CHAPTER V

ON CERTAIN ANALYTIC FUNCTIONS INVOLVING RUSCHEWEYH DERIVATIVES

5.1. Let \( H \) denote the class of functions

\[ f(z) = z + a_2 z^2 + \ldots \]

which are analytic in the unit disc \( E \), and let \( S \) denote the subclass of functions from \( H \) which are univalent in \( E \). A function \( f \in H \) is said to be starlike of order \( \alpha \), if and only if

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha
\]

for some \( \alpha (0 < \alpha < 1) \) and for all \( z \in E \). We denote by \( S^*(\alpha) \), the class of all such functions, and by \( S^*(0) = S^* \) the class of starlike functions.

We denote by \( C \) the class of convex (univalent) functions in the unit disc \( E \), that is; the class of functions for which \( zf'(z) \in S^* \). The last condition may be expressed as

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad z \in E.
\]

For a function \( f \in H \), we say that it belongs to the class \( S^*_n(A,B) \), \(-1 \leq B < A \leq 1\) and \( n \in \mathbb{N}_0 = \{0, 1, 2, \ldots \} \), if and only if
\[
(5.1.3) \quad \frac{D^{n+1}f(z)}{D^n f(z)} < \frac{1 + Az}{1 + Bz}, \quad z \in E
\]

where
\[
D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z).
\]

(Here '⋆' and '<' means the Hadamard product of two analytic functions and subordination respectively).

Geometrically, the condition (5.1.3) means that the image of \( E \) under the function \( \frac{D^{n+1}f(z)}{D^n f(z)} \) is contained inside the open disc centred on the real axis whose diameter has end points \( (1-A)/(1-B) \) and \( (1+A)/(1+B) \). From this it follows that \( \text{Re} \left[ \frac{D^{n+1}f(z)}{D^n f(z)} \right] > (1-A)/(1-B) \).

It is readily seen that by specializing the parameters \( n, A \) and \( B \), we get several subclasses of analytic functions. For instance, \( S_n^*(0,-1) = K_n \) is the class introduced and studied by Ruscheweyh [121] while \( S_0^*(A,B) = S^*(A,B) \) is the class considered by Janowski [51] which in turn reduces to the following subclasses. \( S^*(1,-1) = S^*, S^*(1-2\alpha,-1) = S^*(\alpha), 0 \leq \alpha < 1 \) and \( S^*(\varphi,0) \) is the class defined by
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < \varphi, \quad 0 < \varphi \leq 1, \quad z \in E.
\]

The general theory of differential subordination was introduced by Miller and Mocanu [76]. The first order differential subordination with many interesting applications can be found in [78]. Namely, if \( \Psi : \Omega \rightarrow \mathbb{C} \) is
analytic in a domain $\Omega \subset \mathbb{C}^2$, and if $h(z)$ is univalent in $E$, $p(z)$ is analytic in $E$ with $(p(z), zp'(z)) \in \Omega$, $z \in E$, then $p(z)$ is said to satisfy the first order differential subordination if

$$\psi(p(z), zp'(z)) < h(z).$$

The univalent function $q(z)$ is said to be a dominant of the differential subordination (5.1.4) if $p(z) < q(z)$ for all $p(z)$ satisfying (5.1.4). If $\tilde{q}(z)$ is a dominant of (5.1.4) and $\tilde{q}(z) < q(z)$ for all dominants $q(z)$ of (5.1.4), then $\tilde{q}(z)$ is said to be the best dominant of (5.1.4).

In section 5.2 of this chapter, we discuss some preliminaries lemmas which are needed to establish the main results. We give a criteria for functions in $H$ to be in the class $S_n^*(A, B)$ and a result for the image domain for functions $(D^n f(z)/z)^\mu$, where $\mu \neq 0$ is a complex constant and $f(z) \in S_n^*(A, B)$ in section 5.3. Finally, an estimate for the real part of the function $D^n f(z)/z$ is given for functions $f \in H$ and satisfying the condition

$$\text{Re}\left\{D^n f(z) D^{n+1} f(z)/z^2\right\} > \alpha \ (0 \leq \alpha < 1) \text{ in } E.$$

5.2. In this section, we include some preliminaries lemmas which are needed to establish our main results.
Lemma 5.2.1. Let \( \Phi \) be a complex valued function, \( \Phi : \Omega \rightarrow \mathbb{C} \), \( \Omega \subset \mathbb{C}^2 \) (\( \mathbb{C} \) is the complex plane), and let 
\[ u = u_1 + iu_2, \ v = v_1 + iv_2. \] 
Suppose that the function \( \Phi(u,v) \) satisfies

(i) \( \Phi(u,v) \) is continuous in \( \Omega \),
(ii) \((1,0) \in \Omega \) and \( \text{Re}\{\Phi(1,0)\} > 0 \),
(iii) for all \((iu_2, v_1) \in \Omega \) such that
\[ v_1 \leq -(1 + u_2^2)/2, \ \text{Re}\{\Phi(iu_2, v_1)\} \leq 0. \]

Let \( q(z) = 1 + q_1 z + q_2 z^2 + \ldots \) be analytic in \( \mathbb{E} \) such that \((q(z), zq'(z)) \in \Omega \) for all \( z \in \mathbb{E} \). If \( \text{Re}(\Phi(q(z), zq'(z))) > 0, z \in \mathbb{E} \), then \( \text{Re}(q(z)) > 0 \) in \( \mathbb{E} \).

We owe this lemma to Miller and Mocanu [75].

The following lemma can be found in [78].

Lemma 5.2.2. Let \( q(z) \) be univalent in \( \mathbb{E} \) and let 
\( \Theta(w) \) and \( \Phi(w) \) be analytic in a domain \( \Omega \) containing \( q(\mathbb{E}) \), 
with \( \Phi(w) \neq 0 \) when \( w \in q(\mathbb{E}) \). Set \( Q(z) = zq'(z)\Phi(q(z)) \), 
\( h(z) = \Theta(q(z)) + Q(z) \) and suppose that

(i) \( Q(z) \) is starlike (univalent in \( \mathbb{E} \) with \( Q(0) = 0 \), 
\( Q'(0) \neq 0 \)) and

(ii) \( \text{Re}\left\{ \frac{zh'(z)}{Q(z)} \right\} = \text{Re}\left\{ \frac{Q'(q(z))}{\Phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0, z \in \mathbb{E}. \)

If \( p(z) \) is analytic in \( \mathbb{E} \) with \( p(0) = q(0) \), \( p(\mathbb{E}) \subset \Omega \), and
We note that Lemma 5.2.3 is valid for the function $(1+z)^\beta$, $z \in E$.

**Lemma 5.2.3.** [119] The function $(1-z)^\beta = \exp(\beta \ln(1-z))$, $\beta \neq 0$ is univalent in $E$, if and only if, $\beta$ is either in the closed disc $|\beta - 1| \leq 1$ or in the closed disc $|\beta + 1| \leq 1$.

We note that Lemma 5.2.3 is valid for the function $(1+z)^\beta$, $z \in E$.

**Lemma 5.2.4.** Let $p(z)$ be analytic in $E$ with $p(0) = 1$, and $p(z) \neq 0$ in $0 < |z| < 1$. If $p(z)$ satisfies

$$(5.2.2) \quad (n+1)(1 - \frac{1}{p(z)}) + \frac{zp'(z)}{p(z)^2} < \frac{(A-B)z}{(1 + Az)^2} \left\{ (n+2)+(n+1)Az \right\},$$

$z \in E$ for $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $-1 < B < A < 0$, then

$$p(z) < q(z) = \frac{1+Az}{1+Bz}$$

and $q(z)$ is best dominant.

**Proof.** If we take $q(z) = (1+Az)/(1+Bz)$, $-1 < B < A < 0$, $\theta(w) = (n+1)(1-1/w)$ and $\bar{\Phi}(w) = \frac{1}{w^2}$, then it is easy to check that $q(z)$, $\theta(w)$ satisfy the hypothesis of Lemma 5.2.2. Further, since

$$\theta(z) = zq'(z)\bar{\Phi}(q(z)) = \frac{(A-B)z}{(1+Az)^2}$$

is starlike in $E$ and
\[ h(z) = \theta(q(z)) + Q(z) = \frac{(A-B)z \{(n+2)+(n+1)Az\}}{(1 + Az)^2} \]

the condition (i) and (ii) Lemma 5.2.3 are satisfied. Thus the required result follows from (5.2.1).

5.3. In this section, we give a criteria for functions in \( H \) to be in the class \( S_n(A,B) \) and a result for the image domain for functions of the form \( (D^n f(z)/z)^\mu \), where \( \mu \neq 0 \) is a complex constant and \( f(z) \in S_n(A,B) \) and \( f(z) \in S_n(A,B) \). Also an estimate for the real part of \( (D^n f(z)/z) \) is given for functions \( f(z) \in H \) and satisfying the condition \( \text{Re} \{D^n f(z), D^{n+1} f(z)/z^2\} > a(0 < a < 1) \) in \( E \).

We now prove

**Theorem 5.3.1.** Let \( A \) and \( B \) be fixed numbers such that \(-1 < B < A < 0\). Let \( f(z) \in H \) with \( D^n f(z) \neq 0 \) for \( 0 < |z| < 1 \) and \( n \in \mathbb{N}_0 = \{0,1,2,...\} \). If

\[
(5.3.1) \quad \left| \frac{D^{n+2} f(z)}{D^{n+1} f(z)} \right| < \frac{k_n(A,B)}{(n+2)} \left| \frac{D^{n+1} f(z)}{D^n f(z)} \right| , \quad z \in E
\]

where \( k_n(A,B) = \frac{(A-B)\{(n+2)-(n+1)A\}}{(1+A)^2} \),

then \( f(z) \) is univalent in \( E \) and

\[
(5.3.2) \quad \frac{D^{n+1} f(z)}{D^n f(z)} < \frac{1+Az}{1+Bz} , \quad z \in E.
\]
Proof. Condition (5.3.1) and $D^n f(z) \neq 0$ for $0 < |z| < 1$ implies that $D^{n+1} f(z) \neq 0$ for $0 < |z| < 1$. Let $p(z) = D^{n+1} f(z) / D^n f(z)$. Then $p(z)$ is analytic in $E$ with $p(0) = 1$ and $p(z) \neq 0$ for $0 < |z| < 1$. Taking logarithmic differentiation of $p(z)$ and using the identity

$$z(D^n f(z))' = (n+1)D^{n+1} f(z) - nD^n f(z), \quad n \in \mathbb{N}_0$$

we obtain

$$(n+1)(1 - \frac{1}{p(z)}) + \frac{zp'(z)}{p(z)^2} = (n+2) \frac{D^n f(z)}{D^{n+1} f(z)} \cdot \frac{D^{n+2} f(z)}{D^{n+1} f(z) - 1}$$

which with the aid of (5.3.1) implies that

$$|(n+1)(1 - \frac{1}{p(z)}) + \frac{zp'(z)}{p(z)^2}| < K_n(A, B).$$

To show that relation (5.2.2) is true, it is sufficient to show that the image of $E$ under the function $h_1(z) = (A-B) z \{ (n+2) + (n+1)Az \}^2 / (1+Az)^2$ contains the disc

$$\{ w : |w| < K_n(A, B) \}.$$ Clearly $U \subset h_1(E)$ and the distance from the origin to an arbitrary point of the image domain $h_1(E)$ is given by

$$|h_1(e^{it})| = \frac{(A-B)[(n+2)^2 + 2A(n+1)(n+2)\cos t + (n+1)^2 A^2]^{1/2}}{1 + 2A \cos t + A^2}, \quad 0 < t < 2\pi$$

$$= \frac{(A-B)[(n+2)^2 + 2A(n+1)(n+2)s + (n+1)^2 A^2]^{1/2}}{(1+2As + A^2)^2}, \quad -1 \leq s \leq 1$$
Since the functions $|h_1(e^{it})|$ and $|h_1(e^{it})|^2$ attain the minimum value for the same $t$ or $s$, by letting

$$\Psi(s) = \frac{(A-B)^2[(n+2)^2+2A(n+1)(n+2)s+(n+1)^2A^2]}{(1+2As+A^2)^2}$$

we get

$$\Psi'(s) = -2A(A-B)^2 \frac{[(n+2)(n+3)+4A^2s+n(n+1)A]}{(1+2As+A^2)^2} \geq 0$$

for $-1 < A \leq 0$ and $-1 \leq s \leq 1$. This shows that $\Psi(s)$ is a non-decreasing function of $s$, so that

$$\min_{|s| \leq 1} \Psi(s) = \Psi(-1) = (A-B)^2 \frac{[(n+2)-(n+1)A]^2}{[1-A]^2} = (K_n(A,B))^2.$$

Hence the function $|h_1(e^{it})|$ has the minimum value $K_n(A,B)$ and our assertion that

$$(n+1)(1-\frac{1}{p(z)}) + \frac{zp'(z)}{p(z)^2} < \frac{(A-B)z}{1+Az} \frac{(n+2)+(n+1)Az}{(1+Az)^2}$$

is true. Now, by applying Lemma 5.2.4, we have

$$\frac{D^{n+1}f(z)}{D^nf(z)} < \frac{1+Az}{1+Bz}, \quad z \in E.$$  

This proves (5.3.2). Since,

$$\Re \left(\frac{1+Az}{1+Bz}\right) > \frac{1-A}{1-B} \geq \frac{1}{2}$$
By Rusheweyh's result this proves that $f$ is univalent in $E$.

Next suppose that

$$(5.3.4) \quad \frac{(n+4)A - A^3 - 1}{(n+3) - nA^2} \leq B < A \leq 0,$$

for $n \in \mathbb{N}_0$. Since the relation (5.3.2) implies that

$$\left| \frac{D^{n+1}f(z)}{D^n f(z)} \right| < \frac{1 + A}{1 + B}$$

from (5.3.1), we have for $z \in E$ and $n \in \mathbb{N}_0$

$$\left| \frac{D^{n+2}f(z)}{D^{n+1} f(z)} - 1 \right| < \frac{1}{(n+2)}, \text{ by (5.3.3)}.$$

The above inequality after simplification yields (5.3.4).

This completes the proof of the theorem.

In the case $A = 0$ and $B = -k$, $0 < k \leq 1$, Theorem 5.3.1 yields

**Corollary 5.3.1.** Let $f(z) \in H$ and $D^n f(z) \neq 0$ for $0 < |z| < 1$. If there exists a real number $k$, $0 < k \leq 1$, such that

$$\left| \frac{D^{n+2}f(z)}{D^{n+1} f(z)} - 1 \right| < k \left| \frac{D^{n+1}f(z)}{D^n f(z)} \right|, \quad z \in E$$

for $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots \}$, then $f(z)$ is univalent in $E$ and
\begin{align*}
\frac{D^{n+1} f(z)}{D^n f(z)} & < \frac{1}{1-kz}, \quad z \in E.
\end{align*}

In particular, if \( 0 < k \leq \frac{1}{(n+3)} \), then

\begin{align*}
\frac{D^{n+2} f(z)}{D^{n+1} f(z)} & > \frac{n+1}{n+2}
\end{align*}

for \( z \in E \) and \( n \in \mathbb{N}_0 \).

The first half of this result was also obtained by Obradović and Owa [94].

Taking \( n = 0 \) and \( k = (1-\alpha)/\alpha, \frac{1}{2} \leq \alpha < 1 \) in Corollary 5.3.1, we have the following result obtained by Obradović and Owa [93].

**Corollary 5.3.2.** Let \( f(z) \in H \) with \( f'(z) \neq 0 \) for \( 0 < |z| < 1 \), and let \( \frac{1}{2} \leq \alpha < 1 \). If

\begin{align*}
\left| \frac{z f''(z)}{f(z)} \right| & < \frac{2(1-\alpha)}{\alpha}, \quad \left| \frac{z f'(z)}{f(z)} \right|, \quad z \in E
\end{align*}

then \( f(z) \in S^*(\alpha) \) and

\begin{align*}
\frac{zf'(z)}{f(z)} & < \frac{\alpha}{\alpha-(1-\alpha)z}, \quad z \in E.
\end{align*}

In particular, if \( \frac{3}{4} \leq \alpha < 1 \), then \( f(z) \in C \) and

\begin{align*}
\left| \frac{z f''(z)}{f(z)} \right| & < 1, \quad z \in E.
\end{align*}
Theorem 5.3.2. Let $p(z)$ be analytic in $E$ with $p(0) = 1$ and $p(z) \neq 0$ for $0 < |z| < 1$, and let $-1 < B < A < 1$.

(i) Let $B \neq 0$ and $\mu$ be a complex number with $\mu \neq 0$. Let $A$, $B$ and $\mu$ satisfy either

\begin{equation}
|\mu(n+1) \frac{A-B}{B} - 1| \leq 1
\end{equation}

or

\begin{equation}
|\mu(n+1) \frac{A-B}{B} + 1| \leq 1, \quad n \in \mathbb{N}_0.
\end{equation}

If $p(z)$ satisfies

\begin{equation}
1 + \frac{zp'(z)}{\mu(n+1)p(z)} < \frac{1+Az}{1+Bz}, \quad z \in E
\end{equation}

then

\begin{equation}
p(z) \prec (1+Bz)^{\mu(n+1)(A-B)/B}
\end{equation}

and this is the best dominant.

Proof. (i) We choose $\Theta(w) = 1$, $\Phi(w) = 1/\mu(n+1)w$ and $q(z) = (1+Bz)^{\mu(n+1)(A-B)/B}$. Then in view of Lemma 5.2.3 and the condition (5.3.5), the function $q(z)$ is univalent in $E$. Further

\begin{equation}
Q(z) = zq'(z) \frac{\Phi(q(z))}{(A-B)z} = \frac{1+Az}{1+Bz}
\end{equation}

is starlike in $E$ and

\begin{equation}
h(z) = \Theta(q(z)) + Q(z) = \frac{1+Az}{1+Bz}.
\end{equation}

Thus conditions (i) and (ii) of Lemma 5.2.3 are satisfied and result (5.3.6) follows by applying Lemma 5.2.2.
(ii) The proof is similar to that in case (i),

Theorem 5.3.3. Let \( f(z) \in H \) with \( D^n f(z) \neq 0 \) for \( 0 < |z| < 1 \) and \( n \in N_0 \). Let \(-1 \leq B < A \leq 1\).

(i) If \( B \neq 0 \) and

\[
\frac{D^{n+1} f(z)}{D^n f(z)} < \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{E}
\]

then

(5.3.7) \[
\frac{D^n f(z)}{z} \frac{\mu}{(1 + Bz)} < \frac{\mu(n+1)(A-B)/B}{(1 + Bz)}, \quad z \in \mathbb{E}
\]

where \( \mu 
eq 0 \) is a complex constant satisfying (5.3.5).

(ii) If \( B = 0 \) i.e., \( 0 < A \leq 1 \), and

\[
\frac{D^{n+1} f(z)}{D^n f(z)} < (1 + Az)
\]

then

(5.3.8) \[
\frac{D^n f(z)}{z} \frac{\mu}{(1 + Az)} < e, \quad z \in \mathbb{E}
\]

where \( \mu \neq 0 \) and \( |\mu A| < \pi(n+1) \).

Proof. Let \( p(z) = (D^n f(z)/z)^\mu \). Then \( p(z) \) is analytic in \( \mathbb{E} \) with \( p(0) = 1 \) and \( p(z) \neq 0 \) in \( \mathbb{E} \). For such \( p(z) \) from (i) and (ii) of Theorem 5.3.2, we deduce relations (5.3.7) and (5.3.8) of Theorem 5.3.3.

Putting \( n = 0 \) in Theorem 5.3.3, we have
Corollary 5.3.3. Let \( f(z) \in S^*(A,B), -1 < B < A < 1 \).

(i) If \( B \neq 0 \), then
\[
\left( \frac{f(z)}{z} \right)^\mu < (1+Bz)^{\mu(A-B)/B}
\]
where \( \mu \) is a complex number with \( \mu \neq 0 \) such that
\[
|\mu(A-B)/B + 1| \leq 1
\]

(ii) If \( B = 0 \), then
\[
\left( \frac{f(z)}{z} \right)^\mu < \exp(\muAz)
\]
where \( \mu \neq 0 \) and \( |\mu A| < \pi \).

This is the earlier result due to Obradovic and Owa [93].

Remarks. (1) Taking \( \mu = 1, A = 0 \) and \( B = -1 \) in Corollary 5.3.3, we get
\[
\Re\left( \frac{zf'(z)}{f(z)} \right) > \frac{1}{2} \implies \Re\left( \frac{f(z)}{z} \right) > \frac{1}{2}, \quad z \in E.
\]

2. By suitably choosing the values of parameters \( A, B, \mu \) and \( n \) in Theorem 5.3.3, we get the results obtained by Obradovic and Owa [93].

Theorem 5.3.4. If \( f(z) \in H \) satisfies in \( E \) the condition
\[
(5.3.9) \quad \Re \left\{ \frac{D^n f(z)}{z} \right\}^{2\mu-1} \left\{ \frac{D^{n+1} f(z)}{z} \right\} > \gamma \frac{(2\mu n + 2\mu + 1)\gamma - 1}{2\mu(n+1)} = \gamma_n(\mu)
\]
for $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\mu > 0$ and $\frac{1}{2\mu(n+1)+1} \leq \gamma < 1$, then
\[
\text{Re}\left\{ \left( \frac{D^n f(z)}{z} \right)^\mu \right\} > \gamma, \quad z \in \mathbb{D}.
\]

**Proof.** Let $\gamma_n(\mu) = \gamma \cdot \left\{ (2\mu n + 2n + 1)\gamma - 1 \right\} / 2\mu(n+1)$ and consider the function
\[
p(z) = (1 - \gamma)^{-1} \left\{ \left( \frac{D^n f(z)}{z} \right)^\mu - \gamma \right\}.
\]
Then $p(z)$ is analytic in $\mathbb{D}$ and $p(0) = 1$. A simple computation shows that
\[
\left( \frac{D^n f(z)}{z} \right)^{2\mu-1} \frac{D^{n+1} f(z)}{z} = (\gamma+(1-\gamma)p(z))^2
\]
\[
+ (\gamma+(1-\gamma)p(z)) \frac{(1-\gamma)z p'(z)}{\mu(n+1)}.
\]
where
\[
(5.3.10) \quad \Psi(u,v) = (\gamma+(1-\gamma)u)^2 + (\gamma+(1-\gamma)u) \frac{(1-\gamma)v}{\mu(n+1)}.
\]
Using (5.3.9) and (5.3.10), it follows that
\[
\text{Re}\left\{ \Psi(p(z), zp'(z)) \right\} > \gamma_n(\mu).
\]
So the function $\Phi(u,v)$ defined by
\[
\Phi(u,v) = \Psi(u,v) - \gamma_n(\mu)
\]
satisfies the following properties
\begin{enumerate}
  \item $\Phi(u,v)$ is continuous in $\Omega = \mathbb{D} \times \mathbb{D}$.
  
  \item $(1,0) \in \Omega$ and $\text{Re}\left\{ \Phi(1,0) \right\} = (1 - \gamma_n(\mu)) > 0$

  \item for all real $u_2$, $v_1 \leq -(1 + u_2^2)/2$, we have
\end{enumerate}
\[\text{Re}\{\phi(\mu, v)\} = \gamma^2 - (1 - \gamma)^2 u_2^2 + \frac{\gamma(1-\gamma)v_1}{\mu(n+1)} - \gamma_n(\mu)\]

\[\leq \gamma^2 - \frac{\gamma(1-\gamma)}{2\mu(n+1)} - \gamma_n(\mu) - (1-\gamma) \left\{ (1-\gamma) + \frac{\gamma}{2\mu(n+1)} \right\} u_2^2\]

Thus, by Lemma 5.3.1, \(\text{Re}\{p(z)\} > 0\) in \(E\) and hence

\[\text{Re}\\left\{ \left( \frac{-D^n f(z)}{z} \right)^\mu \right\} > \gamma, \quad z \in E.\]

This completes the proof of Theorem 5.3.4.

**Remark.** Taking \(\gamma = (2\mu_2 + 2\mu + 1)^{-1}\) in Theorem 5.3.4, we obtain

\[(5.3.11) \quad \text{Re} \left( \frac{D^n f(z)}{z} \right)^{2\mu-1} \frac{D^{n+1} f(z)}{z} > 0, \text{ implies that} \]

\[\text{Re} \left( \frac{D^n f(z)}{z} \right)^\mu > \frac{1}{2\mu n + 2\mu + 1}, \quad z \in E\]

and the above result for \(n = 0\) and \(\mu = 1\) gives that, for \(f(z) \in H,\)

\[\text{Re}(f'(z)). \frac{f(z)}{z} > 0 \text{ implies } \text{Re} \left( \frac{f(z)}{z} \right) > \frac{1}{2}, \quad z \in E.\]

This result was also obtained by Obradovic [91].

Similarly, for \(\mu = \frac{1}{2},\) (5.3.11) yields

\[\text{Re} \left( \frac{D^{n+1} f(z)}{z} \right) > 0 \text{ implies } \text{Re} \left( \frac{D^n f(z)}{z} \right) > \frac{1}{n+2}, \quad z \in E\]

which for \(n = 0\) gives

\[\text{Re}(f'(z)) > 0 \text{ implies } \text{Re} \left( \frac{f(z)}{z} \right)^{1/2} > \frac{1}{2} \text{ in } E.\]