Chapter - I

INTRODUCTION
The thesis entitled "Investigations of plane symmetry in the general theory of relativity" consists of seven chapters and deals with the study of plane gravitational waves in Einstein's theory of general relativity on the line of Takeno (1961). He has investigated \((z-t)\)-type and \((t/z)\)-type plane gravitational waves mathematically. His noteworthy work forms the background and motivation of our investigations. Moreover, some geometrical properties of the plane symmetric space-times, plane symmetric inhomogeneous models, plane wave solutions of weakened field equations, the coordinate system in six dimension and generalized line element connected with the plane wave have been discussed. Also in order to make the thesis self explanatory we present brief discussion of general theory of relativity, plane gravitational waves, weakened field equations, the work of Takeno and mathematical tools in introductory chapter.

[1.1] GENERAL THEORY OF RELATIVITY

The success of Newtonian mechanics based on the three laws of motion and Newtonian gravitation based on the universal law of Gravitation is well known. However, at high speeds comparable to the speed of light, Newtonian mechanics is not applicable. Moreover, Newton's equations of motion are invariant under Galilean transformations, but Maxwell's electromagnetic equations do not obey this rule. This has resulted into the special theory of relativity due to the genius Albert Einstein in 1905. His theory provides new approach of space and time to deal with the conventional ideas. The special theory of relativity is based on two postulates:
Einstein principle of Relativity,

Constancy of speed of light.

The first postulate is stated as, "The laws of physics are the same in all inertial systems". The second postulate implies that the speed of light in free space is same for all inertial observers and does not depend on the relative velocity of the source of light and observer. The special theory of relativity fails to study relative motion in accelerated frame of reference and is not applicable to all kinds of motion. Also it does not accommodate gravitation in its structure. Taking into account these limitations, Einstein reformulated the special theory of relativity and put forth a new theory in 1915, known as general theory of relativity or Einstein's theory of gravitation.

This Theory is based on two postulates:

1. The principle of covariance
2. The principle of Equivalence.

The principle of covariance states that "All laws of nature or physical laws are covariant i.e. independent of the coordinate systems." This implies that the laws should be tensorial and hence the line element of special relativity known as Minkowski space-time of (3 + 1) dimension,

\[ ds^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2, \]  

(1.1.1)

which is not invariant under a general coordinate transformation, is replaced by Riemannian metric
\[ ds^2 = g_{ij} dx^i dx^j, \quad i, j = 1, 2, 3, 4, \]  

(1.1.2)

where the metric tensor \( g_{ij} \) is symmetric, represents gravitational potentials and \( g_{ij} = g_{ij}(x^i) \).

Principle of equivalence states that “The gravitational and inertial forces acting on a body are locally equivalent and indistinguishable from one another”. According to Weinberg (1972) the principle of equivalence is stated as “At every space-time point in an arbitrary gravitational field it is possible to choose a locally inertial coordinate system such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated cartesian coordinate system in the absence of gravitation.” According to Newton’s theory of gravitation the inertial mass and the gravitational mass of the same body are equal. This fact was experimentally verified by Eötvös (1889) and Southerus (1910), this served as a tool to Einstein to formulate the principle of equivalence.

[1.2] THE FIELD EQUATIONS

The field equations of general relativity are

\[ G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} = K T_{ij}, \]  

(1.2.1)

where \( G_{ij} \) is Einstein tensor, \( R_{ij} \) is Ricci tensor, \( g_{ij} \) is metric tensor, \( R \) is a scalar curvature, \( T_{ij} \) is an energy momentum tensor describing material distribution and \( K = \text{constant} = \frac{8\pi G}{c^4} \), where \( G \) is the constant of gravitation.
In vacuum, \( T_{ij} = 0 \) hence the field equations (1.2.1) reduce to,

\[
R_{ij} = 0, \quad i, j = 1, 2, 3, 4.
\]

The first exact solution representing static spherically symmetric gravitational field in empty space-time i.e. \( R_{ij} = 0 \) was obtained by Schwarzschild in 1916:

\[
ds^2 = -(1-\frac{2M}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) + (1-\frac{2M}{r})dt^2,
\]

where \( M \) is the mass of a centrally situated particle.

Schwarzschild interior solution

\[
ds^2 = -(1-\frac{r^2}{R^2})^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) + \frac{1}{4} \left[ 3\sqrt{1-\frac{r^2}{R^2}} - \sqrt{1-\frac{r^2}{R^2}} \right]^2 dt^2,
\]

where \( R^2 = 3/8\pi\rho \), \( \rho \) is the proper density of perfect fluid inside the star, \( r_1 \) is the radius of the sphere, assuming stars as a sphere.

**[1.3] CLASSICAL TESTS OF GENERAL RELATIVITY**

Einstein general theory of relativity gives description of gravitation phenomenon in agreement with the observations and is one of the most beautiful structures in all of theoretical physics. The theory has been experimentally accountable and till date there is phenomenon which contradicts general relativity. In general the following tests are compatible with the theory:
(i) Advance of perihelion of the planet Mercury,
(ii) Bending of the light rays coming from distant stars in the gravitational field of the Sun,
(iii) Spectral shift due to the change in gravitational field,
(iv) Time delay in Radar sounding,
(v) Geodesic effect,
(vi) New test,
(vii) Shirokov’s effect.

These constitute the historical and observational evidence of validity of the general theory of relativity.

[1.4] PLANE GRAVITATIONAL WAVES

As is evident that the field equations contain ten potentials \( g_{\alpha} \) and it is a big task to solve these non-linear partial differential equations containing ten unknowns. To make system workable we take recourse to symmetries of space –time. In this way the number of unknown \( g_{\alpha} \) is reduced to three or two. The concept of plane symmetry was introduced by Taub in 1951 in the Riemannian space-time. The plane symmetry is exploited to study the gravitational waves too.
The gravitational field, however small, can be disturbed by changing the position of matter which is responsible for creating it. These propagating disturbances are termed as gravitational waves. These waves are propagated in space with the speed of light. There are three types of gravitational waves in general relativity: Plane gravitational waves, cylindrical gravitational waves and spherical gravitational waves.

**[1.5] WEAKENED FIELD EQUATIONS (wfe)**

Kilmister and Newman, Pirani, Rund, Eddington and Rund suggested the following five field equations respectively as alternative to Einstein's field equations in vacuum which are as

$T_{ijk} = R^p_{ijk,p} = 0,$ \hfill (1.5.1)

$(-g)^{1/4} [g^{ij} R_{ijk} - g^{jk} R_{ij} - (1/6) R_{ij} - (1/6) g_{jk} R^{ij} + R^{ij} C_{jk} + (R / 6) g^{ij} C_{jk} ] = 0,$ \hfill (1.5.2)

$(-g)^{1/2} [g^{ij} g^{kl} \{2 R_{jlim} R^{ml} + g^{ml} R_{ij} - R_{ij} \}
- (1/2) g^{jk} (R_{ml} R^{lm} - g^{lm} R_{ij})] = 0,$ \hfill (1.5.3)

$(-g)^{1/2} \{g^{jk} g^{lt} - (1/2) g^{jt} g^{kl} - (1/2) g^{ht} g^{kt} \} R_{ij}^{lt} + R(R^{jk} - (1/4) g^{jk} R) = 0,$ \hfill (1.5.4)

$Q_{i}^{ij} = R_{i}^{ij} = 0,$ \hfill (1.5.5)
where $C_{j\ell h\kappa}$ is Weyl curvature tensor and semicolon ($;\cdot$) denotes the covariant derivative.

These field equations are weaker than the Einstein vacuum field equations in the sense that they each admit a solution for which $R_{ij} = 0$ and hence they have been called "Weakened field equations".

Thompson (1963) made a detailed study of these equations and concluded that they are too weak. Lovelock (1967) obtained the solutions of these equations in a spherically symmetric space-time. Pandey (1975) studied them in from the point of view of wave solutions.

[1.6] TAKENO AND OTHER'S EXPOSITION, OUR INVESTIGATION ON PLANE SYMMETRY

The pioneer work of Einstein (1916, 1918) and Rosen (1937) form the corner stone of investigations of plane gravitational waves in general relativity. On the same line some promising results are produced by Taub (1951), McVitte (1955), Bondi (1957), Bonner (1957), Bondi, Pirani and Robinson (1959). Further the work on plane symmetry was carried out by Bondi (1957), Shibata (1960), Takeno (1961) and others.

Professor H. Takeno (1961) has obtained the plane wave solutions of the field equations $R_{ij} = 0$ and established the existence of $(z-t)$-type and $(t/z)$-type plane gravitational waves propagating in the positive direction of $z$-axis in four dimensional space-time $V_4$ in general relativity, in his research report by defining plane wave as follows:
**Definition:** Plane gravitational waves $g_{ij}$ are defined as non-flat solutions of the field equations

$$R_{ij} = 0, \quad i, j = 1, 2, 3, 4$$

in an empty region of the space-time with

$$g_{ij} = g_{ij}(Z), \quad Z = Z(x^i), \quad x^i = (x, y, z, t)$$

in some suitable coordinate system with

$$g^{ij}Z_{,i}Z_{,j} = 0; \quad Z_{,i} = \frac{\partial Z}{\partial x^i}$$

and

$$Z = Z(z, t); \quad Z_3 \neq 0, \quad Z_4 \neq 0.$$  

In this definition the signature convention adopted is as follow,

$$g_{aa} < 0, \quad \begin{vmatrix} g_{aa} & g_{ab} \\ g_{ba} & g_{bb} \end{vmatrix} > 0, \quad \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} < 0, \quad g_{44} > 0,$$

(no summation for $a$ and $b; \quad a, b = 1, 2, 3$).

And accordingly $g = \det (g_{ij}) < 0$.

Taking the conditions (1.6.2), (1.6.3) and (1.6.4) Takeno has obtained definite forms of the plane wave solutions by solving (1.6.1). From (1.6.4), (1.6.3) is a quadratic equation for $(Z_3/Z_4)$. It has one positive and one negative root by virtue of the relation (1.6.5). Obviously, the positive or
negative root corresponds to the waves propagating in the negative or positive direction of the z-axis respectively. He considered the waves propagating in the positive direction. Then by (1.6.3),

\[ g^{33} \phi^2 + 2g^{34} \phi + g^{44} = 0 \]  
\[ \Rightarrow \phi = \frac{-g^{34} + \sqrt{(g^{34})^2 - g^{33}g^{44}}}{g^{33}} < 0, \]

where \( \phi = Z_{r3}/Z_{r4} \).

The differential equation, \( \phi = Z_{r3}/Z_{r4} \) can be written as

\[ \frac{\partial Z}{\partial z} - \phi \frac{\partial Z}{\partial t} = 0. \]

Solving this differential equation, its solution is

\[ t + z\phi(Z) = \omega(Z), \]  
\[ (1.6.8) \]

where \( \omega \) is an arbitrary function of \( Z \).

From equation (1.6.8),

\[ Z_{r3} = \phi/M; \ Z_{r4} = 1/M; \ M_{r3} = \phi N/M - \bar{\phi}; \ M_{r4} = N/M, \]  
\[ (1.6.9) \]

a bar (\(-\)) over a letter means the derivative with respect to \( Z \) and

\[ M = \bar{\omega} - z \bar{\phi} \neq 0, \ N = \bar{\omega} - z \bar{\phi}, \]  
\[ (1.6.10) \]
He further defined

\[ \omega^i = \phi g^{3i} + g^{4i}. \]  

(1.6.11)

In terms of \( \omega^i \), (1.6.7) takes the form

\[ \phi \omega^3 + \omega^4 = 0. \]  

(1.6.12)

Putting the values of Christoffel’s symbols in \( R_{ab} \) \( (a, b = 1, 2) \) becomes

\[ R_{ab} = \frac{\overline{g}_{ai} \omega^i \overline{g}_{bj} \omega^j}{2M^2}. \]

Hence \( R_{ab} = 0 \) is same as

\[ \rho_a = \overline{g}_{ai} \omega^i = 0. \]  

(1.6.13)

Further, \( R_{ax} = 0, \ (a = 1, 2; \ \alpha = 3, 4). \)

Lastly, \( R_{\alpha\beta} = \frac{N \rho_{\alpha\beta}}{M^3} + \frac{\sigma_{\alpha\beta}}{M^2}, \ (\alpha, \ \beta = 3, 4), \)

where \( \rho_{\alpha\beta} \) and \( \sigma_{\alpha\beta} \) are the functions of \( Z \) defined by,

\[ \sigma_{33} = -\overline{\rho}_{33} + (\phi^2 L_1 - 4\phi L_2 \rho_3 + 2\rho_3^2)/4, \]
\[ \sigma_{34} = \sigma_{43} = -\overline{\rho}_{34} + [\phi L_1 - 2L_2 (\rho_3 + \phi \rho_4) + 2\rho_3 \rho_4]/4, \]
\[ \sigma_{44} = -\overline{\rho}_{44} + (L_1 - 4L_2 \rho_4 + 2\rho_4^2)/4, \]
\[ \rho_{33} = -\phi^2 L_2 + \phi \rho_3, \]
\[ \rho_{34} = \rho_{43} = -\phi L_2 + \rho_3 / 2 + \phi \rho_4 / 2 , \]
\[ \rho_{44} = -L_2 + \rho_4 , \]

where \( \rho_i = \bar{g}_{ij} \omega^j ; L_1 = g^{ij} g^{kl} \bar{g}_{ik} \bar{g}_{jl} ; L_2 = \log \sqrt{g} . \)

Hence, the field equations (1.6.1) becomes
\[ N \rho_{\alpha \beta} + M \sigma_{\alpha \beta} = 0 \quad (1.6.14) \]

which is equivalent to
\[ \bar{\omega} \rho_{\alpha \beta} + \bar{\omega} \sigma_{\alpha \beta} = 0 = \bar{\phi} \rho_{\alpha \beta} + \bar{\phi} \sigma_{\alpha \beta} . \quad (1.6.15) \]

Takeno summarized the above in following statement.

The plane wave solutions of (1.6.1) are given by the \( g_{ij} \) which satisfies
(1.6.2), (1.6.3), (1.6.5), (1.6.13) and (1.6.15).

Takeno (1961) has studied three types of purely gravitational plane waves
viz. \((z - t)\)-type, \((t / z)\)-type and general type.

With \( Z = (z - t) \) and \( Z = (t / z) \) in (1.6.4) the field equation (1.6.15)
becomes
\[ \sigma_{\alpha \beta} = 0 , \]

which is equivalent to
\[
\bar{L}_2 - \bar{\rho}_4 + \frac{\bar{\rho}_4^2}{2} - L_2 \rho_4 + \frac{L_1}{4} = 0 \quad (1.6.16)
\]

and \( \bar{L}_2 - \bar{\rho}_4 + \frac{\bar{\rho}_4^2}{2} - L_2 \rho_4 + \frac{L_1}{4} = 0, \ L_2 = \rho_4, \ L_1 = 2L_2^2 \) respectively. (1.6.17)

The existence of \((z-t)\)-type plane gravitational wave have been characterized by (1.6.2), (1.6.3), (1.6.5), (1.6.13), (1.6.16) and \((t/z)\)-type wave characterized by (1.6.2), (1.6.3), (1.6.5), (1.6.13), (1.6.17). For the general type wave \(Z\) can not be transformed into the form \((z-t)\) or \((t/z)\) so he concluded that the general type wave cannot exist.

Takeno (1961) has proved that \((z-t)\) and \((t/z)\)-type of waves are transformable into each other. He has obtained the solutions of the field equations for both the waves in the form

\[
\begin{align*}
\frac{ds^2}{ds} &= -A dx^2 - 2 D dxdy - B dy^2 + \phi^2(-C + E) dz^2 \\
&+ 2\phi Edzd\tau + (C + E) d\tau^2 \\
\end{align*}
\]

(1.6.18)

in a suitable coordinate system with

\[
g_{\alpha \alpha} = 0, \quad a = 1, 2; \quad \alpha = 3, 4,
\]

where \( Z = Z(z,t) \) and \( A, B, C, D, E, \phi = Z_{,3}/Z_{,4} \) are the functions of \( Z \) satisfying \( A, B > 0, \ (AB - D^2) = m > 0, \ C > |E| \).

For \( Z = (z-t)\)-type wave, (1.6.18) reduce to
\[ ds^2 = -A dx^2 - 2D dx dy - Bdy^2 - (C - E)dz^2 - 2Edzdt + (C + E)dt^2 , \] (1.6.19)

where \( A, B, C, D, E \) are functions of \( Z = z - t \).

He further considered a special case for \( E = 0, C = 1 \) and gets

\[ ds^2 = -Adx^2 - 2D dx dy - Bdy^2 - dz^2 + dt^2 . \] (1.6.20)

For \( Z = (t/z) \)-type wave, (1.6.18) reduces to

\[ ds^2 = -Adx^2 - 2D dx dy - Bdy^2 - Z^2(C - E)dz^2 - 2ZEdzdt + (C + E)dt^2 , \] (1.6.21)

where \( A, B, C, D, E \) are functions of \( Z = t/z \).

In chapter II, we have studied some geometrical aspects of the plane symmetric metrics (1.6.20) and (1.6.21). In this case, both the space-times (1.6.20) and (1.6.21) are neither a space of constant curvature nor a recurrent space in general.

The analogous study was already carried out by Panigrahi and Patra (2005) with reference to the static spherical symmetry.

In chapter III, we have investigated plane symmetric inhomogeneous models in presence of massive scalar field with perfect fluid distribution in general relativity for a given gravitational field. Their physical and geometrical behaviour have been discussed. These obtained plane symmetric inhomogeneous models are
\[ ds^2 = -t^{4/3}(1 + x^2)^{2a} dx^2 - t^{4/3}(dy^2 + dz^2) + k^2 dt^2 \]  
(1.6.22)

and \[ ds^2 = -t^{-4a+2}(1 + x^2)^{2a} dx^2 - t^{2a} (dy^2 + dz^2) + k^2 dt^2, \]  
(1.6.23)

where \( a, \alpha \) and \( k \neq 0 \) are the real constants.

Some of the researchers, Panigrahi, Patra and Sahu (2005) have constructed the plane symmetric inhomogeneous mesonic perfect fluid models in Einstein’s theory of general relativity.

For \( Z = (t/z)-\)type, the plane wave \( g_{ij} \) in (1.6.21) is the solution of \( R_{ij} = 0 \), which is equivalent to

\[ P' = \frac{m}{2m} - \frac{m^2}{4m^2} - \frac{mC}{2mC} - \frac{AB - D^2}{2m} + \frac{Em}{2mCZ} = 0. \]  
(1.6.24)

Also, the space-time (1.6.21) satisfies the identity

\[ \frac{R_{33}}{Z^2} = -\frac{R_{34}}{Z} = R_{44} = \frac{P'}{z^2}, \]

where \( P' = (Av' + Bu' - 2Dw')/m = 0 \),

\[ u' = \frac{\overline{A}}{2} - \frac{1}{4m} (B\overline{A}^2 + A\overline{D}^2 - 2D\overline{A}\overline{D}) - \frac{\overline{A}}{2C} \left( \overline{C} - \frac{E}{Z} \right), \]

\[ v' = \frac{\overline{B}}{2} - \frac{1}{4m} (A\overline{B}^2 + B\overline{D}^2 - 2D\overline{B}\overline{D}) - \frac{\overline{B}}{2C} \left( \overline{C} - \frac{E}{Z} \right), \]

\[ w' = \frac{\overline{D}}{2} - \frac{1}{4m} (B\overline{A}\overline{D} + A\overline{B}\overline{D} - D\overline{A}\overline{B} - D\overline{D}^2) - \frac{\overline{D}}{2C} \left( \overline{C} - \frac{E}{Z} \right). \]
It is easy to deduce that \( R_{ij} = 0 \) is equivalent to \( P' = 0 \).

Hence, the plane waves \( g_{ij} \) given by (1.6.21) be the solutions of \( R_{ij} = 0 \) are the solutions of \( P' = 0 \).

Extension of Takeno’s work was carried out by Thengane, Zade, Karade (2001), Deshmukh and Karade (2002), for \( Z = Z(x, y, z, t) \). Thengane and Karade (2000) generalized the work of Takeno in which they defined the plane gravitational wave \( g_{ij} \) is a non-flat solution of the field equation

\[
R_{ij} = 0, \quad i, j = 1, 2, 3, 4
\]

in an empty region of the space-time with

\[
g_{ij} = g_{ij}(Z), \quad Z = Z(x'), \quad x' = (x, y, z, t)
\]

in some suitable coordinate system with

\[
g'' Z, i Z, j = 0, \quad Z_{, i} = \frac{\partial Z}{\partial x'}
\]

and \( Z = Z(x, y, z, t), \quad Z_{, i} \neq 0 \).

This is the condition, which is different from that of Takeno.

The signature convention adopted is as follow,

\[
g_{\alpha \alpha} < 0, \quad \begin{vmatrix} g_{\alpha \alpha} & g_{\alpha \beta} \\ g_{\beta \alpha} & g_{\beta \beta} \end{vmatrix} > 0, \quad \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} < 0, \quad g_{44} > 0,
\]
(no summation for $\alpha$ and $\beta$; $\alpha, \beta=1, 2, 3$).

And accordingly $g = \det(g_{ij}) < 0$.

They have shown that the plane wave solutions of the field equations $R_{ij} = 0$ in $V_4$ are the solutions of

$$Q \gamma_{ij} + P \delta_{ij} = 0, \quad i, j = 1, 2, 3, 4, \quad (1.6.25)$$

where $\gamma_{ij}, \delta_{ij}$ are the functions of $Z$.

Which further breaks into

$$\bar{\omega}_1 \gamma_{ij} + \bar{\omega}_1 \delta_{ij} = 0 = \bar{\phi}_1 \gamma_{ij} + \bar{\phi}_1 \delta_{ij},$$

$$\bar{\omega}_2 \gamma_{ij} + \bar{\omega}_2 \delta_{ij} = 0 = \bar{\phi}_2 \gamma_{ij} + \bar{\phi}_2 \delta_{ij},$$

$$\bar{\omega}_3 \gamma_{ij} + \bar{\omega}_3 \delta_{ij} = 0 = \bar{\phi}_3 \gamma_{ij} + \bar{\phi}_3 \delta_{ij},$$

which is again in the Takeno's form.

By imposing $\delta_{ij} = 0$, Zade and Karade (2002) reduced the equation (1.6.25) in the form

$$\bar{L}_2 - \bar{\gamma}_4 + \frac{\gamma_4^2}{2} - L_2 \bar{\gamma}_4 + \frac{L_1}{4} = 0,$$

for $Z = \frac{\sqrt{3} t}{x + y + z}$.
where $\gamma_i = \bar{g}_{ij} \omega^j_i$, $L_1 = g^{ij} g^{kl} \bar{g}_{ik} \bar{g}_{jl}$, $L_2 = \log \sqrt{1 - g}$.

For \( \left( \frac{\sqrt{3} t}{x + y + z} \right) \)-type plane wave they have deduced the line element

\[ ds^2 = -AZ^2(dx^2 + dy^2 + dz^2) + Adt^2, \]

where \( A \) be the function of \( Z = \frac{\sqrt{3} t}{x + y + z} \).

Deshmukh and Karade (2002), has considered \( Z = Z(x, y, z, t) \) and obtained the line element

\[

d s^2 = -\phi_1^2C \, dx^2 - 2\phi_1\phi_2C \, dxdy - \phi_2^2E \, dy^2 - \phi_3^2E \, dz^2 \\
+ 2\phi_3C \, dzdt + C \, dt^2, 
\]

where \( \phi_1, \phi_2, \phi_3, C \) and \( E \) are the functions of \( Z \) and \( \phi_i = \frac{Z_i}{Z} \).

On the line of Takeno (1961), Thengane and Karade (2000), Zade and Karade (2002) and Deshmkh and Karade (2002), we have shown in Chapter IV that the plane gravitational waves \( g_\theta(Z) \), \( Z = \frac{\sqrt{3} t}{x + y + z} \) of the space-time (1.6.26) be the solution of the equation,

\[

P = \frac{\bar{m}}{2m} + \frac{\bar{n}}{2n} - \frac{C}{C} - \frac{\bar{m}^2}{4m^2} - \frac{\bar{n}^2}{4n^2} - \frac{3C^2}{2C^2} - \frac{mn\bar{C}}{2mn\bar{C}} \\
- \frac{Z^4(\bar{C}\bar{E} - C^2)}{18m} + \frac{Z^2(\bar{C}\bar{E} + C^2)}{6n} - \frac{\bar{m}}{Zm} - \frac{C}{ZC} + \frac{C^2}{6n} = 0, \tag{1.6.27}
\]
where, \( m = \frac{Z^4}{9} (EC - C^2) \) and \( n = \frac{-Z^2}{3} (EC + C^2) \),

which is in Takeno form.

Further, the plane wave solution of the field equation \( R_{ij} = 0 \) for \( \left( \frac{\sqrt{3} \ t}{x + y + z} \right) \)-type wave can be obtained by using the concept of curvature tensor and shown that \( g_{ij} = g_{ij}(Z) \) be the solutions of the equation

\[
P^i = 0, \quad i = 1, 2, 3, 4,
\]

where,

\[
P^1 = \frac{CZ^3u'}{3m} - \frac{Cv'}{n} = \frac{Z^3C^2}{6n(x + y + z)^2},
\]

\[
P^2 = P^1 + \frac{E(ZE + 2E) - C(ZC + 2C)}{2(E^2 - C^2)(x + y + z)^2},
\]

\[
P^3 = P^1 + \frac{E(2ZE + 5E) - C(2ZC + 5C)}{2(E^2 - C^2)(x + y + z)^2},
\]

\[
P^4 = \frac{\sqrt{3}}{Z} \left[ P^1 + \frac{E(ZE + 3E) - C(ZC + 3C)}{2(E^2 - C^2)(x + y + z)^2} \right]
\]

and

\[
u' = \frac{Z^4(\overline{E} - \overline{C})}{6(x + y + z)^2} - \frac{Z^4C(\overline{E} - \overline{C})^2}{108m(x + y + z)^2} - \frac{Z^4(\overline{C} \overline{E} - \overline{C}^2)}{6C(x + y + z)^2} - \frac{Z^3(\overline{EC} - CE)}{3C(x + y + z)^2},
\]
\[
v' = \frac{Z^4 (\overline{E} + \overline{C})}{6(x + y + z)^2} + \frac{Z^5 C (\overline{E} + \overline{C})^2}{36n(x + y + z)^2} - \frac{Z^4 (\overline{C} \overline{E} + \overline{C}^2)}{6C(x + y + z)^2} - \frac{Z^3 (E \overline{C} - C \overline{E})}{3C(x + y + z)^2}.
\]

Takeno has obtained the coordinate system in which

\[g_{\alpha \alpha} = 0; \ a = 1, 2, \ \alpha = 3, 4\]

in four dimensional space-time \(V_4\) for \(Z = (z - t)\) and \(Z = (t/z)\).

Adhav and Karade (1994) extended the work of Takeno (1961) to higher dimensional space-time \(V_6\) by choosing coordinate system in which

\[g_{\alpha \alpha} = 0, \ a = 1, 2, 3, 4; \ \alpha = 5, 6.\]

In six dimensional space-time \(V_6\), Adhav and Karade (1994), reformulated the Takeno’s definition of plane wave as follows:

A Plane gravitational waves \(g_{ij}\) as the non-flat solutions of the Einstein’s field equations

\[R_{ij} = 0, \ i, j = 1, 2, 3, \ldots, 6\]

in an empty region of the space-time with

\[g_{ij} = g_{ij}(Z), \ Z = Z(x^i), \ x^i = (x^1, x^2, x^3, x^4, x^5, x^6)\]

in some suitable coordinate system with

\[g^{ij} Z_{,i} Z_{,j} = 0, \ Z_{,i} = \frac{\partial Z}{\partial x^i}\]

and \(Z = (z, t); \ Z_{,5} \neq 0, \ Z_{,6} \neq 0.\)
The signature convention adopted is as

\[
\begin{vmatrix}
g_{ll} & g_{lm} \\
g_{ml} & g_{mm}
\end{vmatrix} > 0, \quad \begin{vmatrix}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33} \\
g_{41} & g_{42} & g_{43} & g_{44} \\
g_{51} & g_{52} & g_{53} & g_{54} & g_{55}
\end{vmatrix} < 0, \quad g_{66} > 0,
\]

(not summed for \( l \) and \( m \); \( l, m = 1, 2, 3, 4, 5 \)).

And accordingly \( g = \det (g_{ij}) < 0 \).

In chapter V, we have determined this chosen coordinate system in which

\[
g_{\alpha\alpha} = 0, \quad \alpha = 1, 2, 3, 4; \quad \alpha = 5, 6
\]

in six dimensional space-time \( V_6 \).

The analogous work was already carried out by Chirde et al. (2005) in five dimensional space-time \( V_5 \) for \( Z = (z - t) \) and \( Z = (t/z) \).

In chapter VI, we have deduced the line element,

\[
ds^2 = -A dx^2 - 2D dx dy - B dy^2 - Z^2 (C - E) dz^2 - 2ZE dz dt + (C + E) dt^2,
\]

where \( A, B, C, D \) are the functions of \( Z \) and \( E = E(x, y, Z), \ Z = (t/z) \) by generalizing Takeno space-time, on the line of K. B. Lal and N. Ali (1970).
In chapter VII, we have shown that the plane gravitational wave $g_{ij}$ of the space-time (1.6.21) be the solutions of the weakened field equations (1.5.1) – (1.5.5) in the general relativity.

Also we have proved the following theorems:

**Theorem 1:** The space-time (1.6.21) is the plane wave solutions of wfe (1.5.1), (1.5.2) and (1.5.4).

**Theorem 2:** A necessary and sufficient condition that $g_{ij}$ given by (1.6.21) be a solutions of wfe (1.5.3) is i) $P = 0$, ii) $M = 0$.

where

$$P = \frac{z^2 R_{33}}{Z^2} \quad \text{and} \quad M = \left( \frac{P}{C} - \frac{2PC}{C^2} - \frac{5PC^2}{C} + \frac{2PE}{ZC} + \frac{5EPC}{ZC^2} + \frac{14PEC}{ZC} + \frac{6PE^2}{Z^2C^2} - \frac{2PE}{Z^2C} \right) / Z^4.$$  

**Theorem 3:** A necessary and sufficient condition that $g_{ij}$ given by (1.6.21) be the solutions of wfe (1.5.5), is $P = -\frac{2P}{C} \left( \frac{E}{Z} - \bar{C} \right)$.

Similar work has been done by S. N. Pandey (1975) and Chirde et al. (2005). They have proved that the purely plane gravitational waves $g_{ij}$ of the space-time given by equation (1.6.20) and (1.6.19) are the solutions of the weakened field equations (1.5.1) – (1.5.5).