images over \( X \) form a lattice.

In our discussion \( X \) is considered as the spatial (geometrical) domain of a digital image. It is assumed that the fuzzy set represents an object \( A \) with gray level adjusted to the range between 0 and 1. This image can be considered as a fuzzy set \( f : X \to [0,1] \).

3.1.3 The Fuzzy Hypercube

Bart Kosko’s work\cite{10} regards fuzzy sets as points in the hypercube \( I^n = [0,1]^n = [0,1] \times [0,1] \times \ldots \times [0,1], (n \text{- times}) \)

The geometry of fuzzy sets involves both the domain \( X = \{x_1, x_2, \ldots, x_n\} \) and the range \([0,1]\) of mappings \( f : X \to [0,1] \).

According to him the fuzzy power set \( F(2^X) \), the set of all fuzzy
subsets of $X$ looks like a cube. Vertices of the cube $I^n$ are crisp (non-fuzzy) sets. So the ordinary power set $2^X$, the set of all $2^n$ crisp subsets of $X$ is the Boolean n-cube $B^n$. i.e, $2^X = B^n$. Fuzzy sets fill in the lattice $B^n$ to produce the solid cube $I^n$. i.e, $F(2^X) = I^n$

The crossover point helps to distinguish the under and overexposed images in the unit hypercube. The fuzzy operator for an image enhancement has been established in [48]. The ordering of fuzzy sets in the unit hypercube is essential for this purpose.

Now we define another ordering $\leq^*$ such that the elements of the unit interval are ranked according to their closeness to the midpoint of the interval.

For any $a, b \in I, a \leq^* b$ iff $a \leq b \leq \frac{1}{2}$ or $\frac{1}{2} \leq b \leq a$. That is, $a$ is considered smaller than $b$ if $a$ is more spaced apart from $\frac{1}{2}$ than $b$.

According to [1, 58], the relation $\leq^*$ can be extended into a lattice ordering.
Since fuzzy sets are functions with co-domain $I$, this new partial ordering can be carried over to $F(X)$. That is, for $f, g \in F(x)$,

$$f \leq^* g \text{ iff } f(x) \leq g(x) \leq \frac{1}{2} \text{ or } \frac{1}{2} \leq g(x) \leq f(x).$$

If $X$ is the spatial domain of two images $f$ and $g$ and $f(x), g(x)$ are thought of as the respective gray level values at point $x \in X$, then the meaning of $f \leq^* g$ is that “$g$ is more blurred than $f$” or equivalently “$g$ has less contrast than $f$”.

The following definition is useful to define entropy on the new order relation.

**3.1.4 Definition (Isomorphic Lattices) [41]**

Let $(L, \leq)$ and $(L’, \leq^*)$ be two lattices and $f : L \rightarrow L’$ be a bijection. Then $L$ is said to be a lattice isomorphism if

$$f(x \land y) = f(x) \land f(y) \text{ and }$$

$$f(x \lor y) = f(x) \lor f(y), \forall x, y \in L$$
3.2 Measure on Lattices.

A measure $m$ typically refers to a function on a Boolean lattice $B$ that takes a lattice element to a real number so that for a given $X \in B, m(X) \in R$. We define below, the notion of valuation on lattices. This becomes a good tool to study entropy.

3.2.1 Definition (Valuation)

Let $L$ be a lattice. A function $V : L \rightarrow R$ is called valuation if

1. $x \geq y \Rightarrow V(x) \geq V(y)$
2. $V(x) + V(y) = V(x \lor y) + V(x \land y)$

The first condition above described as $V$ is increasing. If $V$ is decreasing [i.e. $x \geq y \Rightarrow V(x) \leq V(y)$] then $V$ is called co valuation.

3.2.2 Example (Isotone or positive valuation)

Any finite dimensional vector space $R^n$, becomes lattice where $\leq$ is defined by

$$(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n) \text{ if } x_k \leq y_k, \forall k.$$
Any linear functional \( C(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \) is a valuation. This valuation is positive if and only if \( c_k \) is positive for every \( k \).

### 3.2.3 Proposition [41]

Let \( V_1 \) and \( V_2 \) be valuations on lattices \( L_1 \) and \( L_2 \) respectively. Let \( L_1 \times L_2 \to R \) be defined by \( V(a, b) = V_1(a) + V_2(b), \forall (a, b) \in L_1 \times L_2 \). Then \( V \) is a valuation on \( L_1 \times L_2 \).

The valuation \( V \) in the above proposition is called the sum of \( V_1 \) and \( V_2 \). We write \( V = V_1 + V_2 \). As a finite extension of this, we get the following proposition.

### 3.2.4 Proposition

If \( V_1, V_2, \ldots, V_n \) are valuations on lattices \( L_1, L_2, \ldots, L_n \) respectively.

Then the function \( \sum_{i=1}^{n} V_i \) is a valuation on the product lattice \( L_1 \times L_2 \times \ldots \times L_n \), where \( \sum V_n \) is defined by

\[
\left( \sum_{i=1}^{n} V_i \right)(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} V_i(a_i)
\]
3.2.5 Lemma

A monotonically increasing function $V$ on a relatively complemented lattice with 0 is a valuation provided

$$V(x \lor y) = V(x) + V(y) \text{ whenever } x \land y = 0.$$  

**Proof.** Let $L$ be a relatively complemented lattice with 0. Let $x, y \in L$ and let $t$ be a relative complement of $x \land y$ in $[0, y]$. Then by definition,

$$(x \land y) \land t = 0 \text{ and } (x \land y) \lor t = y$$

Hence

$$V(y) = V(x \land y) + V(t) \quad (1)$$

Moreover, since $t \leq y$,

$$x \land t = x \land (y \land t) = (x \land y) \land t = 0$$

Also

$$x \lor t = [x \lor (x \land y)] \lor t = x \lor [(x \land y) \lor t] = x \lor y$$

$$\therefore V[x \lor y] = V[x] + V[t] \quad (2)$$

From (1) and (2)

$$V[x] + V[y] = V[x \lor y] + V[x \land y]$$

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Hence $V$ is a valuation on $L$. □


3.2.6 Definition (Fuzzy Entropy measure)

Let $E : F(2^X) \to [0,1]$ be a function. Then $E$ is a fuzzy entropy measure if it satisfies the following axioms:

i) $E(A) = 0$ iff $A \in 2^X$.

ii) $E(A)$ is maximum iff $A(x) = 0.5, \forall x \in X$.

iii) $E(A) \leq E(B)$ if $A$ is less than $B$.

iv) $E(A) = E(A^c)$

Motivated by the classical Shannon’s entropy function, De Luca and Termini proposed the following parameterized entropy measures. For $k > 0$,

$$E_k(A) = D_k(A) + D_k(A^c)$$

and

$$D_k(A) = -k \sum A(x_i) \log A(x_i),$$

$$D_k(A^c) = -k \sum (1 - A(x_i)) \log (1 - A(x_i))$$
3.2.7 Remark

Even though the order relation $\leq^*$ is committed to a fixed average value of $\frac{1}{2}$ it is possible to shrink or stretch $\leq^*$ in various ways considering any $m$ in the unit interval $I = [0,1]$. Then there exists a relation $\leq_m$ is obtained from $\leq$ by simply replacing $\frac{1}{2}$ by $m$, where $m$ is any value in $(0,1)$. The set $\left(I, \leq_m\right)$ is a lattice for all $m$ and defines a new relation $\leq_m$.

3.2.8 Theorem

All the lattices $I_m = \left(I, \leq_m\right)$ are isomorphic to one another.

Proof

Define $f_{mn} : I_m \to I_n$ by

$$f_{mn}(a) = \begin{cases} \frac{na}{m}, & a \leq m \\ 1 - \frac{(1-a)(1-n)}{1-m}, & a > m \end{cases}$$

It can be shown that $f_{mn}$ is a lattice isomorphism.
3.3 Fuzzy entropy and Isomorphism

3.3.1 Remark (Formation of entropy by means of isomorphism)

Carry $E(x)$ from $I_{\frac{1}{2}} \rightarrow I_m$ by means of the isomorphism $f_{(1/2),m}$.

Then for each $a \in I$, we consider

$$E_m(f_{(1/2),m}(a)) = E(a)$$

(1)

where $E$ is the Shannon’s function

$$E(x) = -x \ln x - (1-x) \ln(1-x).$$

Since $f_{mn} \cdot f_{nm} = 1 = f_{nm} \cdot f_{mn}$, we have $f_{mn}^{-1} = f_{mn}$. Therefore, condition (1) can be written as $E_m = E f_{m,(1/2)}$.

Then

$$f_{m,(1/2)}(x) = \begin{cases} \frac{x}{2m}, & x \leq m \\ 1 - \frac{(1-x)}{2(1-m)}, & x > m \end{cases}$$

Hence $E_m(x)$ can be written as

$$E_m(x) = \begin{cases} \frac{-x}{2m} \ln \frac{x}{2m} - \frac{(1-x)}{2m} \ln \left(1 - \frac{x}{2m}\right), & x \leq m \\ - \left(1 - \frac{1-x}{2(1-m)}\right) \ln \left(1 - \frac{1-x}{2(1-m)}\right) - \frac{1-x}{2(1-m)} \ln \frac{1-x}{2(1-m)}, & x > m \end{cases}$$
Therefore $E(x)$ carry over from $I_1 \rightarrow I_m$.

The following properties show that there is isotone valuation $E_m$ over $I_m$.

### 3.3.2 Properties

1. $E_m(0) = E_m(1) = 0$
2. $E_m(m) = 1$
3. $E_m(x)$ is strictly increasing in $[0, m]$.
4. $E_m(x)$ is strictly decreasing in $[m, 1]$.
5. $\forall m \in (0, 1), E_m(x)$ are convex

In particular the existence of a family of valuated lattices will be extended to fuzzy sets.

### 3.3.4 Definition (Additive entropy)

Let $F_m(X)$ be the set of all fuzzy maps equipped with $\leq_m$

i.e. $f \leq_m g$ iff $f(x) \leq_m g(x)$.
Then for a finite $X$, the entropy can be defined as a functional $e_m$ on $F_m(X)$ by

$$e_m(f) = \sum_{x \in X} E_m(f(x)), \forall f \in F_m(X)$$

### 3.3.5 Remark

If $E_m(x)$ is an isotone valuation in $I_m$, then $e_m(f)$ is an isotone valuation on the lattice $F_m(X)$.

### 3.3.6 Remark

By applying fuzzy lattice theory in the image enhancement problem, we can find a fuzzy point $P_l$ in the unit hypercube at which the image can be enhanced. For the function approximation in image enhancement problem the suitable values for $t, C(x, l)$ and $f_h$ are shown in table 3.1.

### 3.4 Image Enhancement as an Inverse Problem [36]

In view of [36], we consider the image enhancement problem as an inverse problem. The concept of pixel number and its functional relationship is obtained from [6, 55]. In this work, we assume that an
ideal image $f$ has been corrupted to create the measured image $g$. Here $g$ is a low contrast image where $g = [g_1, g_2, \ldots, g_N]^T$ is the spatial intensity values and $g_i$ denote the $i^{th}$ intensity level in a column vector representation of the image $g$. The enhancement problem is the problem of finding the best estimate of $f$ given the measurement $g$. Thus we have a process which takes an input and produces an output and we wish to infer the output. Here we have taken $g = [g_0, g_1, \ldots, g_{255}]^T$ where $g_0, g_1, \ldots, g_{255}$ are pixel numbers. By using $t, C(x, l), f_h$ we can define the membership function $X(x)$ such that the image $X^*(x)$ is reconstructed. Then the functional $E$ is defined as mapping from $[g_0, g_1, \ldots, g_{255}]^T \rightarrow [g_0^*, g_1^*, \ldots, g_{255}^*]^T$ such that $g^* = E(g)$. Here $E(g)$ indicates entropy based optimization with respect to $t, C(x, l), f_h$. Hence the filtered image $g^*$ is obtained from the application of the new intensification operator.
3.4.1. Definition (well-posed problem) [4]

A problem \( g^* = E(g) \) is said to be well-posed if

i) For each \( g \), a solution \( g^* \) exists.

ii) The solution \( g^* \) is unique.

iii) The solution \( g^* \) continuously depends on the data \( g \).

If the conditions in the definition do not hold, the problem is said to be ill posed.

According to [36], the applications of linear algebra helps to find a good approximation \( g^* \) to the function \( g \) on \( \mathbb{R}^n \) satisfying an approximate equation \( g^* = E(g) \). Generally any regularization method tries to analyze a treated well-posed problem whose solution approximates the original ill-posed problem.

In this problem the aim is to produce an image using fuzzy logic system which has the minimum least square error. That is finding the unknown image by the minimization of \( J_c = \|g^* - E(g)\|^2 \).
Directly minimizing $J_c$ does not work as the problem is ill conditioned. In order to give preference to a particular solution with desirable properties, the regularization term is $\lambda(C_f - C_d)$, where $C_f$ is fuzzy average contrast and $C_d$ is the visual quality for under exposed images (here taken as 0.4) and $\lambda = 0.1$ is assumed as the Lagrange multiplier. Therefore $J_c = \|g^* - E(g)\|^2 + \lambda(C_f - C_d)$. The following training method is useful to minimize the entropy and contrast with respect to the parameters $t, C(x, l), f$. This shows that when contrast is minimized, the function $g^*$ approximates $f$.

### 3.4.2 Proposed Algorithm

Let $f$ be the image at one hand and is a degraded version of the image $g$ representing light objects on a dark background. Our aim is to reconstruct $g^*$ from $f$ by applying the newly designed intensification operator so that the entropy can be minimized. To each pixel position $(i, j)$ we associate a contrast which is proportional to $g - f$. To keep
approximating function \( g \) from fluctuating, we associate the fuzzy point \( C \) in the fuzzy domain and modified membership function. This approach actually produces a function \( g^* \) which is an improved version of \( g \). Select the defuzzification function as

\[
x^* = x_{\text{max}} - [-2 \ln(X^*(x)) f_h^2].
\]

To enhance the image we choose the corresponding \( t \)-value from 12 to 16 and the value of \( t \) increases when brightness increases and \( f_h \) is a deciding factor for the image is over contrast (above 0.5) or lower contrast (below 0.5) in a unit hypercube. Here we change the variable \( (C) \) and produces a sequence of approximation to reach the minimum point. Then applying Gradient Descent Training at \( P_i = (C(x,l), t, f_h) \). In an experimental situation of an image enhancement process, suppose there are \( n \)-searching directions for an objective function \( J_c \) corresponding to the point \( P_i \) then as in [49], the search direction show that how an \( n \)-dimensional unit hypercube \( I^n \) can be formed.
\[
S_i^T = \begin{cases}
(1,0,\ldots,0); & i = 1, n+1, 2n+1 \\
(0,1,\ldots,0); & i = 2, n+2, 2n+2 \\
\cdots & \cdots & \cdots \\
(0,0,\ldots,1); & i = n, 2n, 3n
\end{cases}
\]

3.4.3 Iterative process

Now we choose the point \( P_i = (C(x, t), t, f_{ik}) \) and find \( J_C = J(P_i) \).

Set \( P_{i+1} = P_i + \varepsilon S_i (\partial J_C / \partial P_i) \) and \( (J_C)_{i+1} = J(P_{i+1}) \) and continue this process until there is no significant difference between these points (i.e. a point at which there is no significant difference in the value of the objective function and corners of the hypercube cells). Here \( i \) refers to the iteration number and \( \varepsilon S_i \) is a step size which must be chosen so that it is neither too big nor too small. A definite result of function approximation is obtained by an upper bound for the Jacobian or derivative of the functional \( E \).

3.4.4 Proposition

Let \( I^n \subset R^n \) be a bounded unit hypercube in which the set of all digital images are defined. Then for any given continuous
function $f(x)$ in the hypercube $I^n$ and any $\varepsilon > 0$ there is a fuzzy system $g^*(x)$ such that $\|f(x) - g^*(x)\| < \varepsilon$.

3.5 Application of intensification operator in different images

In Figures 3.1(a), 3.2(a), 3.3(a), 3.4(a), 3.5(a) low contrast and enhanced images and their respective histograms are given. Histograms are used to fuzzify the digital images. It is in Gaussian model. In Figures 3.1(b), 3.2(b), 3.3(b), 3.4(b), 3.5(b) curves corresponding to the three parameters $t, C(x,l), f_h$.

Fig. 3.1: (a) Original boy
(2) Original-boy-histogram
(3) Enhanced boy
(4) Enhanced-boy-histogram

(a)
Fig. 3.1: (b)  (1) boy-t-value  (2) boy-C(x,l)-value  (3) boy-f_h-value

Fig. 3.2: (a)  (1) Original Girl  (2) original-girl-histogram  (3) Enhanced Girl  (4) original-girl-histogram

Fig. 3.2: (b)  (1) girl-t-value  (2) girl-C(x,l)-value  (3) girl-f_h-value
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Fig. 3.3: (a) (1) original light house (2) original-light- histogram (3) enhanced light house (4)enhanced light house histogram

Fig. 3.3: (b) (1)Light house t-value (2)Light house $C(x,l)$-value (3) Light house $f_h$-value

Fig. 3.4: (a) (1) Original Shoe (2) original shoe-histogram (3) Enhanced Shoe (4) enhanced shoe- histogram
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Fig. 3.4: (b)
1. Shoe- $t$-value
2. Shoe- $C(x,t)$ -value
3. Shoe- $f_h$-value

Fig. 3.5: (a)
1. Original-Building
2. Original-Build-histogram
3. Enhanced-build
4. Enhanced-build- histogram

Fig. 3.5: (b)
1. Build-$t$-value
2. Build- $C(x,l)$ -value
3. Build- $f_h$-value
3.5.1 Experimental Result

The fuzzy logic system is applied in five images. The newly designed intensification operator is applied in the colour image enhancement process. The initial values of $t$ and $C$ are fixed but $f_h$ not fixed. A number of iterations were done. Then it was observed that; for an enhanced colour image the value of $C(x, l)$ is approximating to a point in the unit hypercube such that the filtered image approximate the original image. This claim is supported by the following table.

**Table 3.1. Experimental result for Approximate Value of $t, C$ & $f_h$**

<table>
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<th></th>
<th>$t$</th>
<th>$C = C(x, l)$</th>
<th>$f_h$</th>
<th></th>
<th></th>
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<td>Enhanced</td>
<td>Initial</td>
</tr>
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<td>157.0967</td>
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