CHAPTER VI

ON THE QUASI-HADAMARD PRODUCT OF CERTAIN UNIVALENT FUNCTIONS
6.1. Let $H$ denote the family of functions $f$ which are analytic in the unit disc $E = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let $S$ denote the subfamily of $H$ consisting of functions that are univalent in $E$. A function $f \in S$ is in $S^\#(\alpha)$, the class of starlike functions of order $\alpha (0 \leq \alpha < 1)$ if and only if $\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha$, $z \in E$. Further, $f \in S$ is in $C(\alpha)$, the class of convex functions of order $\alpha$ if and only if $zf'(z) \in S^\#(\alpha)$.

Let $T$ denote the subclass of $S$ consisting of functions whose non-zero coefficients, from the second on, are negative; that is, an analytic and univalent function $f \in T$, if and only if, it can be expressed in the form

\begin{equation}
(6.1.1) \quad f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0.
\end{equation}

We denote by $T^\#(\alpha)$ and $C^\#(\alpha)$ ($0 \leq \alpha < 1$) the classes obtained by taking intersections, respectively, of the classes $S^\#(\alpha)$ and $C(\alpha)$ with $T$. These classes were introduced and studied by Silverman [127].
For a function

\[ f(z) = \sum_{k=0}^{\infty} a_k z^k \]

analytic in \( E \), the differential operator \( D^n \), \( n \in \mathbb{N}_0 = \{0,1,2,\ldots\} \) is defined by

(i) \( D^0f(z) = f(z) \)

(ii) \( D^1f(z) = zf'(z) \)

(iii) \( D^n f(z) = D(D^{n-1} f(z)) \), \( n \geq 2 \).

This operator was introduced by Salagean [124]. We note that if \( f \) is defined by (6.1.2), then \( D^n f(z) = \sum_{k=0}^{\infty} k^n a_k z^k \).

Let \( S_n^*(\alpha) \) denote the class of functions \( f \in T \) such that

\[ \text{Re} \left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\} > \alpha, \quad n \in \mathbb{N}_0 \]

for all \( z \in E \) and \( 0 \leq \alpha < 1 \). It is readily seen that \( S_0^*(\alpha) = T^*(\alpha) \) and \( S_1^*(\alpha) = C^*(\alpha) \), \( 0 \leq \alpha < 1 \). Further, we note that the condition (6.1.3) is equivalent to

\[ \left| \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right| < 1 \]

or

\[ \left| \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right| < 2 \left( \frac{D^{n+1}f(z)}{D^n f(z)} - \alpha \right) - \left( \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right) \]
A necessary and sufficient condition for a function $f(z)$ given by (6.1.1) to be in $S^*(a)$ is that

$$ (6.1.5) \quad \sum_{k=2}^{\infty} k^n(k-a)a_k \leq (1-a). $$

A more general form of this result can be found in [105].

From (6.1.5), it follows that for any positive integer $n$

$$ S^*_n(a) \subset S^*_{n-1}(a) \subset \ldots \subset S^*_2(a) \subset C^*(a) \subset T^*(a), $$

and

$$ S^*_n(a) \cap S^*_n(a_1), \quad 0 \leq a_1 \leq a_2 < 1. $$

We also observe that for every $n \in \mathbb{N}_0$, the class $S^*_n(a)$ is non-empty as the functions of the form

$$ f_n(z) = z - \sum_{k=2}^{\infty} \frac{(1-a)^{\lambda_k} z^k}{k^n(k-a)}, $$

where $0 \leq a < 1$, $\lambda_k \geq 0$ and $\sum_{k=2}^{\infty} \lambda_k \leq 1$, satisfy the condition (6.1.5).

If $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ ($a_k \geq 0$) and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ ($b_k \geq 0$), then the quasi-Hadamard product of $f(z)$ and $g(z)$ is defined by

$$ (f \ast g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k. $$
Similarly, we can define the quasi-Hadamard product of more than two functions. We note that Padmanabhan and Manjini [105] used the phrase "Modified Hadamard product" instead of "Quasi-Hadamard product" in this definition.

Problems concerning quasi-Hadamard product of two or more functions have been considered by many researchers [60, 61, 96, 97, 102, 105]. Recently, Owa [98] has proved the following results for the quasi-Hadamard product of more than two functions.

**Theorem 1.** Let the functions $f_i(z)$ be in $T^*(a_i)$, $0 \leq a_i < 1$ for each $i = 1, 2, \ldots, m$, respectively. If $\sum_{i=1}^m a_i \leq 1$, then the quasi-Hadamard product

$$(f_1 \ast f_2 \ast \ldots \ast f_m)(z) \text{ belongs to } T^\# \left( \prod_{i=1}^m a_i \right).$$

**Theorem 2.** Let the functions $f_i(z)$ be in $C^*(a_i)$ for each $i = 1, 2, \ldots, m$, respectively. If $\sum_{i=1}^m a_i \leq 1$, then the quasi-Hadamard product $(f_1 \ast f_2 \ast \ldots \ast f_m)(z)$ belongs to $C^\#(\prod_{i=1}^m a_i)$.

**Theorem 3.** Let the functions $f_i(z)$ be in $T^*(a_i)$ for each $i = 1, 2, \ldots, m$, respectively. Let the functions $g_j(z)$ be in $C^*(\beta_j)$ ($0 \leq \beta_j < 1$) for each $j = 1, 2, \ldots, q$, respectively. If $\sum_{i=1}^m a_i + \sum_{j=1}^q \beta_j \leq 1$, then the
quasi-Hadamard product \((f_1 * f_2 * \ldots * f_m) * (g_1 * g_2 * \ldots * g_q)(z)\) belongs to \(C^k\left(\prod_{i=1}^{m} \alpha_i \cdot \prod_{j=1}^{q} \beta_j \right)\).

**Theorem 4.** Let the functions \(f_i(z)\) be in the same class \(C^k(\alpha)\) for every \(i = 1, 2, \ldots, m\), and let \(0 < \alpha \leq r_0\), where \(r_0\) is a root of the equation \(2^m(1-\alpha^m)-(1-\alpha)^m = 0\) in the interval \((0, \frac{1}{m})\). Then the quasi-Hadamard product \((f_1 * f_2 * \ldots * f_m)(z)\) belongs to \(C^k(m\alpha)\).

However, the stringent restrictions \(\sum_{i=1}^{m} \alpha_i \leq 1\) and \(\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{q} \beta_j \leq 1\) in Theorems 1, 2 and 3 diminish the utility of these results. Vinod Kumar [61] observed that the technique given Owa [98] fail to prove these results if \(\sum_{i=1}^{m} \alpha_i > 1\) and \(\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{q} \beta_j > 1\). By using a different technique, Kumar [61] has proved these results without restricting \(\sum_{i=1}^{m} \alpha_i\) and \(\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{q} \beta_j\) and there by improving Theorems 1, 2 and 3 of Owa [98]. We state the results obtained by Kumar [61].

**Theorem A.** For each \(i = 1, 2, \ldots, m\), let the functions \(f_i(z)\) belong to the classes \(T^k(\alpha_i)\), respectively. Then, the quasi-Hadamard product \((f_1 * f_2 * \ldots * f_m)(z)\) belongs to the class \(S^k_{m-1}(\alpha^k)\), where \(\alpha^k = \max\{\alpha_1, \alpha_2, \ldots, \alpha_m\}\).
Theorem B. For each \(i = 1, 2, \ldots, m\), let the functions \(f_i(z)\) belong to the classes \(C^*(\alpha_i)\), respectively. Then, the quasi-Hadamard product \((f_1 \ast f_2 \ast \ldots \ast f_m)(z)\) belongs to the class \(S_{2m-1}(\alpha^*)\), where \(\alpha^* = \max\{\alpha_1, \alpha_2, \ldots, \alpha_m\}\).

Theorem C. For each \(i = 1, 2, \ldots, m\), let the functions \(f_i(z)\) belong to the classes \(T^*(\alpha_i)\), respectively and for each \(j = 1, 2, \ldots, q\), let the functions \(g_j(z)\) belong to the classes \(C^*(\beta_j)\), respectively. Then, the quasi-Hadamard product \((f_1 \ast f_2 \ast \ldots \ast f_m) \ast (g_1 \ast g_2 \ast \ldots \ast g_q)(z)\) belongs to the class \(S_{m+2q-1}(\gamma)\), where \(\gamma = \max\{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \ldots, \beta_q\}\).

Theorem D. For each \(i = 1, 2, \ldots, m\), let the functions \(f_i(z)\) belong to the class \(C^*(\alpha)\), and let \(0 < \alpha \leq r_0\), where \(r_0\) is a root of the equation \(2^m(1-mr)-(1-r)^m = 0\) in the interval \((0, \frac{1}{m})\). Then, the quasi-Hadamard product \((f_1 \ast f_2 \ast \ldots \ast f_m)(z)\) belongs to the class \(S_{m-1}(\alpha\alpha)\).

The object of the present chapter is to further improve the results of Kumar [61]. In section 6.2, we improve Theorems A, B, C and D of Kumar [61] by employing a different technique. The classes to which the quasi-Hadamard product belongs, determined by us are smaller than those given in [61]. Thus, our results are more inclusive as well as applicable. Finally, in section 6.3,
distortion theorems for the class $S_n^\#(\alpha)$ are proved in terms of a general class of fractional integral operators involving Gauss hypergeometric series. The results obtained here besides generalizing some of the work of Silverman [127] yields a number of new results.

Unless otherwise mentioned, we assume throughout this chapter that the functions of the form

$$f_1(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_{k,i} > 0$$

and

$$g_j(z) = z - \sum_{k=2}^{\infty} b_k z^k, \quad b_{k,j} > 0$$

are analytic in the unit disc $E$. We, further, assume that $0 < \alpha_i < 1$, $0 < \beta_j < 1$ and $n_1 \in N_0 = \{0, 1, 2, \ldots \}$.

6.2. In this section, we prove some sharp results concerning the quasi-Hadamard product of more than two functions in the class $S_n^\#(\alpha)$.

First, we prove

Theorem 6.2.1. Let the functions $f_1(z)$ be in the classes $S_{n_1}^\#(\alpha_i)$ for each $i = 1, 2$, respectively. Then, the quasi-Hadamard product $(f_1 \ast f_2)(z)$ belongs to $S_p^\#(\gamma)$ where $p = n_1 + n_2 + 1$ and
(6.2.1) \[ \gamma = \gamma(a_1, a_2) = \frac{2(a_1 + a_2) - 3a_1a_2}{2 - a_1a_2} \]

The result is best possible.

**Proof.** In view of (6.1.5), it is sufficient to prove that

\[ \sum_{k=2}^{\infty} n_1^{n_1+n_2+1} (k-\gamma) a_{k,1} \cdot a_{k,2} \leq (1 - \gamma). \]

Since \( f_i(z) \in S_n(a_i) \) for \( i = 1, 2 \), we have

\[ \sum_{k=2}^{\infty} n_1^{k}(k-\gamma) a_{k,1} \cdot a_{k,2} \leq 1 - a_i. \]

Therefore, by virtue of Cauchy-Schwarz inequality,

\[ (6.2.2) \sum_{k=2}^{\infty} \left\{ \frac{n_1^{n_1+n_2} (k-\gamma)(k-\gamma)}{(1-a_1)(1-a_2)} \right\}^{1/2} \sqrt{a_{k,1} \cdot a_{k,2}} \leq 1. \]

Thus, we need to find the largest \( \gamma \) such that

\[ \sum_{k=2}^{\infty} n_1^{n_1+n_2} (k-\gamma)(k-\gamma) a_{k,1} \cdot a_{k,2} \leq \sum_{k=2}^{\infty} n_1^{n_1+n_2} (k-\gamma)(k-\gamma) \left\{ \frac{n_1^{n_1+n_2} (k-\gamma)(k-\gamma)}{(1-a_1)(1-a_2)} \right\}^{1/2} \sqrt{a_{k,1} \cdot a_{k,2}} \]

or, equivalently, that

\[ \sqrt{a_{k,1} \cdot a_{k,2}} \leq \left\{ \frac{n_1^{n_1+n_2} (k-\gamma)(k-\gamma)}{(1-a_1)(1-a_2)} \right\}^{1/2} \cdot \frac{1 - \gamma}{k^{n_1+n_2+1} (k-\gamma)}, k \geq 2. \]
In view of (6.2.2), it is enough to find the largest \( \gamma \) such that

\[
\frac{(1-\alpha_1)(1-\alpha_2)}{n_1+n_2} \frac{1}{2} \leq \frac{n_1+n_2}{k} \frac{(k-\alpha_1)(k-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)} \frac{1}{2} \frac{n_1+n_2+1}{k} \frac{(k-\gamma)}{(k-\alpha_1)(k-\alpha_2)}, \quad k \geq 2.
\]

The above inequality is equivalent to

\[
\gamma \leq \frac{(k-\alpha_1)(k-\alpha_2)-k^2(1-\alpha_1)(1-\alpha_2)}{(k-\alpha_1)(k-\alpha_2)-k(1-\alpha_1)(1-\alpha_2)} = \frac{k(\alpha_1 + \alpha_2) - (k+1)\alpha_1 \alpha_2}{k - \alpha_1 \alpha_2}, \quad k \geq 2.
\]

We denote the right hand side of (6.2.3) by \( \Phi(k) \) and show that \( \Phi(k) \) is an increasing function of \( k \geq 2 \). This will be true if for \( k \geq 2 \)

\[
(6.2.4) \quad \Phi(k+1) - \Phi(k) = \frac{(k+1)(\alpha_1 + \alpha_2) - (k+2)\alpha_1 \alpha_2}{k + 1 - \alpha_1 \alpha_2} - \frac{k(\alpha_1 + \alpha_2) - (k+1)\alpha_1 \alpha_2}{k - \alpha_1 \alpha_2}.
\]

On simplifying (6.2.4), we have

\[
\Phi(k+1) - \Phi(k) = \frac{(1-\alpha_1)(1-\alpha_2)}{(k+1-\alpha_1 \alpha_2)(k-\alpha_1 \alpha_2)}.
\]
which is certainly positive for \( k > 2 \) and \( 0 \leq \alpha_1, \alpha_2 < 1 \).

Thus (6.2.4) holds true. Putting \( k = 2 \) in (6.2.3) we deduce (6.2.1).

The result is best possible for functions of the form

\[
f_i(z) = z - \frac{1 - \alpha_i}{2^{n_i}(2 - \alpha_i)} z^2, \quad i = 1, 2.
\]

The above theorem can be extended for more than two functions which is as follows.

**Theorem 6.2.2.** Let the functions \( f_i(z) \) be in the classes \( S^{\#}_{n_i}(\alpha_i) \) for each \( i = 1, 2, \ldots, m \), respectively.

Then, the quasi-Hadamard product \((f_1 \ast f_2 \ast \ldots \ast f_m)(z)\) belongs to \( S^{\#}_{p}(\gamma_m) \), where \( p = n_1 + n_2 + \ldots + n_m + m - 1 \) and \( \gamma_m \) is defined by

\[
(6.2.5) \quad \gamma_m = \gamma_m(\alpha_1, \alpha_2, \ldots, \alpha_m) = \frac{\prod_{i=1}^{m} (2 - \alpha_i) - 2^m \prod_{i=1}^{m} (1 - \alpha_i)}{\prod_{i=1}^{m} (2 - \alpha_i) - 2^{m-1} \prod_{i=1}^{m} (1 - \alpha_i)}
\]

The result is best possible.

**Proof.** We prove by induction on \( m \). From Theorem 6.2.1, it follows that the result is true for \( m = 2 \). Let us assume that (6.2.5) is true for \( m = s-1 \). Then, we shall prove it for \( m = s \). By assumption, the quasi-Hadamard
product \((f_1 * f_2 * \ldots * f_{s-1})(z)\) belongs to the class 
\(S_{p_0}^{s-1}(\gamma_{s-1})\), where \(p_0 = n_1 + n_2 + \ldots + n_{s-1} + (s-2)\) and \(\gamma_{s-1}\) is
given by
\[
\gamma_{s-1} = \frac{\prod_{i=1}^{s-1} (2-\alpha_i) - 2^{s-1} \sum_{i=1}^{s-1} (1-\alpha_i)}{\prod_{i=1}^{s-1} (2-\alpha_i) - 2^{s-2} \sum_{i=1}^{s-1} (1-\alpha_i)}.
\]

(6.2.6)

Since \(f_s(z) \in S_{n_s}^s(\alpha_s)\), by using Theorem 6.2.1, we deduce
that the quasi-Hadamard product \(((f_1 * f_2 * \ldots * f_{s-1})*f_s)(z)\)
belongs to the class \(S_{p_1}^{s}(\gamma_s)\), where \(p_1 = p_0 + n_s + 1\) and \(\gamma_s\)
is given by
\[
\gamma_s = \frac{2(\gamma_{s-1} + \alpha_s) - 3\gamma_{s-1} \cdot \alpha_s}{2 - \gamma_{s-1} \cdot \alpha_s}.
\]

(6.2.7)

Using (6.2.6) in (6.2.7) we deduce the assertion (6.2.5).
This completes the proof of Theorem 6.2.2.

It is easily seen that the result is best possible
for the functions of the form
\[
f_i(z) = z - \frac{1 - \alpha_i}{2^{n_i}(2 - \alpha_i)} z^2, \quad 1 \leq i \leq m.
\]

Remark. From (6.2.1), we note that \(\gamma > \alpha_1\) and
\(\gamma > \alpha_2\). Similarly, from (6.2.5), it follows that for
each \(i = 1, 2, \ldots, m\)
\[
\gamma_i \geq \alpha_j, \quad j = 1, 2, \ldots, i.
\]
This implies that

\[ \gamma_1 \geq \max \{ a_1, a_2, \ldots, a_1 \} = \lambda_1 \quad \text{(say)}. \]

Thus,

\[ S_n^*(\gamma_1) \subseteq S_n^*(\lambda_1) \]

for each \( i = 1, 2, \ldots, m \) and \( n \in \mathbb{N}_0 \). We, further, note that the containment is proper if \( m > 2 \).

Putting \( n_1 = 0 \) for each \( i = 1, 2, \ldots, m \) in Theorem 6.2.2, we have

**Corollary 6.2.1.** Let the functions \( f_i(z) \) be in the classes \( T^*(a_i) \) for each \( i = 1, 2, \ldots, m \), respectively. Then, the quasi-Hadamard product \( (f_1 \ast f_2 \ast \ldots \ast f_m)(z) \) belongs to the class \( S_{m-1}^*(\gamma_m) \subseteq S_{m-1}^*(\lambda) \), where \( \gamma_m \) is defined as in (6.2.5) and \( \lambda = \max \{ a_1, a_2, \ldots, a_m \} \). The result is best possible.

Letting \( n_1 = 1 \) for each \( i = 1, 2, \ldots, m \) in Theorem 6.2.2, we get the following result.

**Corollary 6.2.2.** Let the functions \( f_i(z) \) be in the classes \( C^*(a_i) \) for each \( i = 1, 2, \ldots, m \) respectively. Then, the quasi-Hadamard product \( (f_1 \ast f_2 \ast \ldots \ast f_m)(z) \) belongs to \( S_{2m-1}^*(\gamma_m) \subseteq S_{2m-1}^*(\lambda) \), where \( \gamma_m \) is defined as in (6.2.5) and \( \lambda = \max \{ a_1, a_2, \ldots, a_m \} \).

The result is best possible.
Corollary 6.2.3. For each \( i = 1, 2, \ldots, m \), let the functions \( f_i(z) \) be in the classes \( T^*(a_i) \), respectively; and for each \( j = 1, 2, \ldots, q \), let the function \( g_j(z) \) be in the classes \( C^*(\beta_j) \), respectively. Then, the quasi-Hadamard product \( (f_1 \ast f_2 \ast \ldots \ast f_m \ast g_1 \ast g_2 \ast \ldots \ast g_q)(z) \) belongs to the class \( S^*(\gamma_{m, q}) \subseteq S^*(\lambda) \), where \( p = m + 2q - 1 \), \( \lambda = \max\{a_1, a_2, \ldots, a_m, \beta_1, \beta_2, \ldots, \beta_q\} \) and \( \gamma_{m, q} \) is given by

\[
\gamma_{m, q} = \gamma(a_1, a_2, \ldots, a_m, \beta_1, \beta_2, \ldots, \beta_q) = \prod_{i=1}^{m} (2 - a_i) \prod_{j=1}^{q} (2 - \beta_j) - 2^{m+q} \prod_{i=1}^{m} (1 - a_i) \prod_{j=1}^{q} (1 - \beta_j)
\]

The result is best possible.

The proof of Corollary 6.2.3 follows from Corollaries 6.2.1 and 6.2.2 followed by Theorem 6.2.1.

Remark. In view of the remark following Theorem 6.2.2, we observe that the corollaries 6.2.1, 6.2.2 and 6.2.3 provide better estimate when compared with Theorems A, B and C of Kumar [61].

Theorem 6.2.3. For each \( i = 1, 2, \ldots, m \), let the functions \( f_i(z) \) be in the classes \( C^*_c(\alpha), 0 \leq \alpha < 1 \). Then, the quasi-Hadamard product \( (f_1 \ast f_2 \ast \ldots \ast f_m)(z) \) belongs
Therefore,

(6.2.8) \[ \gamma = \gamma(m, \alpha) = \frac{2 \left\{ (2-\alpha)^m - (1-\alpha)^m \right\}}{2(2-\alpha)^m - (1-\alpha)^m}. \]

The result is best possible.

**Proof.** Since \( f_i(z) \in C_\alpha \) for each \( i = 1, 2, \ldots, m \), we have

\[ \sum_{k=2}^{\infty} k(k-\alpha)a_{k,i} \leq (1-\alpha). \]

Therefore,

(6.2.9) \[ \sum_{k=2}^{\infty} k^m \frac{(k-\alpha)^m}{1-\alpha} \prod_{i=1}^{m} a_{k,i} \leq 1. \]

We have to find the largest \( \gamma = \gamma(m, \alpha) \) such that

(6.2.10) \[ \sum_{k=2}^{\infty} k^{m-1} \frac{(k-\gamma)^m}{1-\gamma} \prod_{i=1}^{m} a_{k,i} \leq 1. \]

In view of (6.2.9), the above inequality is satisfied if

\[ \frac{k-\gamma}{1-\gamma} \leq \frac{k(k-\alpha)^m}{(1-\alpha)^m}, \quad k \geq 2. \]

That is, if

(6.2.10) \[ \gamma \leq \frac{k \left\{ (k-\alpha)^m - (1-\alpha)^m \right\}}{k (k-\alpha)^m - (1-\alpha)^m}, \quad k \geq 2. \]

We shall prove that the right hand side of (6.2.10) is an increasing function of \( k \geq 2 \). This will be true if
the function

(6.2.11) \( \phi_m(k) = (k^2 - 1)(k+1 - \alpha)^m - k^2(k-\alpha)^m + (1-\alpha)^m \)

is non-negative for each \( k \geq 2 \) and \( m \geq 1 \). Now,

(6.2.12) \( \phi_1(k) = k(k-1) > 0. \)

Also, from the recursive formula

\[
\phi_{m+1}(k) = (k-\alpha)\phi_m(k) + (k-1) \left\{ (k+1)(k+1 - \alpha)^m - (1-\alpha)^m \right\}, \quad m=1,2,\ldots,
\]

we have

(6.2.13) \( \phi_{m+1}(k) > (k-\alpha)\phi_m(k), \quad k \geq 2. \)

Thus, by using (6.2.12) and (6.2.13), we deduce that \( \phi_m(k) \) is non-negative for \( k \geq 2 \) and \( m \geq 1 \). Now, by putting \( k = 2 \) in the right hand side of (6.2.10), we get the required result. This completes the proof of Theorem 6.2.3.

The result is best possible for the functions of the form

(6.2.14) \( f_1(z) = z - \frac{(1-\alpha)}{2(2-\alpha)} z^2, \quad 1 \leq 1 \leq m. \)

Taking \( m = 1 \), in Theorem 6.2.3, we get the following comparable result due to Silverman [127].

**Corollary 6.2.4.** For \( 0 < \alpha < 1 \), we have

\[
C^*(\alpha) \subseteq T^*(2\frac{2}{3-\alpha}).
\]

The result is best possible.
Theorem 6.2.4. For each \( i = 1, 2, \ldots, m \), let the functions \( f_i(z) \) belong to the class \( C^s(\alpha) \), and let
\[
0 < \alpha \leq r_0,
\]
where \( r_0 \) is a root of the equation
\[
2^m(1-mr) - (1-r)^m = 0 \text{ in } (0, \frac{1}{m}).
\]
Then, the quasi-Hadamard product \((f_1 * f_2 * \ldots * f_m)(z)\) belongs to the class \( S^s_{m-1}(r) \subseteq S^s_{m-1}(am) \), where \( r \) is defined as in (6.2.8).

The result is best possible.

Proof. The first half of the theorem, that is,
the quasi-Hadamard product \((f_1 * f_2 * \ldots * f_m)(z)\) belongs to the class \( S^s_{m-1}(r) \) follows from Theorem 6.2.3.

It remains to show that
\[
S^s_{m-1}(r) \subseteq S^s_{m-1}(am)
\]
where \( m \geq 1 \), \( am < 1 \) and \( r \) is defined as in (6.2.8).

This will be true if
\[
2 \left\{ (2-\alpha)^m - (1-\alpha)^m \right\} \geq am \cdot \left\{ 2(2-\alpha)^m - (1-\alpha)^m \right\},
\]
or, equivalently, if
\[
2(1-am)(2-\alpha)^m - (2-am)(1-\alpha)^m \geq 0.
\]

Since
\[
(2-\alpha)^m \geq 2^{m-1}(2-am) \quad (m \geq 1, \, am < 1, \, 0 \leq \alpha < 1),
\]
For $0 < a < r_0$, where $r_0$ is a root of the equation $2^m(1- m^r)-(1-r)^m = 0$. This proves the Theorem 6.2.4.

The result is best possible for the functions $f_1(z)$ defined by (6.2.14).

Remark. We observe that Theorem 6.2.4 improves Theorem D of Kumar [61].

6.3. This section deals with the determination of distortion theorems for the class $S^k_n(a)$ in terms of a general class of fractional integral operators involving the Gauss-hypergeometric series.

Using (6.1.5) and Theorem 1.6.1, we now prove

**Theorem 6.3.1.** Let $a$, $b$ and $c$ satisfy the inequalities:

(6.3.1) $a > 0$, $b < 2$, $a+c > -2$, $b-c < 2$ and $2a > b(a+c)$. 
Also, let the function $f(z)$ defined by (6.1.1) be in the class $S_{n}^{a}(\alpha)$. Then

\[
(6.3.2) \quad |I_{0,z} f(z)| \geq \frac{\Gamma(2-b+c)}{\Gamma(2-b)\cdot\Gamma(2+a+c)} |z|^{1-b} \left\{1 - \frac{(1-\alpha)(2-b+c)}{2^{m-1}(2-\alpha)(2-b)(2+a+c)} \right\}.
\]

and

\[
(6.3.3) \quad |I_{0,z} f(z)| \leq \frac{\Gamma(2-b+c)}{\Gamma(2-b)\cdot\Gamma(2+a+c)} |z|^{1-b} \left\{1 + \frac{(1-\alpha)(2-b+c)}{2^{n-1}(2-\alpha)(2-b)(2+a+c)} \right\}
\]

for $z \in \mathbb{E}$ if $b \leq 1$ and $z \in \mathbb{E} - \{0\}$ if $b > 1$.

Equalities in (6.3.2) and (6.3.3) are attained by the function

\[
(6.3.4) \quad f(z) = z - \frac{1 - z}{2^{n}(2-\alpha)} z^{2}
\]

at certain values of $z$, where $b$ is assumed to be a rational number for the case (6.3.3).

**Proof.** By virtue of Theorem 1.6.1, we have

\[
I_{0,z} f(z) = \frac{\Gamma(2-b+c)}{\Gamma(2-b)\cdot\Gamma(2+a+c)} z^{1-b} - \sum_{k=2}^{\infty} \frac{\Gamma(k+1) \cdot \Gamma(k+1-b+c)}{\Gamma(k+1-b)\cdot\Gamma(k+1+a+c)} z^{k-b}
\]
Now consider the function \( \Psi(z) \) defined by

\[
\Phi(z) = \frac{\Gamma(2-b) \cdot \Gamma(2+a+c)}{\Gamma(2-b+c)} \cdot z^b a, b, c
\]

\[
= z - \sum_{k=2}^{\infty} \Psi(k) a_k z^k,
\]

where

\[
\Psi(k) = \frac{\Gamma(2-b) \cdot \Gamma(2+a+c)}{\Gamma(2-b+c)} \cdot \frac{\Gamma(k+1) \cdot \Gamma(k+1-b+c)}{\Gamma(k+1-b) \cdot \Gamma(k+1+a+c)}.
\]

It is easily seen from the assumption (6.3.1) that \( \Psi(k) \) is an non-decreasing function of \( k \geq 2 \), and we have

\[
0 < \Psi(k) \leq \Psi(2) = \frac{2(2-b+c)}{(2-b)(2+a+c)}.
\]

In view of (6.1.5), we also have

\[
\sum_{k=2}^{\infty} a_k \leq \frac{1 - \alpha}{2^n(2-\alpha)}.
\]

Making use of (6.3.6) and (6.3.7) in (6.3.5), we see that

\[
|\Phi(z)| \geq |z| - \left( \sum_{k=2}^{\infty} \Psi(k) a_k \right) |z|^2
\]

\[
\geq |z| - \Psi(2) |z|^2 \cdot \sum_{k=2}^{\infty} a_k
\]

\[
\geq |z| - \frac{1 - \alpha}{2^n(2-\alpha)} \Psi(2) |z|^2
\]

which implies the assertion (6.3.2).

The assertion (6.3.3) can be proved similarly.
Finally, in view of Theorem 1.6.1, it is easy to verify that the function given by (6.3.4) does indeed attain the equality in (6.3.2) for \( z = |z| \). If \( b \) is a rational number, we choose integers \( m_1 \) and \( m_2 \) such that

\[
\frac{(2m_1 + 1)}{m_1 - m_2} = 2(b - 1).
\]

Letting \( \theta = (2m_1+1)\pi / (1-b) = (2m_2+1)\pi / (2-b) \), we can see that (6.3.4) attains the equality in (6.3.3) at the value \( z = |z| e^{i\theta} \).

**Corollary 6.3.1.** Let the function \( f(z) \) defined by (6.1.1) be in the class \( S^*_n(a) \). Then

\[
(6.3.8) \quad |D_z^{-\lambda} f(z)| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{1 - a}{2^{n-1}(2-a)(2+\lambda)} |z| \right\}
\]

and

\[
(6.3.9) \quad |D_z^{-\lambda} f(z)| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 + \frac{(1 - a)}{2^{n-1}(2-a)(2+\lambda)} |z| \right\}
\]

for \( \lambda > 0 \) and \( z \in E \). Equalities in (6.3.8) and (6.3.9) are attained by the function given by (6.3.4) at certain values \( z \), where \( \lambda \) is assumed to be a rational number for the case (6.3.9).

**Proof.** In view of the relationship (1.6.8), Corollary 6.3.1 follows readily from Theorem 6.3.1 by putting \( a = -b = \lambda \).
Remark. Letting \( \lambda \to 0 \) and putting \( n = 0 \) (resp. \( n = 1 \)) in Corollary 6.3.1, we obtain the corresponding distortion theorems due to Silverman [127].

**Corollary 6.3.2.** Let the function \( f(z) \) defined by (6.1.1) be in the class \( S_n^*(\alpha) \). Then \( D_z^{-\lambda} f(z) \) is contained in the disc with centre origin and radius \( R \) given by

\[
R = R(\alpha, \lambda) = \frac{1}{\Gamma(2+\lambda)} \left\{ 1 + \frac{1 - \alpha}{2^{n-1}(2-\alpha)(2+\lambda)} \right\}.
\]

The result is sharp with the extremal function being given by (6.3.4).