SYNOPSIS

The thesis entitled "Investigations Of Plane Symmetry In The General Theory Of Relativity", comprises seven chapters and is mainly devoted to the study of plane gravitational waves in Einstein’s theory of general relativity on the line of Takeno (1961). Furthermore some geometric aspects of plane symmetry space-time, plane symmetric inhomogeneous cosmological models, plane wave solutions of the weakened field equations and the problems connected with plane wave \((z-t)\)-type and \((t/z)\)-type have been discussed.

The first chapter is introductory and contains the information in brief of the concepts such as general theory of relativity, plane gravitational waves, weakened field equations, the work of Takeno (1961) on plane gravitational waves, other expositions and our investigations.

The motivations for our work is that of Takeno’s (1961) contribution in the form of space-times (2) and (3) which can obtained from (1). The space-time (1) represents \((z-t)\)-types plane gravitational waves as

\[
ds^2 = -Adx^2 - 2Ddxdy - Bdy^2 - (C - E)dz^2 - 2Edzdt + (C + E)dt^2,
\]

where \(A, B, C, D, E\) are functions of \(Z = (z-t)\), with \(A, B > 0\), \(m = (AB - D^2) > 0\), \(C > |E|\).

Taking \(E = 0\), \(C = 1\), the space-time (1) is transformed to the form

\[
ds^2 = -Adx^2 - 2Ddxdy - Bdy^2 - dz^2 + dt^2,
\]

for \((t/z)\)-type wave, it assumes the form
\[ ds^2 = -Adx^2 - 2Ddxdy - Bdy^2 - Z^2(C - E)dz^2 \\
- 2ZEdzdt + (C + E)dt^2, \]  \hspace{1cm} (3)

where \( A, B, C, D, E \) are functions of \( Z = (t/z) \).

In chapter II, we have studied some geometrical aspects of (2) and (3) with regard to constant curvature, symmetric space, recurrent spaces on the line of Panigrahi and Patra (2005).

In chapter III, we have investigated plane symmetric inhomogeneous models in presence of massive scalar field with perfect fluid distribution in general relativity for a given gravitational field. Their physical and geometrical behaviour have been discussed. Here we deduced the plane symmetric inhomogeneous models

\[ ds^2 = -t^{4/3}(1 + x^2)^2dx^2 - t^{4/3}(dy^2 + dz^2) + k^2dt^2 \hspace{1cm} \text{and} \]
\[ ds^2 = -t^{-4\alpha+2}(1 + x^2)^2dx^2 - t^{-2\alpha}(dy^2 + dz^2) + k^2dt^2, \]  \hspace{1cm} (4) (5)

where \( a, \alpha, k \neq 0 \) are real constants.


The line element deduced by Deshmukh and Karade (2002) in the form

\[ ds^2 = -\phi_1^2Cd\phi_1^2 - 2\phi_1\phi_2Cd\phi_2 - \phi_2^2Ed\phi_2 + \phi_3^2Edz^2 \\
+ 2\phi_3Cdzdt + Cdt^2, \]  \hspace{1cm} (6)

where \( \phi_1, \phi_2, \phi_3, C, E \) are functions of \( Z = Z(x, y, z, t) \)

and \( \phi_i = \frac{Z_i}{Z_{ij}} \); ( , ) represents a partial derivative.
Taking recourse to this space-time (6) we have shown that the plane wave
\[ g_y(Z), Z = \frac{\sqrt{3} t}{x + y + z} \]
of the space-time (6) are the solutions of the field equations
\[
\frac{m}{2m} + \frac{n}{2n} - \frac{C}{C} - \frac{m^2}{4m^2} + \frac{n^2}{4n^2} - \frac{3C^2}{2C^2} - \frac{mn C}{2mn C} - \frac{Z^4 (C E - C^2)}{18m} + \frac{Z^2 (C E + C^2)}{6n} - \frac{m}{Zm} - \frac{C}{ZC} + \frac{C^2}{6n} = 0 = P \text{ (say),}
\]
where \( m = \frac{Z^4}{9} (EC - C^2) \) and \( n = -\frac{Z^2}{3} (EC + C^2) \).

It is also demonstrated that the plane wave solutions of the field equations
\( R_{ik} = 0 \) for \( \left( -\frac{\sqrt{3} t}{x + y + z} \right) \)-type wave can be obtained by using the concept of curvature tensor \( R_{ijkl} \) and Ricci tensor \( R_{ij} \), the metric tensor \( g_y = g_y(Z) \) be the solutions of the equations
\[ p' = 0, \quad i = 1, 2, 3, 4, \]

\[ p^1 = \frac{CZ^2 u'}{3m} - \frac{Cv'}{n} + \frac{C^2 Z^2}{6n(x + y + z)^2}, \]

\[ p^2 = p^1 + \frac{E(ZE + 2E) - C(ZC + 2C)}{2(E^2 - C^2) (x + y + z)^2}, \]

\[ p^3 = p^1 + \frac{E(2ZE + 5E) - C(2ZC + 5C)}{2(E^2 - C^2) (x + y + z)^2}, \]

\[ p^4 = \frac{\sqrt{3}}{Z} \left[ p^1 + \frac{E(ZE + 3E) - C(ZC + 3C)}{2(E^2 - C^2) (x + y + z)^2} \right] \text{ and } \]

(iii)
$$u' = \frac{Z^4 (\bar{E} - \bar{C})}{6(x + y + z)^2} - \frac{Z^8 C (\bar{E} - \bar{C})^2}{108m(x + y + z)^2} - \frac{Z^4 (\bar{C} E - \bar{C})^2}{6C(x + y + z)^2} - \frac{Z^3 (E \bar{C} - CE)}{3C(x + y + z)^2}$$

$$v' = \frac{Z^4 (\bar{E} + \bar{C})}{6(x + y + z)^2} + \frac{Z^6 C (\bar{E} + \bar{C})^2}{36n(x + y + z)^2} - \frac{Z^4 (\bar{C} E + \bar{C})^2}{6C(x + y + z)^2} - \frac{Z^3 (E \bar{C} - CE)}{3C(x + y + z)^2}.$$

In Chapter V we deduced the co-ordinate system chosen by Adhav and Karade (1994) where $g_{\alpha\alpha} = 0, \alpha = 1, 2, 3, 4; \alpha = 5, 6$.

This choice led Adhav and Karade (1994) to the investigation of the plane gravitational waves in a space-time $V_6$. The said system is explicitly expressed in the form:

$$\begin{align*}
x^1 &= x'^1 - c_1 z', \\
x^2 &= x'^2 - c_2 z', \\
x^3 &= x'^3 - c_3 z', \\
x^4 &= x'^4 - c_4 z', \\
x^5 &= x'^5 = z = z', \\
x^6 &= x'^6 = t = t',
\end{align*}$$

where $g_{\alpha\alpha} = 0, \alpha = 1, 2, 3, 4; \alpha = 5, 6$.

If $Z(x', x^2, x^3, x^4, x^5, x^6)$ are independent of $x^4$ and $x^5, x^6$ then this co-ordinate system transforms to that of Chirde et al. (2005) and Takeno (1961).

In chapter VI, we have deduced the line element,

$$ds^2 = -Adx^2 - 2Ddxdy - Bdy^2 - Z^2 (C - E)dz^2 - 2ZEzdzt + (C + E)dt^2,$$

where $A, B, C, D$ are the functions of $Z$ and $E = E(x, y, Z), Z = (t/z)$ by generalizing Takeno space-time, on the line of K B Lal and N Ali (1970).

In the last chapter, we have proved that the plane gravitational waves $g_{ij}$ given by the space-time

$$(iv)$$
\[ ds^2 = -Adx^2 - 2Ddx dy - Bdy^2 - Z^2(C - E)dz^2 - 2ZEdzdt + (C + E)dt^2 , \]

where \( A, B, C, D, E \) are the functions of \( Z = t/z \), with \( A, B > 0 \), \( m = (AB - D^2) > 0 \) and \( C > |E| \), be the solutions of the following weakened field equations (wfe).

\[ T_{ik} = R^p_{ik;p} = 0, \]

\[ (-g)^{1/4} [g^{gh} R_{k;ih} - g^{ih} R_{g;kh} + (1/6) R_{ik} - (1/6) g_{jk} g^{ih} R_{j;h} - R^{ih} C_{jnk} + (R/6) g^{ih} C_{jnh} ] = 0, \]  

\[ (-g)^{1/2} [g^{ij} g^{kl} \{2 R_{ijkl} R^{ml} + g^{ml} R_{ijkl} - R_{ij} \} - (1/2) g^{hk} (R^l_{ml} R^m_{lj} - g^{lm} R_{ljm})] = 0, \]

\[ (-g)^{1/2} \{g^{hk} g^{nu} - (1/2) g^{ht} g^{ku} - (1/2) g^{hu} g^{kt} \} R_{klt} + R(R^{kh} - (1/4) g^{kh} R)] = 0, \]

\[ Q_{ik}^{ij} = R_{ik}^{ij} = 0, \]

where \( C_{jnk} \) is the Weyl curvature tensor and semicolon \( (; ) \) denotes the covariant derivative.

Some of these results are expressed in the theorems.