CHAPTER VI

STRUCTURES ON ANTI Q-FUZZY LEFT N-SUBGROUP OF NEAR RING UNDER TRIANGULAR NORM

Introduction: Kim Ju Pil and Bae Deok Rak [1997] have contributed some basic concepts of fuzzy congruence, and proved some fundamental properties. Several results that were obtained, were analogs of some basic theorems of group theory. In particular the lattice of some fuzzy normal subgroups is modular in a group *G*.

The notion of a fuzzy generalized subgroup with respect to a t-norm (or T-fuzzy generalized subgroup) gave some related properties, especially, the Representation Theorem for these fuzzy generalized subgroups. Next, using the concept of continuity of t-norms we obtain a correspondence between TF(G), the set of all T-fuzzy generalized subgroups of a generalized group G, and the set of all T-fuzzy generalized subgroups of the corresponding quotient generalized group. Subsequently, the quotient structure of T-fuzzy generalized subgroups was studied: The notion of a T-fuzzy normal generalized subgroup giving some related properties, were constructed the quotient generalized group, and proved the homomorphism theorem. Finally, the lattice of T-fuzzy generalized subgroups and proved that F(G) is a Heyting algebra.

Syransu [1998] got that the class $F_2(D)$ of all idempotent fuzzy subsets of *D*, where *D* is a semi group in which the cancellation laws are valid, forms a complete lattice under the point wise definition of order. Several characterizations for the members of this class were given. Complete distributive of the sup-min product over an arbitrary intersection of fuzzy subsets of *D* is established. This key result is profitably applied to study the order-structure

Tang and Zhang [2001] discussed the concept of category of LF groups, which takes the category of fuzzy groups as its subcategory. The category of LF groups was analyzed, and closed under the product operation. Moreover, the concrete structure and some characteristics of product was given.

Section 6.1 - Literature reviews:

Abd-Allah and Omer [1996] found that the fuzzification of partial monoid and partial group have analyzed in a little different way than that one usually found. The notion of the *T*-partial monoid on *I* has been investigated and used to define different membership values for the partial identities, which is consistent with the hall spirit of the fuzzification. This is the motivation by Abed-Allah and Omer [1996]. Examples have been given to get the generalization reasonable. In fact, the basic results of fuzzy groups and fuzzy monoids have been generalized.

The concepts [Chengyi, 1998] of fuzzy prime ideals and prime fuzzy ideals of a ring *R* were proposed, after analyzing the deficiency of the fuzzy ideals .Some characterizations and properties were discussed.

The relation [Dib and Hassan, 1998] between three definitions of a fuzzy topological group is discussed. Some properties of subgroups of a fuzzy topological group were given. The Q-compactness of a fuzzy topological group is also investigated. To know more of this subject, one can refer Doctorate thesis of Massa'deh [2008].

The notion of the definition of an upper fuzzy subgroup was analyzed by Rosenfeld. Many researchers like as Abd – Allah and Omerb [1996], Chengyi [1998], Dib and Hassan [1998], Tang and Zhang [2001], Syransu [1998], Massa'deh [2008] studied the properties of groups and fuzzy sub groups.

Solairaju and Nagarajan [2009c] introduced the concept of max^{i}- interval valued anti fuzzy left h- ideals in a hemi rings and extension principle of interval valued fuzzy set. They investigated some of their properties and structural characteristics, some theorems for homomorphic image and its inverse image on max^{i} – interval valued anti fuzzy left h- ideals of a hemi rings. Relationship between anti fuzzy left h- ideals in a hemi ring and fuzzy left h- ideals is also given. Using lower level set, they gave a characterization of interval value anti fuzzy left h ideals.

Solairaju and Nagarajan [2010j] introduced the concept of anti Q-fuzzy R- sub modules over a commutative ring with respect to t- norm. Some Properties of anti Q-fuzzy R- sub modules were investigated. In Particular, they considered properties of intersection and direct product for anti Q- fuzzy R- sub modules.

Filep and Maurer [1989] combined the structure of a fuzzy congruence and compatible fuzzy partition. The study of the fuzzy algebraic structures started with the introduction of concepts of fuzzy subgroups and fuzzy (left, right) ideals. Anthony and Sherwood [1979] redefined fuzzy subgroups using the concept of triangular norm.

Dip and Hassan [1998], Kim Ju Pil and Bae Deok Rak [1997], and Massa'deh [2007] followed the Rosenfeld approach in investigating fuzzy algebra where given ordinary algebraic structure on a given set X is assumed then introducing the fuzzy algebraic structure as a fuzzy subset A of X satisfying some suitable conditions.

The theory of fuzzy sets which was introduced by Zadeh [1965] is applied to many mathematical branches. Abou-Zoid [1991] , introduced the notion of a fuzzy sub near ring and studied fuzzy ideals of near ring. This concept discussed by many researchers among Cho & Jun [2005], Kim & Jun [2000], Kim & Jun [2001], and Davvaz et. al. [2006]. Osman Kazari et.al. [2007a, 2007b] considered the intuitionistic fuzzification of a right crisp left R-subgroup in a near ring.

Solairaju and Nagarajan [2009] introduced a notion of Q-fuzzy groups. Also Cho and Jun [2005] introduced the notion of normal intuitionistic fuzzy R-subgroup in a near ring, and some results are investigated. The notion of intuitionistic fuzzy sub-quasigroup of quasi-group is discussed by Kim et. al. [2000].

Section 6.2: Basic concepts and definitions

Definition 6.2.1: A non empty set R with two binary operations '+' and '.' is called a near ring if it satisfies the following axioms;

(i) (R, +) is a group; (ii) (R, .) is a semi-group; (iii) $x.(y + z) = x.y + x.z$ for all $x,y,z \in R$.

Precisely speaking it is a left near ring because it satisfies the left distribution law.

As N-subgroup of a near ring R' is a subset 'H' of R such that

(i) (H, +) is a subgroup of (R, +); (ii) $RH \subset H$; (iii) $HR \subset H$.

If ‘H’ satisfies (i) and (ii) then it is called left N-subgroup of R; if ‘N’ satisfies (i) and (iii), then it is called right N-subgroup of R.

A map f : R→ R1 is called homomorphism if f (x + y) = f(x) + f(y) for all x, y in R.

Definition 6.2.2: Let R be a near ring. A fuzzy set I in R is called anti Q-fuzzy sub-near ring in R if (i) μ(x-y, a) $\leq$ max { μ (x, a), μ (y, a) }l; (ii) μ (xy , a) $\leq$ max { μ(x, a), μ (y, a)} for all x, y in R, and for all a in Q.

Definition 6.2.3: (T-norm) A triangular norm is a function T : [0,1] × [0,1] → [0,1] that satisfies the following conditions for all x, y, z in [0,1].

(T1) T (x,1) = x; (T2) T (x,y) = T (y,x); (T3) T [x, T (y,z)] = T (T(x,y) , z)

(T4) T (x, y) $\leq$ T (x,z) when y $\leq$ z.

Definition 6.2.4: A Q-fuzzy set μ is called a Q-fuzzy left N-subgroup of R over Q if μ satisfies (i) μ (n (x - y, q) $\leq$ S{μ (x, q), μ (y, q)}; (ii) μ (nx, q) $\leq$ { μ(x, q)} for all x, y, n $\in$ R and q $\in$ Q.

Aim: In this chapter, the notion of anti Q-fuzzification of left N-subgroups is introduced in a near ring and investigated some related properties, characterization of anti Q-Fuzzy left N-subgroups with respect to a triangular norm are given.

Section 6.3: Properties of Q-fuzzy left N-subgroups

The ring of cossets of a fuzzy ideal was constructed. Let *A* and *B* be fuzzy subgroups of *G* such that *B* contained *A*. The purpose was to introduce the notion of fuzzy cosets and fuzzy normality of *B* in *A*. If *B* is fuzzy normal in *A*, then the set of all fuzzy cosets of *B* in *A* forms a semi group under a suitable operation. Structure properties of *A*=*B* and *A* were determined.

Mukherjee et. al. [1991] have given independent proof of several theorems on pseudo fuzzy cossets of fuzzy normal subgroups The notion of pseudo fuzzy double cosets, pseudo fuzzy middle cosets of a group and its fundamental properties were studied. Cartesian product fuzzy relations were also discussed. Further by Ramakrishna [2008], vague normal groups, vague normalizer, vague centralizer and vague homologous groups were analyzed. Biswas [2006] explained the concept of Vague group and others like as Demirci [1999], and Ramakrishna [2008] discussed it.

The theorems mentioned below are proved in this section:

Theorem 6.3.1: Let S be an s-norm. Then every imaginable anti Q-fuzzy left N-subgroup 'μ' of a near ring R is an anti Q-fuzzy left N-subgroup of R.

Proof: Assume 'μ' is imaginable anti Q-fuzzy left N-subgroup of R. Then it gives that

$\mu (n(x - y, q) \leq S \{ \mu(x, q), \mu(y, q)\}$

and $\mu (nx, q) \leq \{ \mu(x, q)\}$ for all x, y in R.

Since ‘μ’ is imaginable, it follows that

$$\max\{\mu(x, q), \mu(y, q)\} = S\,[\max\{\mu(x, q), \mu(y, q)\}, \max\{\mu(x, q), \mu(y, q)\}$$

$$\leq S\,\{\mu(x, q), \mu(y, q)\}$$

$$\leq \max\{\mu(x, q), \mu(y, q)\}$$

and so

$$S\,\{\mu(x, q), \mu(y, q)\} = \max\{\mu(x, q), \mu(y, q)\}$$

It follows that $\mu\,(n\,(x - y), q) \leq S\,\{\mu(x, q), \mu(y, q)\}$

$$= \max\{\mu(x, q), \mu(y, q)\} \text{ for all } x, y \in R.$$

Hence ‘μ’ is an anti Q-fuzzy left N-subgroup of R.

Theorem 6.3.2: If ‘μ’ is anti Q-fuzzy left N-subgroups of a near ring R and f is a endomorphism of R, then $\mu_{(f)}$ is a Q-fuzzy left N-subgroup of R.

Proof: For any x, y ϵ R, it implies that

(i) $\mu_{[f]}\,(\,n\,(x - y,q) = \mu\,(f\,n(x\text{-}y,q))$

$$= \mu\{f(x,q),Q(y,q)\}$$

$$\leq S\{\,\mu\,(f(x,q), \mu\,(\,f(y,q)\}$$

$$= S\,\{\,\mu_{[f]}\,(\,(x,q), \mu_{[f]}\,(y,q)\,\}$$

(ii) $\mu_{[f]}\,(\,nx,q) = \mu\,(\,f(nx,q)\,]$

$$\leq \mu\,(\,f(x, q)\,]$$

$$\leq \mu_{[f]}\,(\,x, q)\,]$$

Hence $\mu_{[f]}\,(x, q)$ is an anti Q-Fuzzy left N-Subgroup of R.

Theorem 6.3.3: An onto homomorphism of an anti Q-fuzzy left N-subgroup of near ring R is anti Q-fuzzy left N-subgroup.

Proof: Let f : R → R' be an onto homomorphism of near rings and 'ε be an anti Q-fuzzy left N-subgroup of R' and 'μ' be the pre-image of 'ε' under f .

It follows that

(i) μ ((n(x-y),q)] = ε(f (n(x-y),q)

= ε(f (x,q) ,f(y,q)}

$\leq$ S {μ (f(x,q), μ (f(y,q) }

$\leq$ S { (f(x,q), (f(y,q) }

(ii) μ (nx , q) = ε(f(nx, q) }

$\leq$ ε(f(x, q) }

$\leq$ μ (x, q)

Theorem 6.3.4: An onto homomorphic image of an anti Q-fuzzy left N-subgroup with the supremum property is anti Q-fuzzy left N-subgroup.

Proof: Let f: R → R' be an onto homomorphic of near rings and 'μ' be a supremum property of Q-fuzzy N-subgroup of R'.

Let x', y' $\in$ R' and $x_o \in f^{-1}$ (x') , and $y_o \in f^{-1}$ (y') be such that

$$\mu(x_o, q) = \sup_{(h,a) \in f^{-1}(x')} [\mu(h,q),]$$

$$\mu(y_o, q) = \sup_{(h,q) \in f^{-1}(y')} [\mu(h,q)] \quad \text{respectively.}$$

Then it deduces that

(i) $\mu^f (n(x' - y'), q) = \sup \mu(z, q)$ where $(z,a) \in f^{-1}(n(x'-y'),q)$

$\leq S\{\mu(x_o,q),\mu(y_o,q)\}$

$\leq S\{\sup_{(h,a) \in f(x')} (x_o,q), \sup_{(h,a) \in f(y')} (y_o, q)\}$

$= S\{\mu^t(x',q), \mu^t(y',q)\}$

(ii) $\mu^f (nx,q) = \sup \mu(z, q)$ where $(z,q) \in f^{-1}(nx',q)$

$\leq \mu(y_o,q)$

$= \sup_{(h,a) \in f^{-1}(y',q)} [\mu(t, q)]$

$= \mu t(y',q)$

Hence μ^t is an anti Q-fuzzy left N-subset of S'.

Section 6.4: Other properties of anti Q-fuzzy left N-subgroup

Theorem 6.4.1: Let S be a continuous s-norm and 'f' be a homomorphism on a near ring R. If 'μ' is anti Q-fuzzy left N-subgroup of R, then μ^f is an anti Q-fuzzy left N-subgroup of f(R).

Proof: Let $A_1 = f^{-1}(y_1,q)$, and $A_2 = f^{-1}(y_2,q)$ and

$A_{12} = f^{-1}(n(y_1 - y_2, q)$ where $y_1, y_2 \in f(S)$, and $q \in Q$.

Consider $A_1 – A_2 =\{x \in R : (x, q) = (a_1\text{-}q) – (a_2\text{-}q)\}$ for some $(a_1,q) \in A_1$ and $(a_2,q) \in A_2$

If $(x, q) \in A_1 – A_2$, then $(x, q) = (x_1 - q) – (x_2 - q)$ for some $(x_1, q) \in A_1$ and $(x_2,q) \in A_2$.

Then it gives that $f (x, q) = f (x_1, q) – f (x_2, q) = y_1 – y_2$

So $(x, q) \in f^{-1} [(y_1\text{-}q) – (y_2\text{-}q]) = f^{-1} (n(y_1 – y_2),q) = A_{12}$

Thus $A_1 – A_2 \subset A_{12}$

It follows that

$$\mu f[n(y_1 – y_2), q) = \sup_{(x, q) \in f^{-1} (y_1 – q),(y_2 –q) \}} \{ \mu(x,q) \}$$

$$= \sup \{ \mu (x, q) : (x, q) \in A_{12} \}$$

$$\leq \sup\{ \mu (x,q) : (x, q) \in A_1 – A_2\}$$

$$\leq \sup \{ \mu (x_1\text{-}q) - (x_2 - q) : (x_1, q) \in A_1 \text{ and } (x_2, q) \in A_2 \}$$

S is continuous $\Rightarrow$ for every $\epsilon > 0$, it implies that

$$\sup \{ \mu(x_1, q) : (x_1 , q) \in A_1 \} - (x_1^*, q) \in A_2 \} \leq \delta \text{ and}$$

$$\sup\{ \mu(x_2, q): (x_2 , q) \in A_2 \} - (x_2^*, q) \in A_2 \} \leq \delta$$

$$S \{\sup\{ \mu(x_1, q) : (x_1 , q) \in A_1 \},$$

$$\sup \{ \mu(x_2,. q): (x_2 , q) \in A_2 \}- S \{ (x_1^*, q), (x_2^*,q)\} \leq \epsilon$$

Choose $(a_1, q) \in A_1$ and $(a_2, q) \in A_2$ such that sup $\{\mu(x_1, q) : (x_1 , q) \in A_1 \}$- $\mu(a_1, q) \leq \delta$

Then it follows that S {sup { $\mu(x_1, q)$: $(x_1 , q) \in A_1$ },

sup { $\mu(x_2,.q)$: $(x_2 , q) \in A_2$ } - S { μ (a_1, q) - $\mu(a_2, q)$ } $\leq \epsilon$

Consequently it gives that

μ_t $(n(y_1 - y_2, q) \leq$ sup { S ($\mu(x_1, q)$, μ (x_2 , q) : $(x_1, q) \in A_1$ and $(x_2 , q) \in A_2$ }

$\leq$ S {sup { $\mu(x_1,q)$: $(x_1, q) \in A_2$, sup { μ (x_2 ,q) : $(x_2, q) \in A_2$ }

$\leq$ S{ μ_t (y_1 , q), μ_t (y_2, q) }

Similarly, it is showed μ_f (nx ,q) $\leq \mu_f$ [(x, q)] .

Hence μ^f is an anti Q-fuzzy left N-subgroup of f(S).

Theorem 6.4.2: Let 'μ' be an anti Q-fuzzy left N-subgroup of R. Then anti Q-fuzzy subset $< \mu >$ is anti Q-fuzzy left N-subgroup of R generated by μ. Moreover $< \mu >$ is the smallest anti Q-fuzzy left N-subgroup containing it.

Proof: Let x, y ϵ N and μ (x ,q) = t_1 ,

μ(y ,q) = t_2 and μ (n(x-y ,q) = t

Let $t = <\mu> (n (x - y), q)$

$\leq S \{< \mu > (x, q) , < \mu > (n (y, q)\}$

$= S \{ t_1, t_2 \} = t_1$ (say)

Then $t_1 = < \mu > (x, a) = \sup \{k: x \in < \mu_k > \} \geq t$.

Thus there exist k_1 such that $x \in < \mu_{k1} >$.

Also $t_2 = <\mu>(y,q) = \sup \{k: y \in < \mu_k >\} \geq t$.

So there exist $k_1 > t$ such that $y \in < \mu_k >$.

Without loss of generality, assume that k_1, k_2 so that $< \mu_{k1} > \subset < \mu_{k2} >$.

Then $x, y \in < \mu_k >$. So $\mu (n(x-y ,q) > t$, which is a contradiction since $k_2 > t$. Thus $t \geq t_1$.

Consequently, $\mu (n (x - y, q) \leq S \{ < \mu > (x, q), < \mu > ((y, q)\}$ ---(1)

Now let $t_3 = \{< \mu > (nx, q) \leq < \mu > (x, q)\} = t_1$.

Then $t_1 = <\mu> (x, q) = \sup \{k : x \in < \mu_k >\} > t_3$.

Therefore there exists k such that $x \in < \mu_k >$ and $t_1 > k > t_3$.

So that $nx \in < \mu_k > \subset < \mu_t >$ which is a contradiction.

Hence $t_3 = \{< \mu > (nx, q) \geq \{< \mu > (x, q) = t_2$ ---(2)

Conditions (1), and (2) yield that $< \mu >$ is anti Q-fuzzy left N-subgroup of R.

Finally to show that $< \mu >$ is the smallest anti Q-fuzzy left N-subgroup containing μ, assume that U to be an anti Q fuzzy left N-subgroup of R such that $\mu \subset U$.

Claim: $< \mu > \subset U$.

Let if possible, $t = < \mu > (x, q) \geq U (x, q)$ for some $x \in N$, and $q \in U$.

Let $\epsilon > 0$ be given, then $t = \mu_t = \sup \{ k : x \in < \mu_k >$ and $t - \epsilon \leq k \leq t$

so that $x \in < \mu_k > \subset < \mu_{kt} - \epsilon >$, for all $\epsilon > 0$.

Now $\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$, $\alpha_i \in N$, where x_i belongs to t-ϵ.

$x_i \in \mu_t$-ϵ implies $\mu(x_i , q) \geq t$-ϵ , So $U (x_i, q) \geq t$-ϵ for all $\epsilon > 0$

Thus $U (x, q) \leq S \{ U (x_1,q), U (x_1,q) ,\ldots, U (x_n, q) \} \geq t - \epsilon$ for all $\epsilon > 0$.

Hence $\mu (x, q) = t$ which is a contradiction to our supposition.

Theorem 6.4.3: Let U be an anti Q-fuzzy left N-subgroup near ring R and μ^+ be an anti Q-fuzzy set in N defined by $\mu^+ (x, a) = \mu (x, a) + 1 - \mu(0, q)$ for $x \varepsilon N$. Then μ^+ is a normal anti Q-fuzzy left N-subgroup of S containing 'μ'.

Proof: For any $x, y \in N$ and $q \in Q$, it follows that

(i) μ -1(n (x - y),y) = μ (n (x-y), q) + 1 - μ (0, q)

$\leq \{ \mu(x, q), \mu(y, q)\} + 1 - \mu(0, q)\}$

$\leq S\{\mu (x, q) + 1 - \mu(0, q).\mu(y, q) + 1 - (0, q) \}$

$= S\{ \mu^{-1}(x, q) . \mu^{-1}(y, q)\}$

(ii) $\mu^{+}(nx, q) = \mu(nx, q) + 1 - \mu(0, q)$

$\leq \mu(x, q) + 1 - \mu(0, q)$

$= \mu^{+}(x, q)$

Conclusion: Osman Kozanci et. al [2007] introduced the intuitionistic Q-fuzzy R-subgroups of near rings. In this chapter, the notion of Q-fuzzy left N-subgroup of near ring is investigated with respect to t-norm and some characterization of them.