CHAPTER - I

INTRODUCTION

1. SHOCK WAVES

Shock waves are the most important distinctive features of supersonic flow of a gas, across which the medium undergoes sudden and often considerable changes in velocity, pressure, density and temperature. The occurrence of shock waves is commonly associated with supersonic flight, explosions and electric discharges. The formation of shock wave can be simply visualized by considering the uniform motion of a piston into an open ended tube filled with virgin gas. A simple physical explanation of the shock formation, in this case, is the following.

Suppose the continuous motion of the piston is approximated by a set of forward-moving pulses, each of short duration. When the piston makes the first short movement forward, a small disturbance is propagated forwards into the gas at the speed of sound. This small amplitude wave (or sound wave) heats the gas slightly and because of the square of the local speed of sound is proportional to the temperature, the second pulse will be propagated as another sound wave at a speed slightly in excess of first one. Similarly the third pulse will be propagated at a speed slightly in excess of second and so on. Thus the discrete pulses cause a train of sound waves of ever increasing velocity to be propagated through the gas. The tendency is for faster moving rearmost waves to catch up with the slower moving foremost ones. In so doing the sound waves
coalesce to form a more powerful shock front moving at a speed which is in excess of the local speed of sound.

Shock waves are the most conspicuous phenomena occurring in non-linear wave propagation. Even without being caused by initial discontinuities, they may appear and be propagated. The underlying mathematical fact is that, unlike linear partial differential equations, non-linear equations often do not admit solutions which can be continuously extended whenever the differential equations themselves remain regular. The problem of shock waves has a bearing on many problems outside of supersonic aeronautics, for example, detonation waves, but also has great importance for several practical aeronautical problems. In fact, shock waves may cause sudden change in the aerodynamic behaviour of high speed aerocrafts affecting not only their balance and stability but also control producing undesirable vibrations.

Rayleigh [1] and Hugoniot [2] first pointed that an adiabatic reversible transition in a shock would violate the principle of conservation of energy. In fact, Hugoniot [2] showed that in the absence of viscosity and heat-conduction (outside the shock) conservation of energy implies conservation of entropy in continuous flow and also implies a change of entropy across a shock. Rayleigh [1] pointed out that the entropy must increase in crossing a shock front and that, for this reason, a rarefaction shock can not occur in a perfect gas.

It is true, of course, that a shock wave is not discontinuity in the strict sense. It has a finite thickness across which the physical properties change continuously. If this thickness is small compared with some appropriate macroscopic dimension of the flow-field, such as the radius of curvature of a curved
shock, the physical relationships may be obtained by an analysis which treats the discontinuity as strict. The assumption that the discontinuity thickness is small compared with a macroscopic dimension is a fundamental one. The term 'structure' as applied to a shock wave, refers to the values of the physical properties of the fluid within the small but finite thickness of the discontinuity. If thermodynamic equilibrium in a substance is disturbed a characteristic time must elapse before equilibrium can be approximately re-established. This time and the velocity of the fluid, define a characteristic distance which is of the order of a molecular mean free path or greater. If the physical and chemical changes, occurring in the discontinuity are sufficiently slow, so that the thickness of discontinuity is large compared with this characteristic distance, the concept of their thermodynamic quasi-equilibrium may be considered. In this case, the Navier-Stoke’s equations are applicable. If the discontinuity is thin, with physical and chemical changes occurring rapidly, i.e. detonation front, the essential absence of thermodynamic equilibrium must be taken into account ([3]).

Viscosity and heat conduction are negligible, in most of the cases, but where large gradients of temperature and velocity develop, they become important. Viscosity and heat conduction have the effect of smoothing out the discontinuity. Both viscosity and thermal conductivity present themselves as diffusion phenomena, arising respectively from the molecular transfer of momentum and energy. The former manifests itself in diffusion of vorticity and the later in the diffusion of heat.

Relative to the shock wave, the flow on the upstream side must be supersonic, on the downstream side the flow relative to the shock wave may be either
supersonic or subsonic, depending on the inclination to the incident stream of the normal to the wave. If the normal to the wave is parallel to the incident stream, the flow behind is always subsonic relative to the wave. By now, the theory of shock waves in non-conducting and conducting media has been much developed ([4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]).

2. SHOCK WAVES IN MAGNETOGASDYNAMICS

If a conducting fluid moves in a magnetic field, electric fields are induced and electric currents flow. The magnetic field exerts forces on these currents which considerably modify the flow ([36]). In many problems the energy in the electric field is much smaller than that in magnetic field. In these cases, we may express all the electromagnetic quantities in terms of magnetic field ([37]). As a result, we consider only the interaction between the magnetic field and the gas dynamic field. This analysis forms the subject matter of the well known 'magnetogasdynamics' and this interaction is of prime importance in most of the astrophysical and geophysical problems and in the behaviour of interstellar gaseous masses. As done in many problems, we have ignored Maxwell’s displacement current. We also assume as usual, that the dissipative mechanisms such as viscosity and thermal conductivity are absent.

The equations of motion for one dimensional magnetogasdynamic flow in a perfectly conducting fluid are ([8, 10, 38, 39, 40, 41, 42]) as under:

(i) The continuity equation is

\[
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + \frac{i \rho u}{r} = 0, \tag{2.1}
\]
where \( i = 0, 1, 2 \) for planar, cylindrically and spherically symmetric flows, respectively; and \( u \) and \( \rho \) are fluid velocity and density at time \( t \) and at distance \( r \) from the plane, axis or centre of symmetry.

**(ii)** The momentum equation in planar symmetry with magnetic field \( h \) perpendicular to the flow, is

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\mu h}{\rho} \frac{\partial h}{\partial r} = 0,
\]

where \( \mu \) is the magnetic permeability of the medium, and \( p \) is the pressure.

The momentum equation in cylindrical or spherical symmetry with azimuthal magnetic field \( h \), is

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\mu h}{\rho} \frac{\partial (hr)}{\partial r} = 0.
\]

The momentum equation in cylindrical symmetry with axial magnetic field \( h \), is

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\mu h}{\rho} \frac{\partial h}{\partial r} = 0.
\]

If the fluid is not perfectly conducting, but weakly conducting, the momentum equation in cylindrical symmetry takes the form,

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\sigma \mu^2 h_0^2 u}{\rho} = 0,
\]

where \( \sigma \) is the electrical conductivity and \( h_0 \) is the initial axial or azimuthal magnetic field.

**(iii)** The magnetic field equation is

\[
\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial r} + h \frac{\partial u}{\partial r} + \frac{jhu}{r} = 0,
\]
where, for planar symmetry, $j = 0$; for cylindrical symmetry with azimuthal magnetic field ($h$), $j = 0$; for cylindrical symmetry with axial magnetic field ($h$), $j = 1$; and for spherical symmetry with azimuthal magnetic field ($h$), $j = 1$.

If the fluid is not perfectly conducting, but weakly conducting, the magnetic field equation in cylindrical symmetry takes the form

$$\frac{1}{r^i} \frac{\partial}{\partial r} (r^i h) = \mu \sigma h_0 u,$$

where $i = 0$ for axial magnetic field ($h$) and $i = 1$ for azimuthal magnetic field ($h$).

(iv) The energy equation is

$$\frac{\partial}{\partial t} (\rho \rho^{-\gamma}) + u \frac{\partial}{\partial r} (\rho \rho^{-\gamma}) = 0,$$

where $\gamma$ is the ratio of specific heats of the fluid.

If the fluid is not perfectly conducting, but weakly conducting, the energy equation takes the form

$$\frac{\partial}{\partial t} (\rho \rho^{-\gamma}) + u \frac{\partial}{\partial r} (\rho \rho^{-\gamma}) = \sigma (\gamma - 1) \rho^{-\gamma} \mu^2 h_0^2 u^2.$$

The study of magnetogasdynamic shock wave was systematically begun in the year 1950 with the paper of Hoffmann and Teller [43]. The basic properties of magnetogasdynamic shock waves, as determined by the conservation laws (the Rankine-Hugoniot relations), have been developed further by Lee and Chen [38], Christer and Helliwell [39], Kanwal [44], Pai [45], Greenspan [46], Bhutani [47], Ray [48], Summers [49], Rosenau and Frankenthal [50], Verma and Vishwakarma [51], Singh and Singh [52], Ganguly and Jana [53], Ohsaki and
Yoshida [54], Vishwakarma [55], Vishwakarma and Yadav [56], Vishwakarma and Vishwakarma [57], Vishwakarma et al. [58], Vishwakarma and Pathak [59], Vishwakarma and Pandey [60] and Nath et al. [61] and many others; although the more complex question of their existence in nature has yet to be exhaustively treated. The relations connecting the flow variables on the two sides of the shock surface (the generalized Rankine-Hugoniot relations) in the coordinate system in which the velocity in front of the shock wave is zero, are as follows ([8, 37]):

\[
\begin{align*}
  h_2 (U - u_2) &= h_1 U, \\
  \rho_2 (U - u_2) &= \rho_1 U, \\
  p_2 + \frac{1}{2} \mu h_2^2 + \rho_2 (U - u_2)^2 &= p_1 + \frac{1}{2} \mu h_1^2 + \rho_1 U^2, \\
  \frac{1}{2} (U - u_2)^2 + \frac{\gamma p_2}{\gamma - 1} \rho_2 + \frac{\mu h_2^2}{\rho_2} &= \frac{1}{2} U^2 + \frac{\gamma p_1}{\gamma - 1} \rho_1 + \frac{\mu h_1^2}{\rho_1},
\end{align*}
\]

where the subscripts ‘1’ and ‘2’ correspond to the values of the quantities just ahead and just behind the shock surface, respectively, and \( U \) is the shock velocity.

If the fluid is weakly conducting, the magnetic field is continuous across the shock ([10]).

3. DETONATION WAVES

Under normal circumstances the flame in a tube, filled with combustible gas ignited at one end, is propagated with a low velocity of few meter per second. This is called slow combustion or simply combustion. There is an entirely different mechanism of propagation of combustion, involving shock waves. In it, the shock wave ignites the combustible gas as it moves, i.e. the combustion
is propagated with the velocity of the shock, or much faster than ordinary combustion. This mechanism of propagation of combustion is called detonation.

When the shock wave passes some point in the gas, the reaction begins at that point, and continues until all the gas there is burnt, i.e. for a time $\tau$ which characterizes the kinetics of the reaction concerned. It is therefore clear that the shock wave will be followed by a layer moving with it in which combustion is occurring. The width of this layer is equal to the speed of propagation of the shock multiplied by the time $\tau$. It is of importance that the width does not depend on the dimensions of any bodies that are present. When the characteristic dimensions of the problem are sufficiently large, we can regard the shock wave and the combustion zone following it as a single surface of discontinuity which separates the burnt and unburnt gases. Such surface is called a detonation wave (Landau and Lifshitz [62]).

Detonation have many features common with shocks but there are striking differences too. One of the differences between the shocks and detonations is, the fact that the internal energy function $e(v, p)$ for the burnt gas is different from that for the unburnt gas in addition, we must take into account the energy liberated during the chemical reaction and transformed partly into kinetic and partly into internal energy of the burnt gas. The liberated energy is taken from the molecular binding energy of the unburnt gas. That is, the energy used up in recombining the atoms to form new molecules is smaller for the burnt gas than the binding energy of the unburnt gas. We denote the binding energy per unit mass by $g$ and introduce the complete energy $E$ as (Courant and Friedricks [4])

$$E = e + g.$$
It is assumed that this complete energy is a known function $E = E_0(v, p)$, of specific volume $v$ and pressure $p$ for the unburnt gas, and $E = E_1(v, p)$, for the burnt gas, even though the chemical composition of the burnt gas may actually vary with pressure and specific volume. After these stipulation the relations governing the transition from unburnt to burnt gas can be derived from conservation laws of mass, momentum and energy in the same way as that for the shock transition.

The most stable condition of the detonation is the Chapman-Jouguet state where the velocity of detonation wave relative to the combustion products just behind it is exactly equal to the velocity of sound. This result was put forward as a hypothesis by D. L. Chapman in 1899 and E. Jouguet in 1905, but its complete theoretical justification is due to Ya. B. Zel’dovich in 1940. The theory of detonation waves has now been developed to a great extent, see for example (4, 39, 62, 63, 64, 65).

4. **FUNDAMENTAL EQUATIONS OF A NON-IDEAL GAS**

When a strong explosion takes place, the character of the motion of the substance depends essentially on its equation of state. Such a motion was studied originally for the case of an ideal gas; subsequently, examples of solutions of the explosion problem were given for certain real, thermodynamically imperfect media ([7, 66, 67]). It should however be noted that the study of explosions in the media differing from the ideal gas did, as a rule, involve empirical equations of state, which only describe the behaviour of the medium satisfactorily in a certain, limited interval of densities. Almost every one of these empirical equations was found to be incorrect for the low density region and, in the limit
when $\rho \to 0$, it either did not reduce to the equation of state for the ideal gas, or yield an incorrect first term of the so-called virial expansion for the pressure in powers of density.

As we know from statistical physics, at low densities the equation of state can be written in the form [68]

$$p = \rho RT \left[ 1 + \rho B(T) + \rho^2 C(T) + \ldots. \right],$$

where $R$ is the gas constant, and $B(T)$ and $C(T)$ are virial coefficients which can be determined if the molecules interaction potential is known. In the high temperature range, the coefficients $B(T)$ and $C(T)$ tend to constant values equal to $b$ and $(5/8)b^2$ respectively. For gases $\bar{b} \rho << 1$, $\bar{b}$ being the internal volume of the molecules, and therefore it is sufficient to consider the equation of the state in the form (Anisimov and Spiner [69], Ranga Rao and Purohit [70])

$$p = \rho RT \left( 1 + \bar{b} \rho \right), \quad (4.1)$$

Wu and Roberts [71] and Roberts and Wu [72] have used an equivalent equation of state to study the shock wave theory of sololuminescence.

From thermodynamics, we have

$$\left( \frac{\partial E}{\partial v} \right)_T = T \left( \frac{\partial p}{\partial T} \right)_T - p, \quad (4.2)$$

where $E$ is the internal energy per unit mass of the gas and $v$ is the specific volume.

Using the equation of state (4.1) in equation (4.2), we get

$$\left( \frac{\partial E}{\partial v} \right)_T = 0,$$

which shows that the internal energy $E$ is a function of temperature $T$ only. Therefore,

$$E = c_v T, \quad (4.3)$$
where $c_v$ is the specific heat at constant volume.

Using equation (4.2) in the first law of thermodynamics, we have

$$c_p - c_v = T \left( \frac{\partial p}{\partial T} \right)_v \left( \frac{\partial v}{\partial T} \right)_p; \quad (4.4)$$

where $c_p$ is the specific heats of the gas at constant pressure.

Using equation (4.1) in equation (4.4), we get

$$c_p - c_v = \frac{R (1 + \bar{b} \rho)^2}{1 + 2 \bar{b} \rho} \approx R; \quad (4.5)$$

neglecting second and higher powers of $\bar{b} \rho$.

Equation (4.5) implies that

$$c_v = \frac{R}{\gamma - 1}. \quad (4.6)$$

Then equations (4.1), (4.3) and (4.6) give the internal energy $E$ as a function of $p$ and $\rho$, in the form

$$E = \frac{p}{\rho (1 + \bar{b} \rho) (\gamma - 1)}. \quad (4.7)$$

The speed of sound ‘$a_n$’ may be calculated from equation (4.1) as follows:

$$a_n^2 = \frac{dp}{d\rho} = \frac{(1 + 2 \bar{b} \rho) \gamma p}{(1 + \bar{b} \rho) \rho}. \quad (4.8)$$

The equation of the energy of the non-ideal gas whose equation of state is in the form of equation (4.1) and which is non-conducting or perfectly conducting, is given by ([73])

$$\left( \frac{\partial p}{\partial t} + \frac{u \partial p}{\partial r} \right) - a_n^2 \left( \frac{\partial \rho}{\partial t} + \frac{u \partial \rho}{\partial r} \right) = 0, \quad (4.9)$$
where \(a_n^2\) is given in equation (4.8). If the gas is weakly conducting, it takes the form

\[
\left( \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} \right) - a_n^2 \left( \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} \right) = (\gamma - 1) \sigma \mu^2 h_1^2 u^2; \tag{4.10}
\]

where \(h_1\) is the initial axial or azimuthal magnetic field.

The equation of continuity, magnetic field and momentum are the same as for a perfect gas.

5. FUNDAMENTAL EQUATIONS OF MIXTURE OF A GAS AND SMALL SOLID PARTICLES

In the present thesis, we are concerned only with the two-phase flow of mixture of a gas and small solid particles (Marble [74], Murray [75], Soo [76], Pai [77], Rudinger [78]). When a large number of small solid particles flow in a fluid and the velocity of the fluid is sufficiently high, the behaviour of such solid particles is similar to ordinary fluid. We may consider these solid particles as a pseudo-fluid. The solid particles are assumed to be spheres of identical mass \(m_p\), radius \(r_p\), and specific heat \(c_s\). We may consider the mixture as a mixture of two fluids: one is the real fluid, gas or liquid and the other is the pseudo-fluid of solid particles. For each species \(r\) in the mixture, we would like to know its velocity vector \(\mathbf{q}_r\), its temperature \(T_r\), its pressure \(p_r\) and its density \(\rho_r\). In this case, we have two definitions for density: the species density and the partial density. We consider an element of the mixture of a fluid \(f\) and solid particles \(p\) with total mass \(M = M_f + M_p\) and with total volume \(v = v_f + v_p\), where subscript “\(f\)” refers to the value for the fluid, gas or liquid, and subscript “\(p\)” refers to that of solid particles.

It is convenient to introduce the number density of the solid particles \(n_p\).
which is the number of solid particles per unit volume at a point in the flow-field. The volume occupied by the solid particles $v_p$ is

$$v_p = n_p v \tau_p,$$  \hspace{1cm} (5.1)

where $\tau_p = \frac{4}{3} \pi r_p^3$ is the volume of a solid particle in the mixture. Without subscript, we refer to the value of the mixture as a whole.

The mass of the solid particles in the volume $v$ of the mixture is

$$M_p = m_p n_p v.$$  \hspace{1cm} (5.2)

The species density of the solid particles is defined as

$$\rho_{sp} = \frac{M_p}{v_p} = \frac{m_p}{\tau_p}.$$  \hspace{1cm} (5.3)

Hence the species density of solid particles is a constant for a given problem.

The partial density of pseudo-fluid of solid particles is defined as

$$\bar{\rho}_p = \frac{M_p}{v} = m_p n_p = Z \rho_{sp} = \rho_{sp} \tau_p n_p.$$  \hspace{1cm} (5.4)

where $z$ represents the fraction of volume of solid particles in the mixture which is one of the important variables in the treatment of two-phase flow of a gas and small solid particles. From equation (5.4) we have the volume fraction of solid particles as follows

$$Z = \frac{v_p}{v} = n_p \tau_p.$$  \hspace{1cm} (5.5)

The partial density of the pseudo-fluid of solid particles one is $\bar{\rho}_p$ of the fundamental variables in our analysis and it is proportional to $Z$ or $n_p$. 

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Similarly we have also the species density of the gas and partial density of the gas, too. The species density of the gas or the fluid is defined as

\[
\rho_f = \frac{M_f}{v_f} \quad \text{or} \quad \rho_g = \frac{M_g}{v_g},
\]

(5.6)

and the partial density of gas is defined as

\[
\rho_g = \frac{M_g}{v} = \frac{M_g}{v_g} \left( \frac{v - v_p}{v} \right) = (1 - Z)\rho_g.
\]

(5.7)

(i) Equation of State and Thermodynamics

For each species in the mixture of a gas and a pseudo-fluid of solid particles, we have one equation of state. For the gas, we may use the perfect gas law which is

\[
p_g = R\rho_g T_g = R(1 - Z)\rho_g T_g = (1 - Z)p,
\]

(5.8)

where \( p_g \) is the partial pressure of the gas in the mixture and \( T_g \) is the partial temperature of the gas and \( R \) is the gas constant. The total pressure of the mixture is \( p \) which is obtained, from (5.8), as

\[
p = R\rho_g T_g.
\]

(5.9)

Since the total pressure of the mixture is the sum of the partial pressure of the gas \( p_g \) and partial pressure of the pseudo-fluid of solid particles \( p_p \), we have \( p = p_g + p_p \). With the help of the equation (5.8) and (5.9), we find that the partial pressure of the solid particles must be

\[
p_p = Zp_p.
\]

(5.10)

The equation of state for the pseudo-fluid of solid particles is simply

\[
\rho_{sp} = \text{constant}.
\]

(5.11)
We consider the thermodynamic equilibrium condition such that $T_p = T_g = T$. The density of the mixture as a whole is given by

$$\rho = Z \rho_{sp} + (1 - Z) \rho_g = \bar{\rho}_p + \bar{\rho}_g.$$  \hspace{1cm} (5.12)

The mass concentration of the pseudo-fluid of solid particles is defined as

$$k_p = \frac{\bar{\rho}_p}{\rho} = \frac{Z \rho_{sp}}{\rho}.$$  \hspace{1cm} (5.13)

In equilibrium flow, $k_p$ is a constant in the whole flow field. Therefore, from equation (5.13)

$$\frac{Z}{\rho} = \text{constant},$$  \hspace{1cm} (5.14)

in the whole flow field, for equilibrium flow.

Also, from equations (5.12) and (5.13), we get

$$Z = \frac{k_p}{(1 - k_p) G + k_p},$$  \hspace{1cm} (5.15)

where $G = \frac{\rho_{sp}}{\rho_g}$.

The pressure of the mixture is

$$p = p_p + p_g.$$  \hspace{1cm} (5.16)

Now, we shall write $T_g = T$ for simplicity and $T_p$ may be equal to or different from $T$. In thermodynamic equilibrium condition, however, we have $T_p = T_g = T$. From equations (5.9) and (5.16) we find, the following relation between the pressure and density of the mixture as a whole

$$p = \frac{(1 - k_p)}{(1 - Z)} \rho RT = \frac{\rho R_m T}{(1 - Z)},$$  \hspace{1cm} (5.17)
where $R_m = (1 - k_p) R$; and $R_m$ may be considered as an effective gas constant of the mixture and $\rho$ is given by equation (5.12). It is interesting to notice that if the volume fraction $Z$ of the solid particles is negligibly small, the perfect gas law holds for the mixture as a whole when an effective gas constant $R_m$ is used. For many engineering problems, we do have a very small value of $Z$ but $k_p$ is not negligible in comparison with unity. On the other hand, if $Z$ is not negligibly small in comparison to unity, the volume fraction $Z$ of solid particles does affect the equation of state of the mixture as a whole because $Z$ is a function of $\rho$.

The internal energy of the mixture per unit mass $U_m$ is related to the internal energies of the two species by following relation:

$$\rho U_m = Z \rho_{sp} c_{sp} T_p + (1 - Z) \rho_g c_v T$$

or

$$U_m = k_p c_{sp} T_p + (1 - k_p) c_v T;$$

(5.19)

where $c_v$ is the specific heat of the gas at constant volume, $c_{sp} = c_s + c_{vp}$ and we assume that $c_{sp}$ and $c_v$ are constants for simplicity, $c_{vp}$ being an effective specific heat at constant volume of the pseudo-fluid of solid particles due to random translational motion and $c_s$ is the specific heat of the solid particles due to the internal degree of freedom. For thermodynamic equilibrium condition, we have the specific heat of the mixture at constant volume $c_{vm}$ as follows:

$$c_{vm} = k_p c_{sp} + (1 - k_p) c_v.$$

(5.20)

The enthalpy of the mixture per unit mass is

$$H_m = U_m + \frac{p}{\rho} = k_p \left( c_{sp} T + \frac{p}{\rho_{sp}} \right) + (1 - k_p) c_p T;$$
where \(c_p\) is the specific heat of the gas at constant pressure. For thermodynamic equilibrium condition, the specific heat of the mixture at constant pressure is then

\[
c_{pm} = k_p c_{sp} + (1 - k_p) c_p. \tag{5.21}
\]

The specific heats of the mixture are independent of the volume fraction \(Z\) but depend on the mass concentration \(k_p\) of the solid particles. The ratio of specific heats of the mixture is

\[
\Gamma = \frac{c_{pm}}{c_{vm}} = \frac{(1 - k_p) c_p + k_p c_{sp}}{(1 - k_p) c_v + k_p c_{sp}} = \gamma \frac{1 + \eta \delta}{1 + \eta \delta}, \tag{5.22}
\]

where

\[
\gamma = \frac{c_p}{c_v}, \quad \delta = \frac{c_{sp}}{c_v}, \quad \text{and} \quad \eta = \frac{k_p}{1 - k_p}. \tag{5.23}
\]

The ratio \(\Gamma\) is always smaller than \(\gamma\) of the gas if \(k_p\) is different from zero. As \(k_p = 0\), \(\Gamma = \gamma\).

Subtracting equation (5.20) from equation (5.21) and using \(c_p - c_v = R\), we obtain

\[
c_{pm} - c_{vm} = (1 - k_p) R. \tag{5.24}
\]

From equations (5.22) and (5.24), we get

\[
c_{vm} = \frac{(1 - k_p) R}{(\Gamma - 1)} \quad \text{and} \quad c_{pm} = \frac{(1 - k_p) R \Gamma}{(\Gamma - 1)}.
\]

Therefore,

\[
U_m = c_{vm} T = \frac{(1 - k_p) RT}{(\Gamma - 1)}
\]

or, using equation (5.17),

\[
U_m = \frac{(1 - Z) p}{(\Gamma - 1) \rho}. \tag{5.25}
\]

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If we consider the mixture as a homogeneous medium, the first law of thermodynamics for the mixture gives:

\[ dQ = dU_m - \frac{1}{\rho^2} \rho p \, dp, \]  

(5.26)

where \( dQ \) is the heat addition to the mixture. Equation (5.26) is the energy equation of the mixture as a whole.

For isentropic change of state of gas-particle mixture, we have \( dQ = 0 \), equation (5.26) gives

\[ \frac{1}{\Gamma - 1} \frac{dT}{T} = \frac{1}{1 - Z} \frac{d\rho}{\rho}. \]  

(5.27)

Since \( Z = \frac{k_p \rho}{\rho_{sp}} \), for constant \( k_p \) and \( T_p = T \), from the integration of (5.27), we have:

\[ T \left( \frac{\rho}{1 - Z} \right)^{(\Gamma - 1)} = \text{constant}. \]  

(5.28)

If \( Z << 1 \), the isentropic change of state of the mixture has similar relation as that for a pure gas with an effective ratio of specific heats \( \Gamma \). In general, the volume fraction \( Z \) has some influence on the isentropic change of the mixture.

Similarly from equation (5.17) for a given \( k_p \) and \( T_p = T \), we have

\[ \frac{dp}{\rho} = \frac{dT}{T} + \frac{1}{(1 - Z)} \frac{d\rho}{\rho}. \]  

(5.29)

From equations (5.27) and (5.29), we have

\[ \frac{dp}{\rho} = \frac{\Gamma}{(1 - Z)} \frac{d\rho}{\rho}, \]  

(5.30)

or

\[ p \left( \frac{\rho}{1 - Z} \right)^{-\Gamma} = \text{constant}. \]  

(5.31)
Again, if $Z << 1$, equation (5.31) is identical in form for the corresponding relation of pure gas but with an effective ratio of specific heats.

We may calculate the so-called equilibrium speed of sound of the mixture "$a_m$" from (5.31) as follows:

$$a_m^2 = \frac{dp}{d\rho} = \frac{\Gamma(1 - k_p)RT}{(1 - Z)^2} = \frac{\Gamma R_m T}{(1 - Z)^2}$$

or

$$a_m^2 = \frac{\Gamma p}{\rho(1 - Z)}. \quad (5.32)$$

(ii) Equation of Continuity

For each species in the mixture, we have one equation of continuity which gives the conservation of mass of that species. Combining equations of continuity of both the species, we have the equation of continuity as a whole ([77, 79]) for one dimensional motion

$$\frac{\partial p}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + \frac{iu}{r} = 0, \quad (5.33)$$

where $i = 0, 1$ and 2 correspond to the plane, cylindrical and spherical symmetry, respectively; $\rho$ is the density and $u$ the flow velocity of the mixture.

(iii) Equation of Motion

For each species the conservation of momentum gives the corresponding equation of motion for that species. If we combine equations of motion of both the species, we may obtain an equation of motion for the mixture as a whole.

The equation of motion for the unsteady one-dimensional flow of the mixture can be written as ([77, 79])

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0, \quad (5.34)$$
where \( p = p_g + p_p \) is the total pressure of the mixture. Here, it is assumed that
the viscous stress and heat conduction of the mixture are negligible.

(iv) **Equation of Energy**

For each species, the conservation of energy gives the corresponding equation of energy for that species. Combining the energy equations of both the species we may obtain an energy equation of the mixture as a whole.

The energy equation of unsteady one-dimensional flow of the mixture in which the viscous stress and heat conduction are assumed to be negligible can be written as ([77, 79])

\[
\frac{\partial U_m}{\partial t} + u \frac{\partial U_m}{\partial r} - \frac{p}{\rho^2} \left( \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} \right) = 0,
\]

where \( U_m \) is the internal energy of the mixture of gas and small solid particles.

Shock waves in a mixture of small solid particles and a gas have been studied by several authors, for example, Jena and Sharma [23], Vishwakarma [24], Steiner and Hirschler [28], Vishwakarma and Haidar [29], Pai et al. [79], Miura and Glass [18, 80], Suzuki et al. [81], Higashino and Suzuki [82], Gretler and Regenfelder [83, 84], Vishwakarma and Pandey [31], Vishwakarma and Nath [85, 86], Singh et al. [87] and Singh and Vishwakarma [88]. The generalized shock conditions, in this case, are

\[
\rho_2 (U - u_2) = \rho_1 U,
\]

\[
p_2 + \rho_2 (U - u_2)^2 = p_1 + \rho_1 U^2,
\]

\[
U_{m2} + \frac{p_2}{\rho_2} + \frac{1}{2} (U - u_2)^2 = U_{m1} + \frac{p_1}{\rho_1} + \frac{1}{2} U^2,
\]

\[
\frac{Z_2}{\rho_2} = \frac{Z_1}{\rho_1}.
\]
Here, it is assumed that the viscous stress and heat conduction of the mixture are negligible.

6. THE CONCEPT OF SELF-SIMILARITY

The fluid motion is said to be one-dimensional when all its properties depend on only one geometric coordinate and on the time. The motion in which the distributions of the flow variables remain similar to themselves with time and vary only as a result of change in scale is called self-similar. Self-similar motion is of great importance in gasdynamics. In this case the flow variables do not depend on the coordinates and time separatively, but depend only on particular combinations of them (the fact that we identify one of the independent variables with time is of no significance [89]). The methods of dimensional analysis can be used to find exact solutions of certain problems of one-dimensional unsteady motion of a compressible fluid.

In the Eulerian approach, the basic physical variables are the velocity $u$, the density $\rho$, and the pressure $p$. The characteristic parameters are the linear coordinate $r$, the time $t$, and the constants that enter into the equations, the boundary and the initial conditions of the problems. Since the dimensions of the quantities $\rho$ and $p$ contain the mass, at least one constant ‘$a$’, the dimensions of which also contain the mass must be a characteristic parameter. Hence as in [7] we can assume, without any loss of generality,

$$\dim a = ML^kT^s.$$  \hspace{1cm} (6.1)

We can then write for the velocity, density and pressure as

$$u = \frac{r}{t} v, \quad \rho = \frac{a}{r^{k+3s}t} G, \quad p = \frac{a}{r^{k+1+2s}t} P,$$  \hspace{1cm} (6.2)
where \( v, G \) and \( P \) are arbitrary quantities and therefore, can depend only on one-dimensional combinations of \( r, t \) and other parameters of the problem. In general, they are functions of two non-dimensional variables. But if an additional characteristic parameter ‘\( b \)’ can be introduced with dimensions independent of those of ‘\( a \)’, the number of independent variables which can be formed by combining \( r, t, \) ‘\( a \)’ and ‘\( b \)’ is reduced to one. Since the dimensions of the constant ‘\( a \)’ contain the mass, we choose the constant ‘\( b \)’ in a manner such that its dimensions do not contain the mass, i.e.

\[
\text{dim } b = L^m T^n. 
\]  

(6.3)

The single non-dimensional independent variable, in this case, will be \( r^m t^n / b \) which can be replaced (for \( m \neq 0 \)) by the variable,

\[
\eta = \frac{r}{b^{1/m} t^\alpha}, \quad \text{where} \quad \alpha = -\frac{n}{m}. \tag{6.4}
\]

If \( m = 0 \), \( v, G \) and \( P \) depend only on the time, and the velocity \( u \) is proportional to \( r \).

The solution depending on the independent variables may contain a number of arbitrary constants.

Self-similar motion occurs when \( r \) and \( t \) in the equations occur only in terms of the variable \( rt^{-\alpha} \) (called similarity variable), where the constant \( \alpha \) is determined by the configuration given initially. Self-similar property is worth attention since it reduces the number of independent variables from two to one. Consequently, the partial differential equations, governing a certain problem, are replaced by ordinary differential equations. The solution of these ordinary differential equations can sometimes be obtained exactly in closed form (for
example, violent spherical explosion in a uniform atmosphere at rest [6]) and in other cases, approximately by using numerical integration.

7. **WHITHAM’S METHOD**

Chester [90] has studied the motion of a shock wave down a non-uniform tube on the basis of a linearized theory in which the changes in the tube area and the consequent changes in shock strength have been assumed to be small. In this linearized theory, the solution breaks down when the flow behind the shock wave is nearly sonic. Chisnell [91] and Whitham [8] have shown that the Chester’s work could be simplified and extended. One minor simplification is that whereas Chester worked with the full three dimensional equations and performed an averaging process in the course of his analysis, it is sufficient to work from the outset with the one dimensional formulation. Whitham has shown that the motion of the shock can be found in a simple way, without solving the equations for the flow behind the shock in detail. His method can be summarized as the following rule. The relevant equations of motion are first written in characteristic form (differential relation). Then the rule is to apply the differential relation which must be satisfied by the flow quantities along a characteristic to the flow quantities just behind the shock wave together with the shock relations which express these values in terms of shock strength. In fact, it is found to be accurate even when the total change in shock strength are not small, although a full understanding of this fact still lacking. Some acceptable arguments in the support of this empirical technique are given in detail by Whitham [8].
References


