1.1 Introduction

Zadeh L.A [66] introduced the notion of fuzzy sets and fuzzy logic. The issue of universal approximation is crucial to fuzzy systems. Theoretical works [5, 9, 24, 39 and 64] establish that fuzzy systems can always be constructed to uniformly approximate any desired continuous non linear functions with required approximation accuracy. The studies in [25, 26, 30,5 and 8] establish the necessary and sufficient conditions of fuzzy systems as universal approximators. The establishment of necessary conditions has also provided insight on the strength and limitations of fuzzy systems as functional approximators. It is known that Mamdani type and Takagi Sugeno (TS) fuzzy systems are universal approximators. The notion of entropy for fuzzy subsets was introduced in [24]. Klir [41] discussed the algebraic requirements for fuzzy entropy measures. Kosko [42] studied simple geometric considerations that lead to fuzzy entropy. In
[25] the strength and limitations of MISO fuzzy systems and its approximation capabilities are established.

The geometry of the fuzzy sets which involves both the domain \( X = \{x_1, x_2, \ldots, x_n\} \) and the range \([0,1]\) of the mapping helps us to describe fuzzy concepts and to prove fuzzy theorems. In the geometry of fuzzy sets, the fuzzy power set (i.e., the set of all fuzzy subsets of \( X \)) looks like a cube and a fuzzy set is a point in the cube. The set of all fuzzy subsets is isomorphic to the unit hypercube \( I^n = [0,1]^n \) [44]. So \( (X, I^n) \) defines a fundamental measurable space. Vertices of the cube \( I^n \) are non-fuzzy sets. The ordinary power set \( 2^X \), the set of all non-fuzzy subsets of \( X \) coincides with the Boolean n-cube \( B^n \). Fuzzy sets fill in the lattice \( B^n \) to produce the solid cube \( I^n \). The midpoint of the cube \( I^n \) is maximally fuzzy with all its membership values equal to \( \frac{1}{2} \). The midpoint is unique in two aspects:

1. The midpoint is the only fuzzy set \( A \) such that \( A = A \cap A^c = A \cup A^c = A \).

2. The midpoint is the only point in the cube \( I^n \) equidistant to each of the \( 2^n \) vertices of the cube. These extremes represent an end of the spectrum of logic and set theory. In this sense the midpoint represents the black hole of set theory. \( (X, I^n, m) \) defines the fundamental measure space of fuzzy theory. The measure \( m \)
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generalizes [43] the classical counti measure of combinatorics and measure theory. In general $m$ does not yield integer values. This is a possible learning and reasoning tool for intelligent machines which have a hyperbolic structure and therefore is called a fuzzy hyper cube [31]. These concepts are utilized in colour image enhancement problem.

A large number of contributions have been done to modelling nonlinear systems using Fuzzy logic controllers (FLC). Theoretically FLC or Fuzzy Logic Filter (FLF) have been proved to be universal approximators under very weak assumptions. That is, they are capable of approximating any real continuous function on a compact set to arbitrary accuracy [46]. In most of existing FLS, shapes, internal parameters of membership functions and fuzzy rules are determined and tuned through trial and error operators. Hence it is necessary to design FLF. In [46] L.X Wang observed that fuzzy systems with Gaussian membership functions are universal approximators. The filters are generally developed for grayscale images and it is possible to extend these filters to colour images by applying them on each colour component separately. Noise reduction is the process of removing a noise from a signal. A modified fuzzy logic system is constructed for reducing noise from a signal. The weight of the output is obtained from membership functions of distance measure. The two
sides of Gaussian type membership functions together with the product t-norm is used to get the weights for the FLF [50]. According to [8 and 9] the weights is equal to unity only if the distance between pixels is small.

The main theme of the thesis is “function approximation of fuzzy filters (fuzzy logic systems) in image signals”. In this process there are two approaches considered (i). Colour image enhancement by using newly designed fuzzy intensification operator in the low contrast image signals and (ii). By using Modified Fuzzy Basis Function to reduce noise from the noisy image signal.

The basic concepts used in the work are explained below. Terms and results appearing in the thesis, which are not listed below, are either standard or are explained as and when they first appear in the text.

1.1.1 Definition (Fuzzy set) [16]

Let $X$ be a non-empty set. A fuzzy subset $A$ of $X$ is defined to be a function $A : X \rightarrow [0,1]$. Equivalently, a fuzzy subset $A$ is defined to be the class of objects having the following representation,

$$A = \{(x, A(x)) : x \in X\},$$

where $A(x)$ is the membership level of $x \in X$.  

1.2 Geometrical representation of classical (crisp) set and fuzzy set [33]

**Fig. 1.1:** Set $A$ in classical set theory.  **Fig. 1.2:** Fuzzy set $A$

In classical set theory, a set $A$ is characterized by membership function from $X \rightarrow \{0,1\}$ defined by

$$A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

In fuzzy set theory, $A$ has no boarder line. It is a fuzzy subset of $X$, characterized by a membership function from $X \rightarrow [0,1]$. Here $A(x)$ is the degree of membership of the element $x$ in $A$.

**1.2.1 Definition (Membership function)**

A membership function is a curve that defines how each point in the input space is mapped to a membership value (or degree of membership) between 0 and 1.
1.2.3 Example

Let $X$ be the set of all people in a certain place and $A$ be the fuzzy subset “tall persons”. Then the membership function of $A$ is as follows:

![Membership function of A](image)

*Fig. 1.3: Membership function of A*

1.2.4 Remark

Membership functions can have a variety of shapes such as bell curves, trapezoids, triangles and Gaussian curves.

![Overlapping Triangular membership functions](image)

*Fig. 1.4: Overlapping Triangular membership functions*
In this work the triangle and Gaussian membership functions are used for approximation studies. Usually overlapping membership functions are used to describe a given variable in a fuzzy logic system.
Fuzzy logic programs are made up of a set of rules in the form is given by IF (Conditions) THEN (actions)

If the conditions are true then the control actions are performed.

In the fuzzy inference engine, fuzzy logic principles are used to combine the rules from fuzzy input sets to fuzzy output sets. The calculus of fuzzy sets is based upon a series of operations.

**Fig. 1.8: Operations used in Classical Set Theory and Fuzzy Set Theory**
In an experimental situation with a fuzzy logic system, the first step allows the fuzzification of the input and output values for each tuple. Basically, dividing each domain to several small regions and assigning a fuzzy membership function to each region. In image enhancement problem, these small regions constitute the hypercube structure for the intensification operator and pixel points in the small regions are used to define distance function which enables the construction of MFBF.

1.3. Fuzzy regions and Mappings

1.3.1 Algorithm

The following algorithm is used to derive a set of rules which can be applied to approximate a function described by a dataset. The basic idea is that each tuple in the dataset, containing many input variables and one output variable, represents a rule. The goal is to discover the most significant rules from the available and have the following steps:
1. Divide the input and output spaces into fuzzy regions.

2. Generate fuzzy rules from given tuples.

3. Assign a degree to each rule.

4. Create the combined set of rules.

5. Determine the mapping based on the set of rules.

1.3.2 Remark (Fuzzy rule for MISO case)[7]

Consider the fuzzy system \( g : U \subset R^n \rightarrow V \subset R \) and the fuzzy rule base consists of \( n \) rules in the following forms:

\[ R_i : \text{IF } x_1 \text{ is } A_1 \text{ and } x_2 \text{ is } A_2 \text{ and } \ldots \text{ and } x_n \text{ is } A_n \text{ THEN } y_i \text{ is } C_i, \]

\( i = 1, 2, \ldots, n \). Fuzzy rules can be divided into two.

(i) Fuzzy rule for Linear Mapping

A fuzzy rule for linear mapping is of the form

\[ \text{IF } x_1 \text{ is } A_1 \text{ and } x_2 \text{ is } A_2 \text{ and } \ldots \text{ and } x_n \text{ is } A_n \text{ THEN } y_1 \text{ is } B_1 \text{ and } y_2 \text{ is } B_2 \text{ and} \ldots \text{ and } y_n \text{ is } B_n \]
(ii) Fuzzy rule for Nonlinear Mapping

A fuzzy rule for non-linear mapping is of the form

\[ R^i : \text{IF } x_1 \text{ is } A^i_1 \text{ and } \ldots \ldots \text{ and } x_n \text{ is } A^i_n \text{ THEN } y \text{ is } B^i \]

\[ x_i \in U \text{ is the input variable and } y_i \in V \text{ is the output variable. The}
\]
fuzzy sets \(A_i \in U\) and \(B_i \in U\) are linguistic terms characterized by fuzzy membership functions \(A_i(x)\) and \(B_i(x)\) respectively.

1.3.3 Definition (Aggregation) [16, 17]

The process of combining output fuzzy sets into a single set is called aggregation. That is, aggregation is a process that unifies the outputs of all the rules.

1.3.4 Lemma [33]

The fuzzy logic system is continuous on entire

\[ \Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_n = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \]

if and only if the \(2^n\) fuzzy rules are assigned to each of the \(N_1 \times N_2 \times \ldots \times N_n\) different combinations of sub intervals, where \(\Omega\) is a compact subset of \(R^n\) \(\Box\)
1.3.5 Lemma [25]

Suppose $\Omega$ is divided into $\Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$ cubes each of which is $[C_j^i, C_j^i + 1] \times \ldots \times [C_j^n, C_j^n + 1]$ for $j_i = 0, 1, \ldots, N_i - 1$, where $i = 1, 2, \ldots, n$. Then $f(x)$ is continuous on $\Omega$ if and only if

$$P = \prod_{i=1}^{n} (N_i + 1)$$ fuzzy rules are added.

1.3.6 Some Definitions

Let $A$ be a fuzzy set on $X$

(i) For any $\alpha \in [0, 1]$, the $\alpha$ – cut of $A$ is defined as

$$\alpha_A = \{x \in X / A(x) \geq \alpha\}.$$ 

(ii) The support of $A$ is defined as

$$\text{supp} (A) = \alpha_A = \{x \in X / A(x) > 0\}.$$ 

(iii) The height of $A$ is defined as

$$\text{ht} (A) = \sup \{A(x) / x \in X\}.$$ 

(iv) $A$ is said to be normal if $\text{ht} (A) = 1$
1.3.7 Definition (Fuzzy number)

A fuzzy set $A$ on the real line is called a fuzzy number if

(i) its $\alpha$-cuts are closed intervals, $\forall \alpha > 0$.

(ii) $\text{supp}(A)$ is bounded; and

(iii) $A$ is normal.

1.3.8 Example

Let $A: \mathbb{R} \to I$ be defined by

$$A(x) = \begin{cases} 1 & \text{if } x \in [3,7] \\ 0 & \text{if } x \notin [3,7] \end{cases}$$

Then all the above conditions are satisfied and so $A$ is fuzzy number.
1.3.9 Remark

The notion of fuzzy number can be extended to a vector fuzzy number which is function from \( \mathbb{R}^{n} \rightarrow \mathbb{I}^{n} \) such that each coordinate satisfy the above conditions.

1.4 Fuzzy Inference System (FIS) [35]

FIS is a decision making unit which performs the inference operations on the rules, a fuzzification inference which transform the crisp inputs into degrees of match with linguistic values and a defuzzification inference which transform the fuzzy results of the inference into crisp output. Fuzzy Inference System is also known as fuzzy rule based systems.

An aggregation operation takes place among all the contributing rules of one of fuzzy variable. Aggregation is a method for using different implication results into one final result. An aggregation operation performs either union or an intersection operator among the implication results. The implication method is defined as the shaping of the consequent (the output fuzzy set) based on the antecedent. The
output for the implication process is a single number given by the antecedent and the output is a fuzzy set.

1.4.1 Remark (Fuzzy sets in Image processing) [11, 49]

Let $X$ be the spatial domain of a digital image. i.e. $X$ is the set of all Pixel positions. It is assumed that the image represents an object B with gray level adjusted to range between 0 and 1. Thus the image can be considered as a fuzzy set in image processing as $f : X \rightarrow I$ and for any $x$; $f(x)$ can be thought of as the degree of membership of pixel $x$ in the object represented in the image.

Cube mappings are very useful in the image enhancement problem. In a multi input fuzzy systems there is cube mapping. If we consider this cube as unit hypercube then it is easy to define the entropy. By the Minimization of entropy in the cube, an image can be enhanced. The studies in [27,31] show that the existence of the reasoning mechanism by which a fuzzy hyper cube process all the rules in one clock period in parallel with the property that intelligent machine has a hyperbolic structure and there fore call it as a Fuzzy
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hypercube $H^F$. Fuzzy hyper cube can be applied to control a class of complex and highly non linear system which suffer from vagueness and uncertainty. The fuzzy hyper cube has its origin in the Fuzzy Associative Memory (FAM) [21]. From operational point of view each fuzzy set in $H^F$ preserves its memberships in Mamdani’s sense [25] rather than using fuzzy arithmetic results [2]. The concept of geometric distance is studied in [8,42,43 and 63]. This is applied in image processing. Also the approximate learning (training) and approximate reasoning (inferencing) are the key procedure in $H^F$. The distance between the two points in a cube $I^n$ lead to measure the size and fuzziness of fuzzy sets or entropy. In image denoising process the distance between fuzzy points (image pixels) is used to define rules and hence the function approximation.

A fuzzy system defines mapping between the cubes. Thus a fuzzy system $g$ is a transformation $g : I^n \rightarrow I^p$ where $I^n$ denote the set of all fuzzy subsets of the domain space and $I^p$ denotes the set of all
fuzzy subsets range space. Then the fuzzy power sets $F(2^X)$ and $F(2^Y)$ replace $I^n$ and $I^p$. Here $X$ and $Y$ are subsets of $I^n$ and $I^p$ respectively.

In our study, the concept of n-dimensional unit hypercube is used in chapter III. The union and intersection of the fuzzy sets helps to find out the maximal and minimal element in a unit hypercube. So it is easy to define the valuations (as shown in chapter III) of fuzzy sets with the help of lattice theory. This leads to the additive entropy.

**1.4.2 Definition (Hypercube)**

A hypercube is an $n$-dimensional analogue of a square ($n = 2$) and a cube ($n = 3$). It is a closed, compact, convex figure consisting of groups of opposite parallel line segments aligned in each of the space’s dimensions, at right angles to each other. This can be generalized to any number of dimensions. A point in a hypercube is of dimension zero. If this point moves one unit length, it becomes a line segment, which is a unit hypercube of dimension one. If this line
segment moves its length in a perpendicular direction from itself and it is a two-dimensional square. If the square moves one unit length in the direction perpendicular to the plane it lies on, it will generate a three-dimensional cube. An $n$-dimensional hypercube is also called an $n$-cube.

1.4.3 Definition (Union and Intersection in Unit hypercube) [23]

Let $A$ and $B$ be two fuzzy sets. Then the union and intersection are respectively defined by

\[
(A \cup B)(x) = \max[A(x), B(x)]
\]

\[
(A \cap B)(x) = \min[A(x), B(x)] , \forall x \in X
\]

*Fig. 1.10:* Geometric representation of union and intersection of fuzzy sets in two dimension
1.4.4 Remark

In a unit hypercube, the fuzzy sets \( A, A^c, A \cup A^c, A \cap A^c \) contract to the mid point as \( A \). The same contraction and expansion occurs in \( n \)-dimensions for the \( 2^n \) fuzzy sets defined by all combinations of \( A(x_1) \) and \( A^c(x_1), \ldots, A(x_n) \) and \( A^c(x_n) \).

1.4.5 Definition (Fuzzy Cross over point)[22]

The element \( x \in X \) at which \( A(x) = 0.5 \) in the unit hypercube is called cross over point of \( A \).

1.4.6 Fuzzy set in a unit square [41]

Consider the universal set consisting of two elements (say) \( X = \{x_1, x_2\} \). Each point in the interior of the unit square represents a subset of \( X \). Since the co-ordinates of the representation correspond to the membership values of the elements in the fuzzy set.
\( \phi = (0, 0) \)  
\( \{x_1\} = (1, 0) \)  
\( X = \{x_1, x_2\} = (1, 1) \)  
\( \{x_2\} = (0, 1) \)

**Fig. 1.11: Geometric visualization of fuzzy sets**

The point M in the above figure corresponds to the fuzzy set \( M = \frac{0.5}{x_1} + \frac{0.3}{x_2} \). The point Y in the centre of the cube represents the most diffuse of all possible fuzzy sets of \( X \) (maximal fuzzy).

\( Y = \frac{0.5}{x_1} + \frac{0.5}{x_2} \).

### 1.5 Necessary Approximation Conditions of fuzzy systems [23]

#### 1.5.1 Problem

Designate a family of multi-input single output continuous functions \( C[X] \) that have a finite number of rules defined is the minimum amount of information necessary for characterizing major
features of a well behaved continuous function on the m-dimensional product space
\[ \Omega = \Omega_1 \times \ldots \times \Omega_m = [a_1, b_1] \times \ldots \times [a_m, b_m]. \]

Suppose the following are given.

1. An arbitrary approximation error bound \( \epsilon \).

2. \( f(x) \) is continuous on entire \( \Omega \) if and only if all the \( 2^m \) fuzzy rules are assigned to each of the cube \( \Omega_1 \times \Omega_2 \times \ldots \times \Omega_m \).

3. Values of \( f(x) \) at every vertex of \( \Omega \) (there are \( 2^m \) such values).

Then for some necessary conditions under which there always exists a MISO fuzzy system \( g(x) \) that satisfies
\[ \max_{x \in \Omega} |g(x) - f(x)| < \epsilon. \]

4. Suppose there are continuous functions \( f_k(x) \) and a sequence of uniform approximation error bounds \( \epsilon_k \), where \( \epsilon_k \to 0 \) as \( k \to \infty \) then
\[ \max_{x \in \Omega} |g(x) - f_k(x)| \leq \epsilon_k \forall k. \]
That is fuzzy system with product inference engine, singleton fuzzifier, center average defuzzifier and Gaussian membership functions are universal approximators.

1.5.2 Remark

Suppose $f : X \rightarrow Y$ is measurable and bounded, then the boundedness of $f$ ensures a uniform approximation.

The following theorem shows the continuity of a function when it is transformed from an input space to an output space. This gives an idea about the existence of the partial derivatives of the entropy with respect to the independent variables in the output space of the image enhancement process.

1.5.3 Theorem (Uniform error bound) [35]

Let the necessary approximation conditions in a fuzzy logic system is satisfied. Consider a class $C[X]$ of all continuous functions $f : \Omega \rightarrow R^n$ whose Jacobians are uniformly bounded by a positive real number $K$,
i.e. \( |f'(x)| \leq K, \forall x \in \Omega, f(x) \in C[X] \)

Select control representative values as the samples of \( f(x) \) at the vertices of the hypercube cells,

\[
u^k_{i_1, i_2, \ldots, i_n} = f^k(X^1_{i_1}, X^2_{i_2}, \ldots, X^n_{i_n}).
\]

Then for any \( f(x) \) from the class of bounded functions and \( x \in \Omega \), we have

\[
|f(x) - g(x)| \leq B\delta
\]

where \( g(x) \) is, the reasoning surface of the fuzzy system, \( B \) is the bound of approximation and \( \delta \) is size of the hypercube interval for \( \Omega \) defined by the selected membership functions \( \square \)

1.5.4 Theorem

Suppose input \( x \in U \) is a compact set in \( R^n \). Then for any given real continuous function \( f(x) \) on \( U \) and any small \( \varepsilon > 0 \), there exists a fuzzy system \( g(x) \) such that

\[
\sup_{x \in U} |f(x) - g(x)| < \varepsilon \quad \square
\]
1.5.5 Definition (Convex combination)[41]

A convex combination is a linear combination of data points which can be vectors, scalars, or more generally points in an affine space where all coefficients are non-negative and sum up to 1.

It is called "convex combination", since all possible convex combinations (given the base vectors) will be within the convex hull of the given data points. In fact, the set of all convex combinations constitutes the convex hull.

We consider only convex fuzzy sets whose membership functions are first increasing and then decreasing for increasing values of elements in the universe of discourse. Typical shape of convex normal fuzzy set and a non convex normal fuzzy set are shown below:

![Fig. 1.12: (a) Convex fuzzy set](image1)
![Fig. 1.12: (b) Non Convex Fuzzy Set](image2)
1.6 Noise Reduction

Noise is any undesired information that contaminates an image. It is the result of errors in the image acquisition process that result in pixel values that do not reflect the true intensities of the real scene. Noise can be generated during image capture, transmission, storage, copying, scanning and display. Gaussian noise can be expressed in terms of its mean and variance values. The edges give the image the appearance depth and sharpness. A loss of edges makes the image appear blurred or unfocused. Several filtering techniques have been proposed over the years. The filtering involves the removal or reduction of the Gaussian noise while preserving or enhancing edges. Fuzzy techniques have been applied in several domains of image processing (Menhardt, 1988). Here we have developed Modified Fuzzy Basis Function (MFBF) such that noise in an image is reduced and hence the modified image approximates the original image.

Digital images are valuable and important sources of information for a variety of research and application area. In this case,
we will focus on fuzzy techniques for digital image corrupted with Gaussian noise. The main goal of Gaussian noise reduction method is to suppress the noise while preserving the fine details and edge elements. Three main types of noise exist: impulse noise, additive noise, and multiplicative noise. Impulse noise is usually characterized by some portion of image pixels that are corrupted, leaving the remaining pixels unchanged.

1.6.1 Definition (Centroid Defuzzification)

The issue of combining several rules to determine a single value for the output is known as defuzzification problem. The Centroid method is one of the basic methods for defuzzification. The Centroid method is based on finding a ‘balance point’ of a property that can be the total geometric figure, the weight of each fuzzy set, the area of the largest fuzzy set or the area of the largest intersection. This is directly applied in noise reduction.
1.6.2 Remark

By the iterative process it can be seen that the fuzzy filter approximates the function and according to [50], Gaussian Membership functions are best suited for function approximation in fuzzy image processing. The reconstruction of this membership function leads to the image enhancement problem in chapter and so to the improved version $g^*$ of $g$ such that $g^*$ approximates $f$. But in image noise reduction problem in chapter V, the combination with triangular membership function helps to approximate $g$ with the original function $f$. The major theorem of chapter IV shows that modification can be made in a Fuzzy Logic Filter of a Fuzzy system. So we have designed a modified membership function in chapter V and checked its effectiveness in image noise reduction.

1.6.3 Fuzzy Filters or Fuzzy logic system in image processing

The entire static mapping performed by the fuzzy inference systems with singleton fuzzifier, product inference centroid defuzzifier has a basis function of the form:
The fuzzy membership functions $A^k_i(x_i)$ are normal.

By using this concept, we introduce the new filter with fuzzy basis function (FBF):

$$g_{a,x} = \frac{\sum_{k_x=1}^p \sum_{k_{x-1}=1}^p \cdots \sum_{k_1=1}^p a \prod_{i=1}^n A^k_i(x_i)}{\sum_{k_x=1}^p \sum_{k_{x-1}=1}^p \cdots \sum_{k_1=1}^p \prod_{i=1}^n A^k_i(x_i)}$$

where $\Phi_{i,j}$ are the fuzzy basis functions. This fuzzy filter is applied in the function approximation in chapter V.