Chapter - III

NATURE OF PLANE SYMMETRIC INHOMOGENEOUS MASSIVE SCALAR FIELD AND PERFECT FLUID IN GENERAL RELATIVITY

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[3.1] INTRODUCTION

It is known that general theory of relativity or Einstein’s theory of gravitation is coordinate invariant. It on majority scale serves as a basis for models of the universe. At the present state of evolution, the matter distribution in the universe seems to be spherically symmetric and it is speculated that isotropic and homogeneous as far as large scale structure is concerned. In its early stages of evolution, it could not have had such a smoothed out picture. It is true that plane symmetry is less restrictive than spherical symmetry and provides avenue to study inhomogeneous cosmological models that play an important role in understanding some essential features of the universe. Moreover, the problem of plane symmetric space-time with perfect fluid as the source has been taken up view of possible applications to astrophysics, cosmology and special relativistic hydrodynamics.

In this chapter, for given (particular) values of gravitational field, we study the behaviour of massive scalar field (V), pressure (p) and density (ρ) of perfect fluid. This shows effect of geometry on matter that is how the geometry ask the matter to behave which have been discussed as the physical and geometrical aspects in section [3.4].

The analogous study was already carried out by Ray and Raj Bali (1978), Panigrahi and Sahu (2002) and Panigrahi, Patra and Sahu (2005) with reference to spherical symmetry.
[3.2] EINSTEIN'S FIELD EQUATIONS

We consider the space-time describing plane symmetry

$$ds^2 = -A^2 dx^2 - B^2 (dy^2 + dz^2) + D^2 dt^2,$$

(3.2.1)

where $A, B$ and $D$ are the functions of $x$ and $t$ only.

The Einstein's field equations are

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij}, \quad i, j = 1, \ldots, 4,$$

(3.2.2)

where the symbols have their usual meaning.

In the present case the energy momentum tensor

$$T_{ij} = T_{ij}^\rho + T_{ij}^\nu,$$

(3.2.3)

where $T_{ij}^\rho$ is the energy momentum tensor of gravitating field represented by perfect fluid as

$$T_{ij}^\rho = (\rho + p) u_i u_j - pg_{ij},$$

(3.2.4)

together with

$$g_{ij} u^i u^j = 1, \quad u_i u^i = 1.$$  

(3.2.5)

The quantities $\rho$, $p$ and $u^i$ are energy density, proper pressure and four velocity vector of the perfect fluid respectively.

The tensor $T_{ij}^\nu$ represents the massive scalar field:
\[
T_{ij}^\nu = \frac{1}{4\pi} \left[ V_i V_j - \frac{1}{2} g_{ij} (V^s V^s - M^2 V^2) \right], \quad s = 1, 2, 3, 4, \quad \text{(3.2.6)}
\]

where \( V = V(x, t) \) is a massive scalar field.

The non-vanishing Christoffel's symbols for (3.2.1) are

\[
\begin{align*}
\Gamma^1_{11} &= \frac{A_1}{A}, \quad \Gamma^1_{14} = \frac{A_4}{A}, \quad \Gamma^1_{22} = \Gamma^1_{33} = \frac{-BB_1}{A^2}, \quad \Gamma^1_{44} = \frac{DD_1}{A^2}, \\
\Gamma^2_{12} &= \Gamma^2_{13} = \frac{B_1}{B}, \quad \Gamma^2_{24} = \Gamma^3_{34} = \frac{B_4}{B}, \quad \Gamma^4_{11} = \frac{AA_4}{D^2}, \\
\Gamma^4_{14} &= \frac{D_1}{D}, \quad \Gamma^4_{22} = \Gamma^4_{33} = \frac{BB_1}{D^2}, \quad \Gamma^4_{44} = \frac{D_4}{D}.
\end{align*}
\]

The non-zero components of the Ricci tensor yield

\[
\begin{align*}
R_{11} &= \frac{2B_1}{B} - \frac{AA_4 A_4}{A^2} + \frac{D_1}{D} - \frac{2A_1 B_1}{A B} - \frac{A_1 D_1}{A D} - \frac{2A A_4 B_4}{B D^2} + \frac{A A_4 D_4}{A D^3}, \\
R_{14} &= \frac{2B_4}{B} - \frac{2A_4 B_1}{A B} - \frac{2B_4 D_1}{B D}, \\
R_{22} &= R_{33} = \frac{BB_1}{A^2} - \frac{BB_4}{A^2} - \frac{B A_1 B_1}{A^3} + \frac{B B_1 D_1}{A^2 D} + \frac{B_1^2}{A^2} - \frac{B A_1 B_4}{A D^2} + \frac{B B_4 D_4}{A D^3} - \frac{B_4^2}{D^2}, \\
R_{44} &= \frac{A_4}{A} + \frac{2B_4}{B} - \frac{D D_1}{A^2} + \frac{D A_1 D_1}{A^3} - \frac{2D B_1 D_1}{A^2 B} - \frac{A_4 D_4}{A D} - \frac{2B_1 D_4}{B D}.
\end{align*}
\]

In this case, the scalar curvature \( R \) is

\[
R = 2 \left[ \frac{A_4}{A D^2} - \frac{2B_1}{A^2 B} + \frac{2B_4}{A^2 D} - \frac{D_1}{A^2 D} + \frac{2A_1 B_1}{A^2 B} + \frac{A_1 D_1}{A^3 D} - \frac{2B_1 D_1}{A^2 B D} + \frac{2A_4 B_4}{A B D^2} - \frac{A_4 D_4}{A D^3} - \frac{2B_4 D_4}{A B^2} - \frac{B_1^2}{A^2 B^2} + \frac{B_4^2}{B^2 D^2} \right]. \quad \text{(3.2.9)}
\]
Here the suffixes 1 and 4 indicate partial differentiation w. r. t. x and t respectively.

(Please see appendix [3.1])

Since the coordinates are considered to be co-moving,

\[ u^1 = 0 = u^2 = u^3 \text{ and } u^4 = \frac{1}{D}. \]

From (3.2.1) and (3.2.4), we have

\[ T^P_{11} = pA^2, \]
\[ T^P_{22} = T^P_{33} = pB^2, \]
\[ T^P_{44} = \rho D^2. \]  \hspace{1cm} (3.2.10)

Also (3.2.1) and (3.2.6), give

\[ T^r_{11} = \frac{A^2}{8\pi} \left[ \frac{V_1^2}{A^2} + \frac{V_4^2}{D^2} - M^2 V^2 \right], \]  \hspace{1cm} (3.2.11)
\[ T^r_{22} = T^r_{33} = \frac{B^2}{8\pi} \left[ -\frac{V_1^2}{A^2} + \frac{V_4^2}{D^2} - M^2 V^2 \right], \]
\[ T^r_{44} = \frac{V_1 V_4}{4\pi} \]  \hspace{1cm} (3.2.11)

and \[ T^r_{44} = \frac{D^2}{8\pi} \left[ \frac{V_1^2}{A^2} + \frac{V_4^2}{D^2} + M^2 V^2 \right]. \]
(Please see appendix [3.2]).

Using (3.2.8), (3.2.9), (3.2.10) and (3.2.11) in (3.2.2), the field equations are reduced to

\[
\frac{2}{BD^2} \left[ B_{44} \frac{DB_1 D_1}{A^2} - \frac{B_4 D_4}{D} \right] - \frac{1}{B^2} \left[ \frac{B_1^2}{A^2} - \frac{B_4^2}{D^2} \right] = -8\pi \left[ p + \frac{1}{8\pi} \left( \frac{V_1^2}{A^2} + \frac{V_4^2}{D^2} - M^2 V^2 \right) \right],
\]

(3.2.12)

\[
\frac{2}{B} \left[ B_{14} - \frac{A_4 B_1}{A} - \frac{D_1 B_4}{D} \right] = -V_1 V_4,
\]

(3.2.13)

\[
\frac{1}{BD^2} \left[ B_{44} \frac{DB_1 D_1}{A^2} - \frac{B_4 D_4}{D} \right] - \frac{1}{A^2 B} \left[ B_{11} - \frac{A_4 B_1}{A} - \frac{A A_4 B_4}{D^2} \right]
- \frac{1}{A^2 D} \left[ A A_{44} - \frac{A A_4 D_4}{D} - D D_{11} + \frac{D A_4 D_1}{A} \right] = -8\pi \left[ p + \frac{1}{8\pi} \left( \frac{-V_1^2}{A^2} + \frac{V_4^2}{D^2} - M^2 V^2 \right) \right] \quad \text{and}
\]

(3.2.14)

\[
\frac{2}{A^2 B} \left[ B_{11} - \frac{A_4 B_1}{A} - \frac{A A_4 B_4}{D^2} \right] + \frac{1}{B^2} \left[ \frac{B_1^2}{A^2} - \frac{B_4^2}{D^2} \right] = -8\pi \left[ p + \frac{1}{8\pi} \left( \frac{V_1^2}{A^2} + \frac{V_4^2}{D^2} + M^2 V^2 \right) \right].
\]

(3.2.15)

The massive scalar field \( V \) satisfies the Klein-Gordon equation:

\[
g^{\theta \nu} V_{;\nu} + M^2 V = 0,
\]

(3.2.16)

where the semicolon ( ; ) denotes covariant differentiation.
For the line element (3.2.1) it takes form

\[ \frac{V_{44}}{D^2} + \left( \frac{A_4}{4} + \frac{2B_4}{B} - \frac{D_4}{D} \right) \frac{V_4}{D^2} + \left( \frac{A_1}{A} - \frac{2B_1}{B} - \frac{D_1}{D} \right) \frac{V_1}{A^2} - \frac{V_{11}}{A^2} + M^2 V = 0. \]  

(3.2.17)

Now we have five equations (3.2.12) – (3.2.17) containing six unknowns viz., \( A, B, D, \rho, p \) and \( V \). To make the system determinate we consider an extra condition as is explained in the following section.

### [3.3] SOLUTIONS OF FIELD EQUATIONS

The field equations (3.2.12) – (3.2.15) and (3.2.17) are highly nonlinear in nature. To obtain the solutions of the field equations we introduce some condition as we take \( V \) as a function of ‘\( t \)’ only. We consider following particular case by taking the metric potentials \( A, B, \) and \( D \) are as follow:

**Case I:** \( A = t^\alpha (1 + x^2)^a, B = t^\alpha \) and \( D = k. \)  

(3.3.1)

where \( \alpha, a \) and \( k \neq 0 \) are real constants.

From equations (3.2.17) and (3.3.1), we obtain

\[ \frac{V_{44}}{k^2} + \frac{3\alpha V_4}{k^2} + M^2 V = 0. \]  

(3.3.2)

On integration, in particular, for \( \alpha = 2/3 \), we have
\[ V = \frac{c_1 \cos kMt}{t} + \frac{c_2 \sin kMt}{t}, \quad (3.3.3) \]

where \( c_1 \) and \( c_2 \) are constants.

Using equations (3.3.1) and (3.3.3) in equations (3.2.12) – (3.2.15), we get

\[ p = -\frac{1}{8\pi} \left[ \frac{V_4^2}{k^2} - M^2 V^2 \right] \quad (3.3.4) \]

and

\[ \rho = -\frac{1}{6\pi k^2 t^2} - \frac{1}{8\pi} \left[ \frac{V_4^2}{k^2} + M^2 V^2 \right], \quad (3.3.5) \]

where \( V \) is given by (3.3.3).

For this, the cosmological model for space-time (3.2.1) in Einstein’s theory of general relativity is given by

\[ ds^2 = -t^{4/3} (1 + x^2)^{2a} dx^2 - t^{4/3} (dy^2 + dz^2) + k^2 dt^2. \quad (3.3.6) \]

**Case II:** \( A = t^{-2\alpha+1}, B = t^{\alpha} \) and \( D = k \),

where \( \alpha, a \) and \( k \neq 0 \) are real constants.

From equations (3.2.17) and (3.3.7), we have

\[ \frac{V_{44}}{k^2} + \frac{V_4}{k^2 t} + M^2 V = 0. \quad (3.3.8) \]
On integration, we have

\[ V = c_3 J_0(kMt) + c_4 Y_0(kMt), \]  

(3.3.9)

where \( c_3 \) and \( c_4 \) are constants and

\[ J_0(kMt) = 1 - \frac{(kMt)^2}{2^2} + \frac{(kMt)^4}{2^4 4^2} - \frac{(kMt)^6}{2^6 4^2 6^2} + \ldots \]

and

\[ Y_0(kMt) = J_0(kMt) \log kMt + \sum_{n=1}^{\infty} \frac{(-1)^n b_n}{2^n (n!)^2} (kMt)^{2n}, \]

(3.3.10)

where \( b_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \).

Using equations (3.3.7) and (3.3.9) in equations (3.2.12) – (3.2.15), we get

\[ p = \frac{\alpha(2-3\alpha)}{8\pi k^2 t^2} - \frac{1}{8\pi} \left[ \frac{V_4^2}{k^2} - M^2 V^2 \right] \]

(3.3.10)

and

\[ \rho = \frac{\alpha(2-3\alpha)}{8\pi k^2 t^2} - \frac{1}{8\pi} \left[ \frac{V_4^2}{k^2} + M^2 V^2 \right], \]

(3.3.11)

where \( V \) is given by (3.3.9).

For this, the cosmological model for space-time (3.2.1) in Einstein’s theory of general relativity is given by

\[ ds^2 = -t^{-4\alpha+2}(1 + x^2)^{2\alpha} dx^2 - t^{2\alpha} (dy^2 + dz^2) + k^2 dt^2. \]

(3.3.12)
[3.4] PHYSICAL AND GEOMETRICAL PROPERTIES OF THE MODELS

Case I: The equations (3.3.3), (3.3.4) and (3.3.5) gives massive scalar field $V$, parameters pressure $p$ and density $\rho$ respectively, for the model (3.3.6). From these equations it is clear that $V, p, \rho$ are decreasing functions of time $t$.

Now the reality conditions given by Ellis (1971) (i) $\rho + p > 0$, (ii) $\rho + 3p > 0$ and (iii) $\rho > 0$ are satisfied.

Also the dominant energy conditions given by Hawking and Ellis (1973) (i) $\rho - p \geq 0$ (ii) $\rho + p \geq 0$ are satisfied.

As $t \to \infty$, then $V \to 0$, $p \to 0$ and $\rho \to 0$. In this case, the space-time represents a vacuum universe.

As $t \to 0$, then $V \to \infty$ and $p, \rho$ are in indeterminate form and it depends on choices of constants $c_1$ and $c_2$.

The scalar expansion $\theta = \frac{2\alpha}{kt}$. Now $\theta \to 0$ as $t \to \infty$ and $\theta \to \infty$ as $t \to 0$. Thus the universe is expanding with increase of time where as the rate of expansion is slow with increase of time.

For the model (3.3.6), the shear scalar $\sigma^2 = 0$ and thus $\lim_{t \to \infty} \frac{\sigma}{\theta} = 0$.

(Please see appendix [3.3]).

Hence the model is shear free and isotropic in nature.
**Case II:** The equations (3.3.9), (3.3.10) and (3.3.11) give scalar massive field $V$, parameters pressure $p$ and density $\rho$ respectively, for the model (3.3.12).

As $t \to 0$, $\rho \to \infty$, $p \to \infty$ and $V \to \infty$ if $c_4 < 0$. In this case the space-time collapses and admits a singularity, which may be a Big-bang singularity.

As $t \to \infty$, $\rho \to 0$ and $p \to 0$, $V$ has singularity ($c_3 \neq 0$ or $c_4 = 0$). Thus the model of the universe does not exist. In this case the space-time represents an empty universe.

The scalar expansion $\theta = \frac{1}{kt}$. Here $\theta \to 0$ as $t \to \infty$ which shows that the model is expanding in nature.

The shear scalar for the model (3.3.12) is given by $\sigma^2 = \frac{2}{3} \left( \frac{1-3\alpha}{t} \right)^2$.

(Please see appendix [3.4]).

Here $\sigma^2 \to \infty$ as $t \to 0$ and $\sigma^2 \to 0$ as $t \to \infty$. Also the rate of change of the shape of the universe is slow with increase of time. Further we obtain $\lim_{t \to \infty} \frac{\sigma}{\theta} = \frac{\sqrt{2}}{\sqrt{3}} k (1-3\alpha)$.

If $\alpha = 0$ and $k = 1$ then $\lim_{t \to \infty} \frac{\sigma}{\theta} = \frac{\sqrt{2}}{\sqrt{3}}$ and it indicates that the universe remains anisotropy through out the evolution.
APPENDIX [3.1]

The formula for Ricci tensor is given by

\[ R_{mn} = \Gamma^r_{mr,n} - \Gamma^r_{mn,r} + \Gamma^p_{mr} \Gamma^r_{pn} - \Gamma^p_{mn} \Gamma^r_{Pr}. \]

Therefore,

\[ R_{11} = \Gamma^r_{1r,1} - \Gamma^r_{11,r} + \Gamma^p_{1r} \Gamma^r_{p1} - \Gamma^p_{11} \Gamma^r_{Pr} \]

\[ = \Gamma^2_{12,1} + \Gamma^3_{13,1} + \Gamma^4_{14,1} - \Gamma^4_{11,4} + (\Gamma^1_{12})^2 + (\Gamma^1_{13})^2 + (\Gamma^1_{14})^2 + 2\Gamma^1_{14}\Gamma^4_{11} - \Gamma^1_{11} [\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13} + \Gamma^4_{14}] - \Gamma^4_{11} [\Gamma^1_{14} + \Gamma^2_{24} + \Gamma^3_{34} + \Gamma^4_{44}] \]

\[ = \left( \frac{B_1}{B} \right)_{1} + \left( \frac{B_2}{B} \right)_{1} + \left( \frac{D_1}{D} \right)_{1} - \left( \frac{A A_4}{D^2} \right)_{1,4} + \frac{A_1^2}{A^2} + \frac{B_1^2}{B^2} + \frac{B_2^2}{B^2} + \frac{D_1^2}{D^2} \]

\[ + 2 \left( \frac{A_4}{A} \right) \left( \frac{A A_4}{D^2} \right) - \frac{A_1}{A} \left[ \frac{B_1}{B} + \frac{B_2}{B} + \frac{D_1}{D} \right] - \frac{A A_4}{D^2} \left[ \frac{A_3}{A} + \frac{B_4}{B} + \frac{B_4}{B} + \frac{D_4}{D} \right] \]

\[ = \frac{2B_1}{B} - \frac{A A_4}{D^2} - \frac{B_2}{B} + \frac{2A_1B_1}{AB} - \frac{A_1D_1}{AD} - \frac{2A A_4 B_4}{B D^2} + \frac{A A_4 D_4}{D^2}. \]

\[ R_{14} = \Gamma^r_{1r,4} - \Gamma^r_{14,r} + \Gamma^p_{1r} \Gamma^r_{p4} - \Gamma^p_{14} \Gamma^r_{Pr} \]

\[ = \Gamma^1_{11,4} + \Gamma^2_{12,4} + \Gamma^3_{13,4} - \Gamma^3_{11,4} + \Gamma^1_{11} \Gamma^1_{14} + \Gamma^2_{11} \Gamma^4_{14} + \Gamma^2_{12} \Gamma^2_{24} + \Gamma^3_{13} \Gamma^3_{34} + \Gamma^4_{14} \Gamma^4_{44} - \Gamma^1_{14} [\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13} + \Gamma^4_{14}] - \Gamma^4_{14} [\Gamma^1_{14} + \Gamma^2_{24} + \Gamma^3_{34} + \Gamma^4_{44}] \]

\[ = \left( \frac{A_1}{A} \right)_{4} + \left( \frac{B_1}{B} \right)_{4} + \left( \frac{B_2}{B} \right)_{4} - \left( \frac{A_1}{A} \right)_{4} + \frac{A_1}{A} \left( \frac{A_4}{A} \right) + \frac{A A_4}{D^2} \left( \frac{D D_1}{A^2} \right) \]

\[ + \frac{B_1}{B} \left( \frac{B_1}{B} \right)_{4} + \frac{B_2}{B} \left( \frac{B_1}{B} \right)_{4} + \frac{A_1}{A} \left( \frac{D_1}{D} \right) + \frac{D_1}{D} \left( \frac{D_1}{D} \right) \]
\[
- \frac{A_4}{A} \left[ \frac{A_1}{A} + \frac{B_1}{B} + \frac{B_4}{B} + \frac{D_1}{D} \right] - \frac{D_1}{D} \left[ \frac{A_4}{A} + \frac{B_4}{B} + \frac{D_4}{D} \right]
= \frac{2B_{14}}{B} - \frac{2A_4B_1}{AB} - \frac{2B_4D_1}{BD}.
\]

\[
R_{22} = \Gamma_{2r,2}^r - \Gamma_{22,r}^r + \Gamma_{2r}^p \Gamma_{r2}^r - \Gamma_{22}^p \Gamma_{rT}^r
= -\Gamma_{22,l}^1 - \Gamma_{22,4}^4 + 2\Gamma_{22}^4 \Gamma_{12}^2 + 2\Gamma_{24}^2 \Gamma_{22}^4
- \Gamma_{22}^4 \left[ \Gamma_{11}^l + \Gamma_{12}^3 + \Gamma_{13}^l + \Gamma_{14}^l \right] - \Gamma_{24}^4 \left[ \Gamma_{14}^l + \Gamma_{24}^3 + \Gamma_{34}^3 + \Gamma_{44}^4 \right]
\]

\[
= -\left( \frac{-BB_1}{A^2} \right) - \left( \frac{BB_1}{D^2} \right)_4 + 2 \left( \frac{-BB_1}{A^2} \right) \left( \frac{B_1}{B} \right) + 2 \left( \frac{B_4}{B} \right) \left( \frac{BB_4}{D^2} \right)
- \left( \frac{-BB_1}{A^2} \right) \left[ \frac{A_1}{A} + \frac{B_1}{B} + \frac{B_4}{B} + \frac{D_1}{D} \right] - \frac{BB_4}{D^2} \left[ \frac{A_4}{A} + \frac{B_4}{B} + \frac{D_4}{D} \right]
= \frac{BB_1}{A^2} - \frac{BB_4}{D^2} - \frac{BA_1B_1}{A^2D} + \frac{BB_1D_1}{A^2D} + \frac{B_2}{A^2} - \frac{BA_4B_4}{AD^2} + \frac{BB_4D_4}{D^2} - \frac{B_2}{D^2}.
\]

Similarly,

\[
R_{33} = R_{22} = \frac{BB_1}{A^2} - \frac{BB_4}{D^2} - \frac{BA_1B_1}{A^2D} + \frac{BB_1D_1}{A^2D} + \frac{B_2}{A^2} - \frac{BA_4B_4}{AD^2} + \frac{BB_4D_4}{D^2} - \frac{B_2}{D^2}.
\]

\[
R_{44} = \Gamma_{4r,4}^r - \Gamma_{44,r}^r + \Gamma_{4r}^p \Gamma_{r4}^r - \Gamma_{44}^p \Gamma_{rT}^r
= \Gamma_{44,4}^l + \Gamma_{34,4}^3 + \Gamma_{14,4}^4 + \left( \Gamma_{14}^1 \right)^2 + \left( \Gamma_{24}^2 \right)^2 + \left( \Gamma_{34}^3 \right)^2 + \left( \Gamma_{44}^4 \right)^2
+ 2\Gamma_{44}^4 \Gamma_{44}^4 \left[ \Gamma_{11}^4 + \Gamma_{12}^3 + \Gamma_{13}^l + \Gamma_{14}^l \right] - \Gamma_{44}^4 \left[ \Gamma_{14}^4 + \Gamma_{24}^3 + \Gamma_{34}^3 + \Gamma_{44}^4 \right]
\]

\[
= \left( \frac{A_4}{A} \right) + \left( \frac{B_4}{B} \right) + \left( \frac{B_4}{B} \right) - \left( \frac{DD_4}{A^2} \right)_4 + \frac{A_2}{A^2} + \frac{B_2}{B^2} + \frac{B_4}{B^2} + \frac{D_4}{D^2}.
\]
\[
+ 2 \left( \frac{D_1}{D} \right) \left( \frac{DD_1}{A^2} \right) - \frac{DD_1}{A^2} \left[ \frac{A_1}{A} + \frac{B_1}{B} + \frac{D_1}{D} \right] - \frac{D_4}{D} \left[ \frac{A_4}{A} + \frac{B_4}{B} + \frac{D_4}{D} \right] \\
= \frac{A_{r4}}{A} + \frac{2B_{r4}}{B} - \frac{DD_{r1}}{A^2} + \frac{DA_1D_1}{A^3} - \frac{2DB_1D_1}{A^2B} - \frac{A_4D_4}{AD} - 2B_4D_4. \\
\]

The scalar curvature \( R \) is given by

\[
R = g^{ij} R_{ij}.
\]

\[
R = g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33} + g^{44} R_{44}.
\]

\[
= g^{11} R_{11} + 2g^{22} R_{22} + g^{44} R_{44} \quad (\because g^{22} R_{22} = g^{33} R_{33})
\]

\[
= -\frac{1}{A^2} \left[ \frac{2B_{r1}}{B} - \frac{AA_{r4}}{D^2} + \frac{D_{r1}}{D} - \frac{2A_1B_1}{AB} - \frac{A_4D_4}{AD} - \frac{2AA_4B_4}{BD^2} + \frac{AA_4D_4}{D^3} \right] \\
- \frac{2}{B^2} \left[ \frac{BB_{r1}}{A^2} - \frac{BB_{r4}}{D^2} - \frac{BA_1B_1}{A^3} + \frac{BB_1D_1}{A^2D} + \frac{B_1^2}{A^2} - \frac{BA_4B_4}{AD^2} + \frac{BB_4D_4}{D^3} - \frac{B_4^2}{D^2} \right] \\
+ \frac{1}{D^2} \left[ \frac{A_{r4}}{A} + \frac{2B_{r4}}{B} - \frac{DD_{r1}}{A^2} + \frac{DA_1D_1}{A^3} - \frac{2DB_1D_1}{A^2B} - \frac{A_4D_4}{AD} - \frac{2B_4D_4}{BD} \right]
\]

\[
= 2 \left[ \frac{A_{r4}}{AD^2} - \frac{2B_{r1}}{A^2B} + \frac{2B_4}{BD^2} - \frac{D_{r1}}{A^2D} + \frac{2A_1B_1}{A^2B} + \frac{A_4D_4}{A^2D} - \frac{2B_4D_1}{A^2BD} - \frac{2A_4B_4}{ABD^2} - \frac{A_4D_4}{AD^3} - \frac{2B_4D_4}{BD^3} - \frac{B_4^2}{A^2B^2} + \frac{B_4^2}{B^2D^2} \right].
\]
APPENDIX [3.2]

We have
\[ T^{\nu}_{ij} = (\rho + P)u_i u_j - P g_{ij} \]
\[ = (\rho + P)u_i u^k g_{kj} - P g_{ij} \]
\[ = (\rho + P)u_i [u^1 g_{1j} + u^2 g_{2j} + u^3 g_{3j} + u^4 g_{4j}] - P g_{ij}. \]

Therefore,
\[ T^{\nu}_{11} = (\rho + P)u_1 [u^1 g_{11} + u^2 g_{21} + u^3 g_{31} + u^4 g_{41}] - P g_{11} \]
\[ = (\rho + P)u_1 u^1 g_{11} - P g_{11} \]
\[ = -P g_{11} \quad (\because u^1 = 0) \]
\[ = PA^2. \]

Similarly, we have
\[ T^{\nu}_{22} = PB^2, \]
\[ T^{\nu}_{33} = PB^2, \]
\[ T^{\nu}_{44} = PD^2. \]

Next, we have
\[ T^{\nu}_{ij} = \frac{1}{4\pi} \left[ V_i V_j - \frac{1}{2} g_{ij} (V_s V^s - M^2 V^2) \right], \quad s = 1, 2, 3, 4, \]

where \( V_i = \frac{\partial V}{\partial x^i}, \quad V \) is a functions of \( x \) and \( t \).
Consider

\[ V_s V^i = V_s g^{ij} V_p \]

\[ = [V_1 g^{1p} + V_2 g^{2p} + V_3 g^{3p} + V_4 g^{4p}] V_p \]

\[ = V_1 [V_p g^{1p}] + V_4 [V_p g^{4p}] \]

\[ = V_1 [V_1 g^{11} + V_2 g^{12} + V_3 g^{13} + V_4 g^{14}] + V_4 [V_1 g^{41} + V_2 g^{42} + V_3 g^{43} + V_4 g^{44}] \]

\[ = V_1^2 g^{11} + V_4^2 g^{44} \]

\[ = -\frac{V_1^2}{A^2} + \frac{V_4^2}{D^2}. \]

\[ \therefore \quad T_{ij} = \frac{1}{4\pi} \left[ V_i V_j - \frac{1}{2} g_{ij} \left( \left( -\frac{V_1^2}{A^2} + \frac{V_4^2}{D^2} \right) - M^2 V^2 \right) \right] . \]

\[ T_{ii} = \frac{1}{4\pi} \left[ V_i V_i - \frac{1}{2} g_{ii} \left( \left( -\frac{V_1^2}{A^2} + \frac{V_4^2}{D^2} \right) - M^2 V^2 \right) \right] \]

\[ = \frac{1}{4\pi} \left[ V_1^2 - \frac{1}{2} (-A^2) \left( \left( -\frac{V_1^2}{A^2} + \frac{V_4^2}{D^2} \right) - M^2 V^2 \right) \right] \]

\[ = \frac{A^2}{8\pi} \left[ \frac{V_1^2}{A^2} + \frac{V_4^2}{D^2} - M^2 V^2 \right] . \]

Similarly, we get

\[ T_{i4} = \frac{V_1 V_4}{4\pi} , \]
\[
T_{22}^V = \frac{B^2}{8\pi} \left[ -\frac{V_1^2}{A^2} + \frac{V_4^2}{D^2} - M^2 V^2 \right],
\]

\[
T_{33}^V = \frac{B^2}{8\pi} \left[ -\frac{V_1^2}{A^2} + \frac{V_4^2}{D^2} - M^2 V^2 \right]
\]

\[
T_{44}^V = \frac{D^2}{8\pi} \left[ \frac{V_1^2}{A^2} + \frac{V_4^2}{D^2} + M^2 V^2 \right].
\]
APPENDIX [3.3]

Case I:

Expansion scalar ($\theta$) $= U^i_j = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} [\sqrt{-g} U^i]$, 

where $U^i = (g_{44})^{-1/2} \delta^i_4$, 

$\Rightarrow U^4 = \frac{1}{D}$ and $U^1 = U^2 = U^3 = 0$.

\[ \theta = \frac{1}{\sqrt{-g}} \left[ \frac{\partial}{\partial x^1} (\sqrt{-g} U^1) + \frac{\partial}{\partial x^2} (\sqrt{-g} U^2) + \frac{\partial}{\partial x^3} (\sqrt{-g} U^3) + \frac{\partial}{\partial x^4} (\sqrt{-g} U^4) \right] \]

\[ = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial t} (\sqrt{-g} U^4) \]

\[ = \frac{1}{A B^2 D} \frac{\partial}{\partial t} \left( A B^2 D \frac{1}{D} \right) \]

\[ = \frac{1}{A B^2 D} [A_4 B^2 + 2 A B B_4] \]

\[ = \frac{A_4}{A D} + \frac{2 B_4}{B D} \]

\[ = \frac{\alpha t^{x-1} (1 + x^2)^a}{t^a (1 + x^2)^a k} + \frac{2 \alpha t^{x-1}}{t^a k} \]

\[ = \frac{3 \alpha}{t^a k}. \]
The shear scalar $\sigma$ is given by

$$
\sigma^2 = \frac{1}{12} \left[ \left( \frac{g_{11,4} - g_{22,4}}{g_{11}} \right)^2 + \left( \frac{g_{22,4} - g_{33,4}}{g_{22}} \right)^2 + \left( \frac{g_{33,4} - g_{11,4}}{g_{33}} \right)^2 \right]
$$

$$
= \frac{1}{12} \left[ \left( \frac{2AA_4}{A^2} - \frac{2BB_4}{B^2} \right)^2 + \left( \frac{2BB_4}{B^2} - \frac{2BB_4}{B^2} \right)^2 + \left( \frac{2BB_4}{B^2} - \frac{2AA_4}{A^2} \right)^2 \right]
$$

$$
= \frac{4}{12} \left[ \left( \frac{A_4}{A} - \frac{B_4}{B} \right)^2 + \left( \frac{B_4}{B} - \frac{B_4}{A} \right)^2 \right]
$$

$$
= \frac{1}{3} \left[ \left( \frac{\alpha t^\alpha - \alpha t}{t^\alpha (1 + x^2)^{\alpha}} \right)^2 + \left( \frac{\alpha t^\alpha}{t^\alpha (1 + x^2)^{\alpha}} - \frac{\alpha t^\alpha - (1 + x^2)^{\alpha}}{t^\alpha (1 + x^2)^{\alpha}} \right)^2 \right]
$$

$$
= \frac{1}{3} \left[ \left( \frac{\alpha}{t} - \frac{\alpha}{t} \right)^2 + \left( \frac{\alpha}{t} - \frac{\alpha}{t} \right)^2 \right]
$$

$$
= \frac{1}{3} (0)
$$

$$
= 0.
$$

**Case II:**

As above, we have

Expansion scalar $\theta = \frac{1}{tk}$.

$$
\sigma = \frac{2}{3} \left( \frac{1 - 3\alpha}{t} \right)^2.
$$