8.1 Introduction

In this chapter, we analyse two classes of batch arrival retrial queueing system with priority service, two kinds of vacations, balking and feedback. The single server provides service to the two classes of units, namely high priority and low priority customers, under a non-preemptive priority rule. As soon as the completion of service for the high priority customers may join the tail of the original queue as a feedback customer. If the server is busy or on vacations at the time of arriving low priority customers they join the orbit or balk (does not join the orbit). The arriving batches of low-priority customers in which not all batches are allowed to join the system at all times. We further assume that the server may take a type-I vacation after serving the last customer in the priority unit present in the system. After completion of the type-I vacation, if the priority...
queue is empty the server have a option to take a type-II vacation. During the type-II vacation if the priority customers arrives then the server may interrupt the vacation or continue the vacation. After completion of type-II vacation the server being available in the system. The retrial time, service time, two types of vacation times are all follows general(arbitrary) distribution. The time dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results are obtained explicitly. Also the average number of customers in the high priority queue, low priority in the orbit and the average waiting time in the queue and orbit are derived. Numerical results are computed and graphical representations are given.

An interesting control problem for queues is about admission control in systems. Control of queues is also one of the most remarkable and interesting areas of queueing theory. Crabill, Gross and Magazine (1977) classified the control models over number of servers, control of service rate, control of admission of customers and control of queue discipline. In this context, there are many researchers including Rue and Rosenberg (1981), Neuts (1984), Stidham (1985), Lee and Srinivasan (1989) and Huang and Mc-Donald (1998) dealt with control policies on arrivals in queues and queueing networks. Choudhury, Ke and Taadj (2009a) studied N policy for an unreliable server with delaying repair and two phases of service. Madan and Choudhury (2004c, 2006), Alnowibet and Tadj (2007) studied many queueing models in this perspective. Badamchi Zadeh (2012) studied a batch arrival queueing system with two phases of heterogeneous service with optional second service and restricted admissibility with single vacation policy. Lotfi Tadj and Paul Yoon (2014) studied $M/G/1$ queue with binomial schedule with k vacations and break downs under N-policy. The concept of vacation interruption was discussed by Jihong Li and Naishuo Tian (2007) in the working vacation under classical queueing models as, “the server can come back from the vacation to the normal working level once some indices of the system, such as the number of customers, achieve a certain value in the vacation period. The server may come back from the vacation without completing the vacation. Such policy is called vacation interruption.”

Badamchi Zadeh (2012) studied a queueing system with Coxian-2 server vacations and admissibility restriction for batch input. Ayyappan et al. (2014) have studied a
transient behavior of $M^{[X]}/G/1$ retrial queueing model with non persistence customers, random breakdown, delaying repair and Bernoulli vacation. Atencia et al (2005) have studied a single-server retrial queue with general retrial times and Bernoulli schedule, Atencia (2016) discussed a discrete-time queueing system with changes in the vacation times, Chaudhry et al. (1983) have discussed a batch arrival queueing system with non-preemptive priority services, Chesoong Kima et.al. (2016) have discussed Priority tandem queueing system with retrials and reservation of channels as a model of call center, Gautam Choudhury et al (2012) have studied a batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair, Gomez Corral. A.(1999) analyzed stochastic analysis of a single server retrial queue with general retrial times, Jinting Wang et al. (2010) have discussed a batch arrival retrial queue with starting failures, feedback and admission control, Monita Baruah et.al. (2013) have discussed a two stage batch arrival queue with reneging during vacation and breakdown periods and Rajadurai et al. (2014) have analysed an $M^{[X]}/(G_1,G_2)/1$ retrial queueing system with balking, optional re-service under modified vacation policy and service interruption.

This chapter is organized as follows: Mathematical description of our model in section (8.2). Equations governing of our model and the time dependent solution have been obtained in section (8.3). The corresponding steady state results have been derived explicitly in section (8.4). Average queue size and the average waiting time are computed in section (8.5) and (8.6). Some particular cases have been discussed in section (8.7). Numerical results with graphical illustrations in section (8.8).

### 8.2 Mathematical description of our model

(i) High priority and low priority units arrive at the system in batches of variable sizes in a compound Poisson process and they are provided service one by one on a FCFS basis. Let $\lambda_1 c_i \, dt$ and $\lambda_2 c_i \, dt (i = 1, 2, 3, ...)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt)$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$, and $\lambda_1 > 0$, $\lambda_2 > 0$ are the average arrival rates of high priority and low priority customers and the high priority customers only form queue, if the server is busy.
(ii) Retrial customers are considered as low priority customers. If the server is available, it begins the service to one of the customer immediately from the arriving batch of low-priority customers and the remaining customers leave the service area and hence join the orbit. Assume that only the customer at the head of the orbit is allowed for access to the server. If the server is busy upon retrial, the customer join the orbit again. Such a process is repeated until the retrial customer finds the server idle and gets the requested service at the time of a retrial. Further assume that the restricted admissibility of arriving batches of low-priority customers in which not all batches are allowed to join the system at all times with probability \((1 - \alpha)\). Also upon arrival, if the customers finds the server busy or on vacation they join the orbit with probability \(b\) or balk the orbit with probability \((1-b)\). The retrial time, that is time between successive repeated attempts of each customer in the orbit is assumed to be generally distributed with distribution function \(A(x)\), density function \(a(x)\). The conditional completion rate for retrials is given by \(\eta(x) = \frac{a(x)}{1-A(x)}\).

(iii) Each customer under high priority and low priority service provided by a single server under non-preemptive priority service rule on a first come-first served basis. The service time for both priority units follows general(arbitrary) distribution with distribution functions \(B_i(v)\) and the density functions \(b_i(v)\), \(i = 1, 2\).

(iv) Let \(\mu_i(x)dx\) be the conditional probability of completion of the high priority and low priority unit service during the interval \((x, x + dx]\), given that the elapsed service time is \(x\), so that

\[
\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \quad i = 1, 2.
\]

and therefore,

\[
b_i(v) = \mu_i(v)e^{- \int_0^v \mu_i(x)dx}, \quad i = 1, 2.
\]

(v) We further assume that as soon as the service of the last high priority unit present in the system is completed, the server has the option to take a type-I vacation of random length with probability \(\theta\), in which case the vacation starts immediately or else with probability \((1 - \theta)\) he may decide to stay in the system. In the later case,
if there is no high priority and low priority unit present in the system, the server
remains idle in the system waiting for the new units to arrive. After completing
type-I vacation, if the high priority queue is empty the server may take a type-II
vacation with probability $r$ or else with probability $(1 - r)$ he may decide to stay
in the system. During the type-II vacation if the high priority customers arrives
the server may interrupt the vacation with probability $\theta_1$ or continue the vacation
with probability $(1 - \theta_1)$. On returning from type-II vacation, the server instantly
starts serving the customer at the head of the priority queue, if any or the server
stays in the system idle.

(vi) These two types of vacation times follow general (arbitrary) distribution with distri-
bution function $V_i(t)$ and the density function $v_i(t)$. Let $\gamma_i(x)dx$ be the conditional
probability of a completion of a vacation during the interval $(x, x + dx]$, given that
the elapsed vacation time is $x$, so that

$$\gamma_i(x) = \frac{v_i(x)}{1 - V_i(x)}$$

and therefore,

$$v_i(t) = \gamma_i(t)e^{-\int_0^t \gamma_i(x)dx}.$$

(vii) As soon as the completion of a service for a high priority customers, if they
are not satisfied with their service they join the tail of the queue as a feedback
customer with probability $p_1$ or may leave the system with probability $q_1$ for
satisfied customers.

(viii) Various stochastic processes involved in the system are assumed to be independent
of each other.

8.3 Equations governing the system

The forward Kolmogorov differential difference equations for the queue size distributions
for the above assumptions:

$$\frac{d}{dt}I_{0,0}(t) = -(\lambda_1 + \lambda_2)I_{0,0}(t) + (1 - \theta)q_1 \int_0^\infty P_{0,0}(x,t)\mu_1(x)dx$$
\[ \begin{align*}
+ \int_0^\infty q_{0,0}(x,t)\mu_2(x)dx + (1-r) \int_0^\infty V_{1,0,0}(x,t)\gamma_1(x)dx \\
+ \int_0^\infty V_{2,0,0}(x,t)\gamma_2(x)dx + \lambda_2(1-\alpha)I_{0,0}(t),
\end{align*} \]

\[ \frac{\partial}{\partial x}I_{0,n}(x,t) + \frac{\partial}{\partial t}I_{0,n}(x,t) = - (\lambda_1 + \lambda_2 + \eta(x))I_{0,n}(x,t) + \lambda_2(1-\alpha)I_{0,n}(x,t); \\
n \geq 1, \quad \text{(8.3.1)} \]

\[ \begin{align*}
\frac{\partial}{\partial x}P_{m,n}(x,t) + \frac{\partial}{\partial t}P_{m,n}(x,t) &= - (\lambda_1 + \lambda_2 + \mu_1(x))P_{m,n}(x,t) \\
&\quad + \lambda_1(1-\delta_{m0}) \sum_{i=1}^m C_i P_{m-i,n}(x,t) + \lambda_2 \alpha b(1-\delta_{0n}) \sum_{i=1}^n C_i P_{m,n-i}(x,t) \\
&\quad + \lambda_2 \alpha(1-b)P_{m,n}(x,t) + \lambda_2(1-\alpha)P_{m,n}(x,t); \quad m, n \geq 0, \quad \text{(8.3.2)}
\end{align*} \]

\[ \begin{align*}
\frac{\partial}{\partial x}q_{m,n}(x,t) + \frac{\partial}{\partial t}q_{m,n}(x,t) &= - (\lambda_1 + \lambda_2 + \mu_2(x))q_{m,n}(x,t) \\
&\quad + \lambda_1(1-\delta_{m0}) \sum_{i=1}^m C_i q_{m-i,n}(x,t) + \lambda_2 \alpha b(1-\delta_{0n}) \sum_{i=1}^n C_i q_{m,n-i}(x,t) \\
&\quad + \lambda_2 \alpha(1-b)q_{m,n}(x,t) + \lambda_2(1-\alpha)q_{m,n}(x,t); \quad m, n \geq 0, \quad \text{(8.3.3)}
\end{align*} \]

\[ \begin{align*}
\frac{\partial}{\partial x}V_{1,m,n}(x,t) + \frac{\partial}{\partial t}V_{1,m,n}(x,t) &= - (\lambda_1 + \lambda_2 + \gamma_1(x))V_{1,m,n}(x,t) \\
&\quad + \lambda_1(1-\delta_{m0}) \sum_{i=1}^m C_i V_{1,m-i,n}(x,t) + \lambda_2 \alpha b(1-\delta_{0n}) \sum_{i=1}^n C_i V_{1,m,n-i}(x,t) \\
&\quad + \lambda_2 \alpha(1-b)V_{1,m,n}(x,t) + \lambda_2(1-\alpha)V_{1,m,n}(x,t); \quad m, n \geq 0, \quad \text{(8.3.4)}
\end{align*} \]

\[ \begin{align*}
\frac{\partial}{\partial x}V_{2,m,n}(x,t) + \frac{\partial}{\partial t}V_{2,m,n}(x,t) &= - (\lambda_1 + \lambda_2 + \gamma_2(x))V_{2,m,n}(x,t) \\
&\quad + \lambda_1(1-\theta_1)(1-\delta_{m0}) \sum_{i=1}^m C_i V_{2,m-i,n}(x,t) \\
&\quad + \lambda_2 \alpha b(1-\delta_{0n}) \sum_{i=1}^n C_i V_{2,m,n-i}(x,t) + \lambda_2 \alpha(1-b)V_{2,m,n}(x,t) \\
&\quad + \lambda_2(1-\alpha)V_{2,m,n}(x,t); \quad m, n \geq 0, \quad \text{(8.3.5)}
\end{align*} \]

The above set of differential equations are to be solved by using the following boundary conditions

\[ I_{0,n}(0,t) = (1-\theta)q_1 \int_0^\infty P_{0,0}(x,t)\mu_1(x)dx + \int_0^\infty q_{0,0}(x,t)\mu_2(x)dx \]

\[ + (1-r) \int_0^\infty V_{1,0,0}(x,t)\gamma_1(x)dx + \int_0^\infty V_{2,0,0}(x,t)\gamma_2(x)dx; \quad n \geq 1, \quad \text{(8.3.7)} \]

\[ P_{0,n}(0,t) = \lambda_1 C_1 I_{0,n}(t) + q_1 \int_0^\infty P_{1,n}(x,t)\mu_1(x)dx + \int_0^\infty q_{1,n}(x,t)\mu_2(x)dx \]

\[ + \int_0^\infty V_{1,1,n}(x,t)\gamma_1(x)dx + \lambda_1 C_1 \theta_1 \int_0^\infty V_{2,0,n}(x,t)dx \]

\[ + \int_0^\infty V_{2,1,n}(x,t)\gamma_1(x)dx + P_1 \int_0^\infty P_{0,n}(x,t)\mu_1(x)dx; \quad m \geq 1, \quad \text{(8.3.8)} \]
\[ P_{m,n}(0,t) = \lambda_1 C_{m+1} I_{0,n}(t) + q_1 \int_0^\infty P_{m+1,n}(x,t) \mu_1(x) \, dx \]
\[ + \int_0^\infty q_{m+1,n}(x,t) \mu_2(x) \, dx + \int_0^\infty V_{1,m+1,n}(x,t) \gamma_1(x) \, dx \]
\[ + \lambda_1 C_{m+1} \theta_1 \int_0^\infty V_{2,0,n}(x,t) \, dx + \int_0^\infty V_{2,m+1,n}(x,t) \gamma_1(x) \, dx \]
\[ + P_1 \int_0^\infty P_{m,n}(x,t) \mu_1(x) \, dx; \ m \geq 1, \ n \geq 0, \]  
\[ (8.3.9) \]

\[ q_{0,0}(0,t) = \alpha \lambda_2 C_1 I_{0,0}(t) + \int_0^\infty I_{0,1}(x,t) \eta(x) \, dx, \]  
\[ (8.3.10) \]

\[ q_{0,n}(0,t) = \alpha \lambda_2 C_{n+1} I_{0,0}(x,t) + \int_0^\infty I_{0,n+1}(x,t) \eta(x) \, dx \]
\[ + \alpha \lambda_2 \sum_{i=1}^n C_i \int_0^\infty I_{0,n-i+1}(x,t) \, dx; \ n \geq 1, \]  
\[ (8.3.11) \]

\[ V_{1,0,n}(0,t) = \theta q_1 \int_0^\infty P_{0,n}(x,t) \mu_1(x) \, dx; \ n \geq 0, \]  
\[ (8.3.12) \]

\[ V_{2,0,n}(0,t) = r \int_0^\infty V_{1,0,n}(x,t) \gamma_1(x) \, dx; \ n \geq 0. \]  
\[ (8.3.13) \]

Initial conditions are:
\[ I_{0,0}(0) = 1, \ P_{m,n}(0) = q_{m,n}(0) = V_{i,m,n}(0) = 0. \]  
\[ (8.3.14) \]

We define the following probability generating functions are:
\[ I_0(Z_2) = \sum_{n=1}^\infty Z_2^n I_{0,n}(x), \ P(Z_1, Z_2) = \sum_{m=0}^\infty \sum_{n=0}^\infty Z_1^m Z_2^n P_{m,n}(x), \]
\[ q(Z_1, Z_2) = \sum_{m=0}^\infty \sum_{n=0}^\infty Z_1^m Z_2^n q(x), \ V_i(Z_1, Z_2) = \sum_{m=0}^\infty \sum_{n=0}^\infty Z_1^m Z_2^n V_i(x). \]  
\[ (8.3.15) \]

Taking Laplace transforms to the equations (8.3.1) to (8.3.13), we get
\[ (s + \lambda_1 + \lambda_2 \alpha) I_{0,0}(s) - 1 = (1 - \theta) q_1 \int_0^\infty \mathcal{P}_{0,0}(x,s) \mu_1(x) \, dx \]
\[ + \int_0^\infty \mathcal{q}_{0,0}(x,s) \mu_2(x) \, dx + (1 - r) \int_0^\infty \mathcal{V}_{1,0,0}(x,s) \gamma_1(x) \, dx \]
\[ + \int_0^\infty \mathcal{V}_{2,0,0}(x,s) \gamma_2(x) \, dx, \]  
\[ (8.3.16) \]

\[ \frac{\partial}{\partial x} I_{0,n}(x,s) + (s + \lambda_1 + \lambda_2 \alpha + \eta(x)) I_{0,n}(x,s) = 0, \]  
\[ (8.3.17) \]

\[ \frac{\partial}{\partial x} \mathcal{P}_{m,n}(x,s) + (s + \lambda_1 + \lambda_2 + \mu_1(x)) \mathcal{P}_{m,n}(x,s) = \lambda_1(1 - \delta_{m0}) \sum_{i=1}^m C_i \mathcal{P}_{m-i,n}(x,s) \]
\[ + \lambda_2 \alpha b(1 - \delta_{m0}) \sum_{i=1}^n C_i \mathcal{P}_{m-n-i}(x,s) + \lambda_2 \alpha (1 - b) \mathcal{P}_{m,n}(x,s) \]
\[ + \lambda_2 (1 - \alpha) \mathcal{P}_{m,n}(x,s); \ m, \ n \geq 0, \]  
\[ (8.3.18) \]

\[ \frac{\partial}{\partial x} q_{m,n}(x,t) + (s + \lambda_1 + \lambda_2 + \mu_2(x)) \mathcal{q}_{m,n}(x,s) = \lambda_1(1 - \delta_{m0}) \sum_{i=1}^m C_i \mathcal{q}_{m-i,n}(x,s) \]
\[ + \lambda_2 \alpha b (1 - \delta m) \sum_{i=1}^{n} C_i \bar{q}_{m,n-i}(x,s) + \lambda_2 \alpha (1 - b) \bar{q}_{m,n}(x,s) \]
\[ + \lambda_2 (1 - \alpha) \bar{q}_{m,n}(x,s); m, n \geq 0, \] (8.3.19)
\[ \frac{\partial}{\partial x} \bar{V}_{1,m,n}(x,s) + (s + \lambda_1 + \lambda_2 + \gamma_1(x)) \bar{V}_{1,m,n}(x,s) \]
\[ = \frac{\lambda_1 (1 - \delta m)}{\sum_{i=1}^{m} C_i \bar{V}_{1,m,i,n}(x,s) + \lambda_2 \alpha b (1 - \delta m) \sum_{i=1}^{n} C_i \bar{V}_{1,m,n-i}(x,s) \]
\[ + \lambda_2 \alpha (1 - b) \bar{V}_{1,m,n}(x,s) + \lambda_2 (1 - \alpha) \bar{V}_{1,m,n}(x,s); m, n \geq 0, \] (8.3.20)
\[ \frac{\partial}{\partial x} \bar{V}_{2,m,n}(x,s) + (s + \lambda_1 + \lambda_2 + \gamma_2(x)) \bar{V}_{2,m,n}(x,s) = \lambda_1 (1 - \theta_i) \]
\[ \times (1 - \delta m) \sum_{i=1}^{m} C_i \bar{V}_{2,m,i,n}(x,s) + \lambda_2 \alpha b (1 - \delta m) \sum_{i=1}^{n} C_i \bar{V}_{2,m,n-i}(x,s) \]
\[ + \lambda_2 \alpha (1 - b) \bar{V}_{2,m,n}(x,s) + \lambda_2 (1 - \alpha) \bar{V}_{2,m,n}(x,s); m, n \geq 0, \] (8.3.21)

The boundary conditions are

\[ I_{0,n}(0,s) = (1 - \theta) q_1 \int_{0}^{\infty} \bar{P}_{0,n}(x,s) \mu_1(x) dx + \int_{0}^{\infty} \bar{q}_{0,n}(x,s) \mu_2(x) dx \]
\[ + (1 - r) \int_{0}^{\infty} \bar{V}_{1,0,n}(x,s) \gamma_1(x) dx + \int_{0}^{\infty} \bar{V}_{2,0,n}(x,s) \gamma_2(x) dx; n \geq 1, \] (8.3.22)
\[ \bar{P}_{0,n}(0,s) = \lambda_1 C_i \bar{I}_{0,n}(s) + q_1 \int_{0}^{\infty} \bar{P}_{1,n}(x,s) \mu_1(x) dx + \int_{0}^{\infty} \bar{q}_{1,n}(x,s) \mu_2(x) dx \]
\[ + \int_{0}^{\infty} \bar{V}_{1,1,n}(x,s) \gamma_1(x) dx + \lambda_1 C_1 \bar{I}_{0,n}(s) \int_{0}^{\infty} \bar{V}_{2,0,n}(x,s) dx \]
\[ + \int_{0}^{\infty} \bar{V}_{2,1,n}(x,s) \gamma_1(x) dx + P_1 \int_{0}^{\infty} \bar{P}_{0,n}(x,s) \mu_1(x) dx; m \geq 1, \] (8.3.23)
\[ \bar{P}_{m,n}(0,s) = \lambda_1 C_{m+1} \bar{I}_{0,n}(s) + q_1 \int_{0}^{\infty} \bar{P}_{m+1,n}(x,s) \mu_1(x) dx \]
\[ + \int_{0}^{\infty} \bar{q}_{m+1,n}(x,s) \mu_2(x) dx + \int_{0}^{\infty} \bar{V}_{1,m+1,n}(x,s) \gamma_1(x) dx \]
\[ + \lambda_1 C_{m+1} \bar{I}_{0,n}(s) + \theta_i \int_{0}^{\infty} \bar{V}_{2,m,n}(x,s) dx + \int_{0}^{\infty} \bar{V}_{2,m+1,n}(x,s) \gamma_1(x) dx \]
\[ + P_1 \int_{0}^{\infty} \bar{P}_{m,n}(x,s) \mu_1(x) dx; m \geq 1, n \geq 0, \] (8.3.24)
\[ \bar{q}_{0,0}(0,s) = \alpha \lambda_2 C_i \bar{I}_{0,0}(s) + \int_{0}^{\infty} \bar{I}_{0,1}(x,s) \eta(x) dx, \] (8.3.25)
\[ \bar{q}_{0,n}(0,s) = \alpha \lambda_2 C_{n+1} \bar{I}_{0,0}(x,s) + \int_{0}^{\infty} \bar{I}_{0,n+1}(x,s) \eta(x) dx \]
\[ + \alpha \lambda_2 \sum_{i=1}^{n} C_i \int_{0}^{\infty} \bar{I}_{0,n-i+1}(x,s) dx; n \geq 1, \] (8.3.26)
\[ \bar{V}_{1,0,n}(0,s) = \theta q_1 \int_{0}^{\infty} \bar{P}_{0,n}(x,s) \mu_1(x) dx; n \geq 0, \] (8.3.27)
\[
V_{2,0,n}(0,s) = r \int_0^\infty V_{1,0,n}(x,s) \gamma_1(x) dx; \ n \geq 0.
\] (8.3.28)

Now we multiply equation (8.3.18) to (8.3.21) by \( z_1^m \) summing over \( m \) from 0 to \( \infty \), we get

\[
\frac{\partial}{\partial x} \mathcal{P}_n(x,z_1,s) + (s + \lambda_1 + \lambda_2 + \mu_1(x)) \mathcal{P}_n(x,z_1,s) = \lambda_1 C(z_1) \mathcal{P}_n(x,z_1,s)
\]
\[
+ \lambda_2 \alpha b \sum_{i=1}^{n} (1 - \delta_{i0}) C_i \mathcal{P}_{n-i}(x,z_1,s) + \lambda_2 \alpha (1 - b) \mathcal{P}_n(x,z_1,s)
\]
\[
+ \lambda_2 (1 - \alpha) \mathcal{P}_n(x,z_1,s); \ n \geq 0,
\] (8.3.29)

\[
\frac{\partial}{\partial x} \mathcal{Q}_n(x,z_1,s) + (s + \lambda_1 + \lambda_2 + \mu_2(x)) \mathcal{Q}_n(x,z_1,s) = \lambda_1 C(z_1) \mathcal{Q}_n(x,z_1,s)
\]
\[
+ \lambda_2 \alpha b \sum_{i=1}^{n} (1 - \delta_{i0}) C_i \mathcal{Q}_{n-i}(x,z_1,s) + \lambda_2 \alpha (1 - b) \mathcal{Q}_n(x,z_1,s)
\]
\[
+ \lambda_2 (1 - \alpha) \mathcal{Q}_n(x,z_1,s); \ n \geq 0,
\] (8.3.30)

\[
\frac{\partial}{\partial x} \mathcal{V}_{1,n}(x,z_1,s) + (s + \lambda_1 + \lambda_2 + \gamma_1(x)) \mathcal{V}_{1,n}(x,z_1,s) = \lambda_1 C(z_1) \mathcal{V}_{1,n}(x,z_1,s)
\]
\[
+ \lambda_2 \alpha b \sum_{i=1}^{n} (1 - \delta_{i0}) C_i \mathcal{V}_{1,n-i}(x,z_1,s) + \lambda_2 \alpha (1 - b) \mathcal{V}_{1,n}(x,z_1,s)
\]
\[
+ \lambda_2 (1 - \alpha) \mathcal{V}_{1,n}(x,z_1,s); \ n \geq 0,
\] (8.3.31)

\[
\frac{\partial}{\partial x} \mathcal{V}_{2,n}(x,z_1,s) + (s + \lambda_1 + \lambda_2 + \gamma_2(x)) \mathcal{V}_{2,n}(x,z_1,s) = \lambda_1 C(z_1) (1 - \theta_1) \mathcal{V}_{2,n}(x,z_1,s)
\]
\[
+ \lambda_2 \alpha b \sum_{i=1}^{n} (1 - \delta_{i0}) C_i \mathcal{V}_{2,n-i}(x,z_1,s) + \lambda_2 \alpha (1 - b) \mathcal{V}_{2,n}(x,z_1,s)
\]
\[
+ \lambda_2 (1 - \alpha) \mathcal{V}_{2,n}(x,z_1,s); \ n \geq 0.
\] (8.3.32)

Now we multiply equation (8.3.17) by \( \mathcal{I}_2^m \) summing over \( n \) from 1 to \( \infty \) and equations (8.3.29) to (8.3.32) by \( \mathcal{I}_2^m \) summing over \( n \) from 0 to \( \infty \), we get

\[
(\frac{\partial}{\partial x} + s + \lambda_1 + \lambda_2 \alpha + \eta(x)) \mathcal{I}_0(x,z_2,s) = 0,
\] (8.3.33)

\[
(\frac{\partial}{\partial x} + s + \lambda_1 (1 - C(z_1)) + \lambda_2 \alpha b (1 - C(z_2)) + \mu_1(x)) \mathcal{P}(x,z_1,z_2,s) = 0,
\] (8.3.34)

\[
(\frac{\partial}{\partial x} + s + \lambda_1 (1 - C(z_1)) + \lambda_2 \alpha b (1 - C(z_2)) + \mu_2(x)) \mathcal{Q}(x,z_1,z_2,s) = 0,
\] (8.3.35)

\[
(\frac{\partial}{\partial x} + s + \lambda_1 (1 - C(z_1)) + \lambda_2 \alpha b (1 - C(z_2)) + \gamma_1(x)) \mathcal{V}_1(x,z_1,z_2,s) = 0,
\] (8.3.36)

\[
(\frac{\partial}{\partial x} + s + \lambda_1 (1 - [1 - \theta_1]C(z_1)) + \lambda_2 \alpha b (1 - C(z_2)) + \gamma_2(x)) \mathcal{V}_2(x,z_1,z_2,s) = 0.
\] (8.3.37)
Multiply the equation (8.3.24) by \( z_1^m \) add to \( z_1 \times (8.3.23) \) and then summing \( m \), we get

\[
z_1 P_{n}(0,z_1,s) = \lambda_1 C(z_1) I_{0,n}(s) + q_1 \int_{0}^{\infty} P_{n}(x,z_1,s) \mu_1(x) dx
\]

\[
- q_1 \int_{0}^{\infty} P_{0,n}(x,s) \mu_1(x) dx + \int_{0}^{\infty} \bar{q}_0(x,z_1,s) \mu_2(x) dx
\]

\[
- \int_{0}^{\infty} \bar{q}_0(x,s) \mu_2(x) dx + \int_{0}^{\infty} V_{1,n}(x,z_1,s) \gamma_1(x) dx
\]

\[
- \int_{0}^{\infty} \gamma_1(x) dx + \lambda_1 C(z_1) \theta_1 \int_{0}^{\infty} V_{2,n}(x,s) dx
\]

\[
+ \int_{0}^{\infty} \gamma_2(x) dx - \int_{0}^{\infty} V_{2,n}(x,s) \gamma_2(x) dx
\]

\[
+ P_1 z_1 \int_{0}^{\infty} P_{n}(x,z_1,s) \mu_1(x) dx; \quad n \geq 0.
\] (8.3.38)

Now multiply the boundary conditions (8.3.22), (8.3.25) to (8.3.28) and (8.3.38) by appropriate powers of \( z_2^n \), we get

\[
I_{0,0}(0,z_2,s) = (1 - \theta) q_1 \int_{0}^{\infty} P_{0,0}(x,z_2,s) \mu_1(x) dx - (1 - \theta) q_1
\]

\[
\times \int_{0}^{\infty} P_{0,0}(x,z_2,s) \mu_1(x) dx + \int_{0}^{\infty} \bar{q}_0(x,z_2,s) \mu_2(x) dx
\]

\[
- \int_{0}^{\infty} \bar{q}_0(x,s) \mu_2(x) dx + (1 - r) \int_{0}^{\infty} V_{1,0}(x,z_2,s) \gamma_1(x) dx
\]

\[
- \int_{0}^{\infty} \gamma_1(x) dx + \int_{0}^{\infty} V_{2,0}(x,z_2,s) \gamma_2(x) dx
\]

\[
- \int_{0}^{\infty} V_{2,0}(x,s) \gamma_2(x) dx,
\] (8.3.39)

\[
z_2 \bar{q}_0(0,z_2,s) = \alpha \lambda_2 C(z_2) I_{0,0}(s) + \int_{0}^{\infty} I_{0}(x,z_2,s) \eta(x) dx
\]

\[
+ \lambda_2 \alpha C(z_2) \int_{0}^{\infty} I_{0}(x,z_2,s) dx,
\] (8.3.40)

\[
\bar{V}_{1,0}(0,z_2,s) = \theta q_1 \int_{0}^{\infty} P_{0}(x,z_2,s) \mu_1(x) dx,
\] (8.3.41)

\[
\bar{V}_{2,0}(0,z_2,s) = r \int_{0}^{\infty} \bar{V}_{1,0}(x,z_2,s) \gamma_1(x) dx
\]

\[
z_1 P_0(0,z_1,z_2,s) = \lambda_1 C(z_1) I_{0}(z_2,s) + q_1 \int_{0}^{\infty} \bar{P}(x,z_1,z_2,s) \mu_1(x) dx
\]

\[
- q_1 \int_{0}^{\infty} \bar{P}(x,z_2,s) \mu_1(x) dx + \int_{0}^{\infty} \bar{q}_1(x,z_1,z_2,s) \mu_2(x) dx
\]

\[
- \int_{0}^{\infty} \bar{q}_1(x,z_2,s) \mu_2(x) dx + \int_{0}^{\infty} \gamma_1(x) dx + \lambda_1 C(z_1) \theta_1 \int_{0}^{\infty} V_{2,0}(x,z_2,s) dx
\]

\[
+ \int_{0}^{\infty} \gamma_2(x) dx - \int_{0}^{\infty} V_{2,0}(x,z_2,s) \gamma_2(x) dx
\]
we integrate equations \(8.3.33\) to \(8.3.37\), we get

\[
I_0(x,z_2,s) = I_0(0,z_2,s) e^{-(s+\lambda_1+\lambda_2\alpha)x} \int_0^x \eta(t)dt,
\]

\(8.3.44\)

\[
\bar{P}(x,z_1,z_2,s) = \bar{P}(0,z_1,z_2,s) e^{-(s+\lambda_1(1-C(z_1))+\lambda_2\alpha b(1-C(z_2)))x} \int_0^x \mu_1(t)dt,
\]

\(8.3.45\)

\[
\bar{q}(x,z_1,z_2,s) = \bar{q}(0,z_1,z_2,s) e^{-(s+\lambda_1(1-C(z_1))+\lambda_2\alpha b(1-C(z_2)))x} \int_0^x \mu_2(t)dt,
\]

\(8.3.46\)

\[
\bar{V}_1(x,z_1,z_2,s) = \bar{V}_1(0,z_1,z_2,s) e^{-(s+\lambda_1(1-C(z_1))+\lambda_2\alpha b(1-C(z_2)))x} \int_0^x \gamma_1(t)dt,
\]

\(8.3.47\)

\[
\bar{V}_2(x,z_1,z_2,s) = \bar{V}_2(0,z_1,z_2,s) e^{-(s+\lambda_1(1-|\theta_1|C(z_1))+\lambda_2\alpha b(1-C(z_2)))x} \int_0^x \gamma_2(t)dt.
\]

\(8.3.48\)

Next, we multiply equations \(8.3.44\) to \(8.3.48\) by \(\eta(x)\), \(\mu_1(x)\), \(\mu_2(x)\), \(\gamma_1(x)\) and \(\gamma_2(x)\) respectively and then integrate with respect to \(x\), we get

\[
\int_0^\infty \bar{I}_0(x,z_2,s)\eta(x)dx = \bar{I}_0(0,z_2,s)M(s+\lambda_1+\lambda_2\alpha),
\]

\(8.3.49\)

\[
\int_0^\infty \bar{P}(x,z_1,z_2,s)\mu_1(x)dx = \bar{P}(0,z_1,z_2,s)\bar{B}_1(A(z,s)),
\]

\(8.3.50\)

\[
\int_0^\infty \bar{q}(x,z_1,z_2,s)\mu_2(x)dx = \bar{q}(0,z_1,z_2,s)\bar{B}_2(A(z,s)),
\]

\(8.3.51\)

\[
\int_0^\infty \bar{V}_1(x,z_1,z_2,s)\gamma_1(x)dx = \bar{V}_1(0,z_1,z_2,s)\bar{V}_1(A(z,s)),
\]

\(8.3.52\)

\[
\int_0^\infty \bar{V}_2(x,z_1,z_2,s)\gamma_2(x)dx = \bar{V}_2(0,z_1,z_2,s)\bar{V}_2(B(z,s)),
\]

\(8.3.53\)

where \(A(z,s) = (s+\lambda_1(1-C(z_1))+\lambda_2\alpha b(1-C(z_2)))\), and \(B(z,s) = (s+\lambda_1(1-|\theta_1|C(z_1))+\lambda_2\alpha b(1-C(z_2)))\)

at \(z_1 = 0\) to the equations \(8.3.49\) to \(8.3.53\), we get

\[
\int_0^\infty \bar{P}_0(x,z_2,s)\mu_1(x)dx = \bar{P}_0(0,z_2,s)\bar{B}_1(A_1(z,s)),
\]

\(8.3.54\)

\[
\int_0^\infty \bar{q}_0(x,z_2,s)\mu_2(x)dx = \bar{q}_0(0,z_2,s)\bar{B}_2(A_1(z,s)),
\]

\(8.3.55\)

\[
\int_0^\infty \bar{V}_{1,0}(x,z_2,s)\gamma_1(x)dx = \bar{V}_{1,0}(0,z_2,s)\bar{V}_1(A_1(z,s)),
\]

\(8.3.56\)
\[
\int_{0}^{\infty} V_{2,0}(x,z_2,s)\gamma_2(x)dx = V_{2,0}(0,z_2,s)V_2(B_1(z,s)), \tag{8.3.57}
\]

where \(A_1(z,s) = (s + \lambda_1 + \lambda_2 \alpha b(1 - C(z_2)))\) and \(B_1(z,s) = (s + \lambda_1 + \lambda_2 \alpha b(1 - C(z_2)))\)

Now substitute the equations (8.3.49) to (8.3.57) into the boundary conditions we get,

\[
I_0(0,z_2,s) = \bar{P}_0(0,z_2,s)\bar{B}_1(A_1(z,s),q_1 \left\{ 1 - \theta + (1 - r)\theta V_1(A_1(z,s)) + r\theta \right\} \left(1 - \bar{M}(s + \lambda_1 + \lambda_2 \alpha) + \bar{\eta}_0(0,z_2,s)\{\bar{B}_2(A(z,s)) - \bar{B}_2(A_1(z,s))\} \right) - \{s + \lambda_1 + \lambda_2 \alpha\}I_0(0,s) - 1, \tag{8.3.58}
\]

\[
\bar{P}(0,z_1,z_2,s)\{z_1 - (p_1 z_1 + q_1)\bar{B}_1(A(z,s))\} = \lambda_1 C(z_1)\bar{I}_0(0,z_2,s)
\frac{1 - \bar{M}(s + \lambda_1 + \lambda_2 \alpha)}{s + \lambda_1 + \lambda_2 \alpha} + \bar{\eta}_0(0,z_2,s)\{\bar{B}_2(A(z,s)) - \bar{B}_2(A_1(z,s))\}
- \bar{P}_0(0,z_2,s)\bar{B}_1(A_1(z,s),q_1 \left\{ 1 + \theta V_1(A_1(z,s))\{1 + r\theta V_2(B_1(z,s))\} \right\} - \theta V_1(A(z,s)))
- r\theta_1 \lambda_1 C(z_1)\frac{1 - \bar{V}_2(B_1(z,s))}{B_1(z,s)} - r\theta V_2(B(z,s)) \right\} - V_1(A(z,s))) \right\},
\tag{8.3.59}
\]

\[
z_2\bar{\eta}_0(0,z_2,s) = \alpha \lambda_2 C(z_2)I_{0,0}(s) + \bar{I}_0(0,z_2,s)\{\bar{M}(s + \lambda_1 + \lambda_2 \alpha)
+ \alpha \lambda_2 C(z_2)\frac{1 - \bar{M}(s + \lambda_1 + \lambda_2 \alpha)}{s + \lambda_1 + \lambda_2 \alpha}\},
\tag{8.3.60}
\]

\[
V_1,0(0,z_2,s) = \theta q_1 \bar{P}_0(0,z_2,s)\bar{B}_1(A_1(z,s)),
\tag{8.3.61}
\]

\[
V_2,0(0,z_2,s) = r\theta q_1 \bar{P}_0(0,z_2,s)\bar{B}_1(A_1(z,s))V_1(A_1(z,s)),
\tag{8.3.62}
\]

let \(z_1 = g(z_2)\) in equation (8.3.59), we get

\[
\bar{P}_0(0,z_2,s)\bar{B}_1(A_1(z,s),q_1
\left\{ \lambda_1 C([g(z_2)])I_0(0,z_2,s)\frac{1 - \bar{M}(s + \lambda_1 + \lambda_2 \alpha)}{s + \lambda_1 + \lambda_2 \alpha} \right\}
\frac{1 - \bar{\eta}_0(0,z_2,s)\{\bar{B}_2(A_2(z,s)) - \bar{B}_2(A_1(z,s))\}}{d,s}
\right\},
\tag{8.3.63}
\]

where

\[
(d,s) = \{ 1 + \theta V_1(A_1(z,s))\{1 + r\theta V_2(B_1(z,s))\} - r\theta_1 \lambda_1 C([g(z_2)])
- \frac{1 - \bar{V}_2(B_1(z,s))}{B_1(z,s)} - r\theta V_2(B_2(z,s)) \right\} - \theta V_1(A_2(z,s)) \right\},
\tag{8.3.63}
\]

substitute equation (8.3.63) into (8.3.58), we get

\[
I_0(0,z_2,s) = \frac{(h,s)}{(g,s)},
\tag{8.3.64}
\]

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where
\[
(h, s) = \varphi_0(0, z_2, s)\{B_2(A_2(z, s)) - B_2(A_1(z, s))\}\left\{1 - \theta + (1 - r)\theta V_1(A_1(z, s)) + r\theta V_1(A_1(z, s))V_2(B_1(z, s))\right\} + \varphi_0(0, z_2, s)B_2(A_1(z, s))\left\{1 + \theta \right\} \\
V_1(A_1(z, s))\{1 + rV_2(B_1(z, s)) - r\theta_1\lambda_1C([g(z_2)])\left[\frac{1 - V_2(B_1(z, s))}{B_1(z, s)}\right] - rV_2(B_2(z, s))\} - \theta V_1(A_2(z, s))\} - \{(s + \lambda_1 + \lambda_2\alpha)\bar{T}_{0,0}(s) - 1\} \\
\left\{1 + \theta V_1(A_1(z, s))\{1 + rV_2(B_1(z, s)) - r\theta_1\lambda_1C([g(z_2)])\left[1 - \frac{V_2(B_1(z, s))}{B_1(z, s)}\right] - rV_2(B_2(z, s))\} - \theta V_1(A_2(z, s))\right\}, \\
(g, s) = \left\{1 + \theta V_1(A_1(z, s))\{1 + rV_2(B_1(z, s)) - r\theta_1\lambda_1C([g(z_2)])\left[1 - \frac{V_2(B_1(z, s))}{B_1(z, s)}\right]ight. \\
\left. - rV_2(B_2(z, s))\} - \theta V_1(A_2(z, s))\right\} - \left\{1 - \theta + (1 - r)\theta V_1(A_1(z, s)) + r\theta V_1(A_1(z, s))V_2(B_1(z, s))\right\}\lambda_1C([g(z_2)])\left[\frac{1 - \bar{M}(s + \lambda_1 + \lambda_2\alpha)}{s + \lambda_1 + \lambda_2\alpha}\right].
\]

Substitute the above equation into (8.3.60), we get
\[
\varphi_0(0, z_2, s) = \frac{(h_2, s)}{(g_1, s)},
\]
where
\[
(h_2, s) = \alpha\lambda_2C(z_2)\bar{T}_{0,0}(s)\left\{1 + \theta V_1(A_1(z, s))\{1 + rV_2(B_1(z, s)) - r\theta_1\lambda_1C([g(z_2)])\left[1 - \frac{V_2(B_1(z, s))}{B_1(z, s)}\right]ight. \\
\left. - rV_2(B_2(z, s))\} - \theta V_1(A_2(z, s))\right\} - \left\{1 - \theta + (1 - r)\theta V_1(A_1(z, s)) + r\theta V_1(A_1(z, s))V_2(B_1(z, s))\right\}\lambda_1C([g(z_2)])\left[\frac{1 - \bar{M}(s + \lambda_1 + \lambda_2\alpha)}{s + \lambda_1 + \lambda_2\alpha}\right], \\
(g_1, s) = z_2\left\{1 + \theta V_1(A_1(z, s))\{1 + rV_2(B_1(z, s)) - r\theta_1\lambda_1C([g(z_2)])\left[1 - \frac{V_2(B_1(z, s))}{B_1(z, s)}\right]ight. \\
\left. - rV_2(B_2(z, s))\} - \theta V_1(A_2(z, s))\right\} - z_2\left\{1 - \theta + (1 - r)\theta V_1(A_1(z, s)) + r\theta V_1(A_1(z, s))V_2(B_1(z, s))\right\}\lambda_1C([g(z_2)])\left[\frac{1 - \bar{M}(s + \lambda_1 + \lambda_2\alpha)}{s + \lambda_1 + \lambda_2\alpha}\right] - \\
\left\{\bar{M}(s + \lambda_1 + \lambda_2\alpha) + \alpha\lambda_2C(z_2)\left[1 - \frac{M(s + \lambda_1 + \lambda_2\alpha)}{s + \lambda_1 + \lambda_2\alpha}\right]\left\{B_2(A_2(z, s))\right\} \right. \\
\left. \{1 - \theta + (1 - r)\theta V_1(A_1(z, s)) + r\theta V_1(A_1(z, s))V_2(B_1(z, s))\} + B_2(A_1(z, s))\theta \right\}
\]
\[
\left\{ 1 - (1 - r)\bar{V}_1(A_1(z, s)) - r\theta_1\bar{V}_1(A_1(z, s))\lambda_4 C([g(z_2)])\left[\frac{1 - \bar{V}_2(B_1(z, s))}{B_1(z, s)}\right] \\
+ \bar{V}_1(A_1(z, s)) - r\bar{V}_1(A_1(z, s))\bar{V}_2(B_2(z, s)) - \bar{V}_1(A_2(z, s)) \right\}.
\]

substitute equation (8.3.65) into equation (8.3.64), we get

\[
I_0(0, z_2, s) = \frac{(h_1, s)}{(g_1, s)}, \quad (8.3.66)
\]

\[
(h_1, s) = \alpha\lambda_2 C(z_2)\bar{I}_{0,0}(s)\left\{ 1 - \theta + (1 - r)\theta\bar{V}_1(A_1(z, s)) + r\theta \\
\bar{V}_1(A_1(z, s))\bar{V}_2(B_1(z, s)) \right\} + \bar{B}_2(A_2(z, s))(1 - (1 - r)\bar{V}_1(A_1(z, s)) - r\theta_1
\]

\[
\bar{V}_1(A_1(z, s))\lambda_1 C([g(z_2)])\left[\frac{1 - \bar{V}_2(B_1(z, s))}{B_1(z, s)}\right] + \bar{V}_1(A_1(z, s)) - r\bar{V}_1(A_1(z, s))
\]

\[
\bar{V}_2(B_2(z, s)) - \bar{V}_1(A_2(z, s)) \right\} - z_2\{(s + \lambda_1 + \lambda_2\alpha)\bar{I}_{0,0}(s) - 1\}{1 + \theta
\]

\[
\bar{V}_1(A_1(z, s))\left\{ 1 + \bar{V}_2(B_1(z, s)) - r\theta_1\lambda_1 C([g(z_2)])\left[\frac{1 - \bar{V}_2(B_1(z, s))}{B_1(z, s)}\right] \\
r\bar{V}_2(B_2(z, s)) - \theta\bar{V}_1(A_2(z, s)) \right\},
\]

substitute equations (8.3.63), (8.3.65) and (8.3.66) into (8.3.59), we get

\[
\mathcal{P}(0, z_1, z_2, s) = \frac{\left\{ 1 - A(z, s) \right\} I_0(0, z_2, s)\left[ 1 - \bar{M}(s + \lambda_1 + \lambda_2\alpha) \right]}{s + \lambda_1 + \lambda_2\alpha} \\
+ \bar{q}_0(0, z_2, s)\left\{ z_2 - \bar{B}_2(A(z, s)) \right\}, \quad (8.3.67)
\]

where

\[
f_2(z, s) = z_1 - \bar{B}_1(A(z, s))\{q_1 + z_1\rho_1\},
\]

**Theorem**

The inequality \( \rho < 1 \) is a necessary and sufficient condition for the system to be stable, under this condition the marginal PGF of the server’s state queue size distribution are given by

\[
I_0(z_2, s) = I_0(0, z_2, s)\left[ 1 - \bar{M}(s + \lambda_1 + \lambda_2\alpha) \right], \quad (8.3.68)
\]

\[
\mathcal{P}(z_1, z_2, s) = \mathcal{P}(0, z_1, z_2, s)\left[ 1 - \bar{B}_1(A(z, s)) \right] \left( A(z, s) \right), \quad (8.3.69)
\]

\[
\bar{q}(z_1, z_2, s) = \bar{q}(0, z_1, z_2, s)\left[ 1 - \bar{B}_2(A(z, s)) \right] \left( A(z, s) \right), \quad (8.3.70)
\]

\[
\bar{V}_1(z_1, z_2, s) = \bar{V}_1(0, z_1, z_2, s)\left[ 1 - \bar{V}_1(A(z, s)) \right] \left( A(z, s) \right), \quad (8.3.71)
\]

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\[ V_2(z_1, z_2, s) = V_2(0, z_1, z_2, s)[1 - V_2(B(z, s))]. \] 

**Proof:** Integrating equations (8.3.44) to (8.3.48) with respect to \( x \) and using the well-known result of renewal theory

\[
\int_0^\infty [1 - H(x)] e^{-sx} dx = \frac{1 - \overline{h}(s)}{s},
\] 

where \( \overline{h}(s) \) is the LST of the distribution function of a random variable \( H(x) \), we get the formulae (8.3.68) to (8.3.72). Thus we obtained the complete solution of the probability generating function for the states are, \( I_0(z_2, s), \, P(z_1, z_2, s), \, q(z_1, z_2, s), \, V_1(z_1, z_2, s) \) and \( V_2(z_1, z_2, s) \).

### 8.4 Steady state Analysis: Limiting behaviour

In this section, we derive the steady state probability distribution for our queueing model. By applying the well-known Tauberian property,

\[
\lim_{s \to 0} s \overline{f}(s) = \lim_{t \to \infty} f(t).
\]

to the equations (8.3.68) to (8.3.72). In order to determine \( I_{0,0} \) we use the normalizing condition

\[ I_{0,0} + I_0(1) + P(1, 1) + q(1, 1) + V_1(1, 1) + V_2(1, 1) = 1. \]

The steady state probability for high priority and low priority customers with retrial, two kinds of vacations, balking and feedback are

\[ I_0(z_2) = I_0(0, z_2)\left[\frac{1 - M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha}\right], \] 

(8.4.1)

\[ P(z_1, z_2) = P(0, z_1, z_2)\left[\frac{1 - B_1(A(z))}{A(z)}\right], \] 

(8.4.2)

\[ q(z_1, z_2) = q_0(0, z_2)\left[\frac{1 - B_2(A(z))}{A(z)}\right], \] 

(8.4.3)

\[ V_1(z_1, z_2) = \theta q_1 P_0(0, z_2) B_1(A_1(z))\left[\frac{1 - V_1(A(z))}{A(z)}\right], \] 

(8.4.4)

\[ V_2(z_1, z_2) = r \theta q_1 P_0(0, z_2) B_1(A_1(z)) V_1(A_1(z))\left[\frac{1 - V_2(B(z))}{B(z)}\right], \] 

(8.4.5)

with, 228
\[ I_0(0, z_2) = \frac{h_1}{g_1}, \quad (8.4.6) \]
\[ P(0, z_1, z_2) = \frac{-A(z)I_0(0, z_2) \left\{ 1 - \frac{M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha} \right\} + q_0(0, z_2)\{ z_2 - B_2(A(z)) \} }{f_2(z)}, \quad (8.4.7) \]
\[ q_0(0, z_2) = \frac{h_2}{g_1}, \quad (8.4.8) \]
\[ P_0(0, z_2)B_1(A(z))q_1 \]
\[ \left\{ \frac{\lambda_1 C([g(z_2)])[I_0(0, z_2) \left\{ 1 - \frac{M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha} \right\} ]}{d} \right\} \]
\[ = \frac{+ q_0(0, z_2)\{ B_2(A_2(z)) - B_2(A_1(z)) \} }{d}, \quad (8.4.9) \]

where
\[ h_1 = \alpha \lambda_2 C(z_2) I_{0,0} \left\{ B_2(A_2(z)) \left\{ 1 - \theta + (1 - r)\theta V_1(A_1(z)) + r\theta V_1(A_1(z))V_2(B_1(z)) \right\} \]
\[ + B_2(A_1(z)) \theta \left\{ 1 - (1 - r)V_1(A_1(z)) - r\theta_1 V_1(A_1(z))\lambda_1 C([g(z_2)]) \left\{ \frac{1 - V_2(B_1(z))}{B_1(z)} \right\} \]
\[ + V_1(A_1(z)) - rV_1(A_1(z))V_2(B_2(z)) - V_1(A_2(z)) \} \right\} - z_2\left\{ (\lambda_1 + \lambda_2 \alpha) I_{0,0} \right\} \]
\[ \left\{ 1 + \theta V_1(A_1(z)) \right\} \left\{ 1 + rV_2(B_1(z)) - r\theta_1 \lambda_1 C([g(z_2)]) \left\{ \frac{1 - V_2(B_1(z))}{B_1(z)} \right\} \]
\[ - rV_2(B_2(z)) - \theta V_1(A_2(z)) \} \right\}, \]
\[ g_1 = z_2\left\{ 1 + \theta V_1(A_1(z)) \right\} \left\{ 1 + rV_2(B_1(z)) - r\theta_1 \lambda_1 C([g(z_2)]) \left\{ \frac{1 - V_2(B_1(z))}{B_1(z)} \right\} \]
\[ - rV_2(B_2(z)) - \theta V_1(A_2(z)) \} - z_2\left\{ 1 - \theta + (1 - r)\theta V_1(A_1(z)) + r\theta V_1(A_1(z)) \right\} \]
\[ V_2(B_1(z)) \left\{ \lambda_1 C([g(z_2)]) \left\{ \frac{1 - M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha} \right\} \right\} - \left\{ M(\lambda_1 + \lambda_2 \alpha) + \alpha \lambda_2 C(z_2) \right\} \]
\[ \frac{1 - M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha} \left\{ B_2(A_2(z)) \left\{ 1 - \theta + (1 - r)\theta V_1(A_1(z)) + r\theta V_1(A_1(z)) \right\} \right\} \]
\[ V_2(B_1(z)) \left\{ B_2(A_1(z)) \theta \left\{ 1 - (1 - r)V_1(A_1(z)) - r\theta_1 V_1(A_1(z))\lambda_1 C([g(z_2)]) \right\} \]
\[ \left\{ \frac{1 - V_2(B_1(z))}{B_1(z)} \right\} + V_1(A_1(z)) - rV_1(A_1(z))V_2(B_2(z)) - V_1(A_2(z)) \} \right\}, \]
\[ f_2(z) = z_1 - B_1(A(z))\{ q_1 + z_1 p_1 \}, \]
\[ h_2 = \alpha \lambda_2 C(z_2) I_{0,0} \left\{ 1 + \theta V_1(A_1(z)) \right\} \left\{ 1 + rV_2(B_1(z)) - r\theta_1 \lambda_1 C([g(z_2)]) \right\} \]
\[ \left\{ \frac{1 - V_2(B_1(z))}{B_1(z)} \right\} - rV_2(B_2(z)) - \theta V_1(A_2(z)) \} \left\{ M(\lambda_1 + \lambda_2 \alpha) + \alpha \lambda_2 C(z_2) \right\} \]
\[
\left\{ 1 - \theta + (1 - r) \theta V_1(A(z)) + r \theta V_1(A(z)) V_2(B_1(z)) \right\}
\]
\[
\lambda_1 C([g(z)]) \left[ \frac{1 - M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha} \right],
\]
\[
d = \left\{ 1 + \theta V_1(A(z)) \left\{ 1 + r V_2(B_1(z)) - r \theta_1 \lambda_1 C([g(z)]) \left[ \frac{1 - V_2(B_1(z))}{B_1(z)} \right] - r V_2(B_2(z)) \right\} - \theta V_1(A_2(z)) \right\}.
\]

Let \( W_q(z_1, z_2) \) be the probability generating function of the queue sizes irrespective of the state of the system.

\[
W_q(z_1, z_2) = I_0(z_2) + P(z_1, z_2) + q(z_1, z_2) + V_1(z_1, z_2) + V_2(z_1, z_2),
\]

after simplification, we get

\[
W_q(z_1, z_2) = \left\{ \frac{N_r}{A(z) B(z) d(z) f_2(z)} \right\},
\]

where

\[
N_r = I_0(0, z_2) \left\{ \frac{1 - M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha} \right\} \left\{ A(z) B(z) d(z) f_2(z) - A(z) B(z) d(z) [1 - B_1(A(z))] + \theta \lambda_1 C([g(z)]) f_2(z) \left\{ B(z) [1 - V_1(A(z))] + r V_1(A_1(z)) A(z) [1 - V_2(B(z))] \right\} \right\}
\]
\[
+ q_0(0, z_2) \left\{ [z_2 - B_2(A(z))] [1 - B_1(A(z))] d(z) + [1 - B_2(A(z))] B(z) f_2(z) \left\{ B(z) [1 - V_1(A(z))] + r V_1(A_1(z)) A(z) [1 - V_2(B(z))] \right\} \right\}.
\]

By using the normalizing condition, we find the unknown probability \( I_{0,0} \).

\[
I_{0,0} = \frac{d \theta_1 \lambda_1 \left\{ 1 - p_1 - E[B_1](\lambda_1 + \lambda_2 \alpha b) E[I] \right\}}{D},
\]

where

and the normalising factor is

\[
\rho = \frac{N}{D},
\]

where

\[
N = I_0(0, 1) \left\{ \frac{1 - M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha} \right\} \left\{ d \theta_1 \lambda_1 \left\{ 1 - p_1 - (p_1 + q_1) E[B_1](\lambda_1 + \lambda_2 \alpha b) E[I] \right\} + d \theta_1 \lambda_1 E[B_1](\lambda_1 + \lambda_2 \alpha b) E[I] + \theta \lambda_1 \left\{ 1 - p_1 - (p_1 + q_1) E[B_1](\lambda_1 + \lambda_2 \alpha b) E[I] \right\} \right\}
\]
\[
E[I] \left\{ \theta_1 \lambda_1 E[I] + r V_1(\lambda_1) [1 - V_2(\lambda_1 \theta_1)] \right\} + q_0(0, 1) \left\{ d \theta_1 \lambda_1 E[B_1] \left\{ 1 - E[B_2] \right\} E[I] \right\} + \left\{ 1 - p_1 - (p_1 + q_1) E[B_1](\lambda_1 + \lambda_2 \alpha b) E[I] \right\}
\]
Equation (8.4.12) gives the probability that the server is idle. Substituting equation (8.4.12) in equation (8.4.11), we have completely and explicitly determined $Wq(z_1, z_2)$, the probability generating function of the queue size.

### 8.5 The average queue length:

The mean number of customers in the high priority queue and in the orbit under the steady state condition is

$$L_{q_1} = \frac{d}{dz_1} W_{q_1}(z_1, 1)|_{z_1=1}, \quad (8.5.1)$$

$$L_{q_2} = \frac{d}{dz_2} W_{q_2}(1, z_2)|_{z_2=1}, \quad (8.5.2)$$

respectively, then

$$L_{q_1} = \frac{Dr''_1(1, 1)N_{r''_1}(1, 1) - Nr''_1(1, 1)Dr''_1(1, 1)}{3(Dr''_1(1, 1))^2}, \quad (8.5.3)$$

$$L_{q_2} = \frac{Dr''_2(1, 1)N_{r''_2}(1, 1) - Nr''_2(1, 1)Dr''_2(1, 1)}{3(Dr''_2(1, 1))^2}, \quad (8.5.4)$$

where

$$N_{r''_1}(1, 1) = -2\lambda_1 E[I]\left\{ I_0(0, 1)\left[1 - \frac{M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha}\right]\{d\theta_1 \lambda_1 \{1 - p_1 - (p_1 + q_1)E[B_1]\} \right\}$$

$$+ \lambda_1 E[I] + d\theta_1 \lambda_1 E[B_1] \{1 - p_1 - (p_1 + q_1)E[B_1]\} \right\} \theta_1 \lambda_1 E[V_1]$$

$$+ rV_1(\lambda_1)|1 - V_2(\lambda_1 \theta_1)| + q_0(0, 1)\left\{ - d\theta_1 \lambda_1 E[B_1]E[B_2] + d\theta_1 \lambda_1$$

$$+ rV_1(\lambda_1)|1 - V_2(\lambda_1 \theta_1)| \right\}.$$
\[
E[B_2\{1 - p_1 + (p_1 + q_1)E[B_1]\lambda_1 E[I]\} + \theta[1 - \overline{B}_2(\lambda_1)]\{1 - p_1 + (p_1 + q_1)E[B_1]\lambda_1 E[I] + r\overline{V}_1(\lambda_1)[1 - \overline{V}_2(\lambda_1)]\}]
\]

\[
N\rho''(1, 1) = I_0(0, 1)\left\{\frac{-M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha}\right\}\left\{-3d\theta_1\lambda_1\{\lambda_1 E[I(I - 1)]\{1 - p_1 + (p_1 + q_1)\}
E[B_1]\lambda_1 E[I]\} + f_2''(1)\lambda_1 E[I]\} + 6d\lambda_1 E[I]\lambda_1(1 - \theta_1)E[I]\{1 - p_1 + (p_1 + q_1)\}
E[B_1]\lambda_1 E[I]\} + 3d\lambda_1\{ - \lambda_1 E[I]\lambda_1 E[I]E[I(I - 1)]E[B_1] + \{1 - p_1 + (p_1 + q_1)\}
E[B_1]\lambda_1 E[I]\} \{E[B_1]^2(\lambda_1 E[I])^2 + E[B_1]\lambda_1 E[I(I - 1)]\} - 6d\lambda_1 E[I]\lambda_1(1 - \theta_1)E[I]
\{1 - p_1 + (p_1 + q_1)E[B_1]\lambda_1 E[I]\} - 3\theta_1 \lambda_1 f_2''(1)\{\theta_1 \lambda_1 E[V_1]\lambda_1 E[I] + r\overline{V}_1(\lambda_1)\lambda_1 E[I][1 - \overline{V}_2(\lambda_1)]\} + 3\theta_1\{1 - p_1 + (p_1 + q_1)E[B_1]\lambda_1 E[I]\} \{2\lambda_1
(1 - \theta_1)E[I]E[V_1]\lambda_1 E[I] - \lambda_1 \theta_1\{E[V_1]^2(\lambda_1 E[I])^2 + E[V_1]\lambda_1 E[I(I - 1)]\} - r\overline{V}_1(\lambda_1)\lambda_1 E[I(I - 1)][1 - \overline{V}_2(\lambda_1)] + 2r\overline{V}_1(\lambda_1)\lambda_1 E[I]E[V_2]\lambda_1(1 - \theta_1)E[I]\}
\} + q_0(0, 1)\left\{3d\lambda_1\theta_1\{\{E[B_1]^2(\lambda_1 E[I])^2 + E[B_2]\lambda_1 E[I(I - 1)]\}E[B_1]\lambda_1 E[I]
E[B_2]\lambda_1 E[I]\} \{E[B_1]^2(\lambda_1 E[I])^2 + E[B_1]\lambda_1 E[I(I - 1)]\} - 6E[B_1]\lambda_1 E[I](\lambda_1 E[I])^2d\lambda_1(1 - \theta_1)E[I] - 3d\lambda_1\theta_1\{E[B_2]^2(\lambda_1 E[I])^2 + E[B_2]\lambda_1 E[I(I - 1)]\}
\{1 - p_1 + (p_1 + q_1)E[B_1]\lambda_1 E[I]\} + E[B_2]\lambda_1 E[I]f_2''(1)\} + 6dE[B_2]\lambda_1 E[I]\lambda_1
(1 - \theta_1)E[I]\{1 - p_1 + (p_1 + q_1)E[B_1]\lambda_1 E[I]\} - 3\theta_1[1 - \overline{B}_2(\lambda_1)]f_2''(1)\{E[V_1]\lambda_1 E[I] - \lambda_1 \theta_1 + r\overline{V}_1(\lambda_1)\lambda_1 E[I][1 - \overline{V}_2(\lambda_1)]\} + 3\theta_1[1 - \overline{B}_2(\lambda_1)]\{1 - p_1 + (p_1 + q_1)E[B_1]\lambda_1 E[I]\}
\} - \lambda_1 \theta_1\{E[V_1]^2(\lambda_1 E[I])^2 + E[V_1]\lambda_1 E[I(I - 1)]\} + 2
E[V_1]\lambda_1 E[I]E[I(1 - \theta_1)E[I] - r\overline{V}_1(\lambda_1)\lambda_1 E[I(I - 1)]][1 - \overline{V}_2(\lambda_1)] - 2r
\overline{V}_1(\lambda_1)\lambda_1 E[I](1 - \theta_1)E[I]\overline{V}_2(\lambda_1)\}
\},
\]

\[
D\rho''(1) = 2d\lambda_1\theta_1\{1 - p_1 + (p_1 + q_1)E[B_1]\lambda_1 E[I]\},
\]

\[
D\rho''''(1) = -3d\lambda_1\theta_1\{\lambda_1 E[I]f_2''''(1) + \lambda_1 E[I(I - 1)]\{1 - p_1 + (p_1 + q_1)E[B_1]\lambda_1 E[I]\}\}
+ 6d\lambda_1 E[I]\lambda_1(1 - \theta_1)E[I]\{1 - p_1 + (p_1 + q_1)E[B_1]\lambda_1 E[I]\},
\]

\[
N\rho'''(1, 1) = -2\lambda_2 \alpha bE[I]\left\{I_0(0, 1)\left\{\frac{-M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha}\right\}\right\}\left\{-\lambda_1 \theta_1 + E[B_1]\lambda_2 \alpha bE[I]\lambda_1 \theta_1 - \theta \lambda_1 (p_1 + q_1)E[B_1]\lambda_2 \alpha bE[I]\{1 - \lambda_1 E[V_1] + r\overline{V}_1(\lambda_1)(1 - \overline{V}_2(\lambda_1))\}\right\} + q_0(0, 1)\left\{d\lambda_1\lambda_1 E[B_1]\{1 - E[B_2]\lambda_2 \alpha bE[I]\} - d
\lambda_1\theta_1 (p_1 + q_1)E[B_1]\lambda_2 \alpha bE[I]E[B_2] - \theta[1 - \overline{B}_2(\lambda_1)](p_1 + q_1)E[B_1]\lambda_2 \alpha bE[I]\right\}
\]
\[
\left\{ \theta_1 \lambda_1 E[V_1] + rV_1(\lambda_1)[1 - V_2(\lambda_1 \theta_1)] \right\},
\]

\[
Nr''_{2}(1, 1) = 6\rho_0(0, 1)\left[ 1 - \overline{M}(\lambda_1 + \lambda_2 \alpha) \right][\lambda_2 \alpha bE[I]\left\{ 3d\lambda_1 \theta_1 \left\{ \lambda_2 \alpha bE[I] \right\}(p_1 + q_1)E[B_1]d - dE[B_1] \lambda_1 \theta_1 + \theta \lambda_1
\]

\[
(p_1 + q_1)E[B_1] \left\{ \theta_1 \lambda_1 E[V_1] + rV_1(\lambda_1)[1 - V_2(\lambda_1 \theta_1)] \right\} \right\} (\lambda_2 \alpha bE[I])^2 + I_0(0, 1)
\]

\[
\left\{ \frac{1 - \overline{M}(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha} \right\}[\lambda_2 \alpha bE[I]\left\{ 3d\lambda_1 \theta_1 \left\{ \lambda_2 \alpha bE[I] \right\}(p_1 + q_1)E[B_1] + (p_1 + q_1)\left\{ E[B_1^2](\lambda_1 E[I])^2 + E[B_1] \lambda_1 E[I](I - 1) \right\} \right\} + 6(p_1 + q_1)E[B_1] \lambda_2 \alpha bE[I]
\]

\[
bE[I]\left\{ d' \lambda_1 \theta_1 - d \lambda_2 \alpha bE[I] \right\} + 3d\lambda_1 \theta_1 \left\{ \lambda_2 \alpha bE[I] \right\}(p_1 + q_1)E[B_1] \left\{ \theta_1 \lambda_1 E[V_1] \lambda_2 \alpha bE[I] + rV_1(\lambda_1) \lambda_2 \alpha bE[I]
\]

\[
[1 - V_2(\lambda_1 \theta_1)] \right\} + 3\theta \lambda_1 \left\{ (p_1 + q_1) \left\{ \lambda_2 \alpha bE[I] \right\}(p_1 + q_1) \left\{ E[B_1^2](\lambda_1 E[I])^2 + E[B_1] \lambda_1 E[I](I - 1) \right\} \right\}
\]

\[
\left\{ \theta_1 \lambda_1 E[V_1] + rV_1(\lambda_1)[1 - V_2(\lambda_1 \theta_1)] \right\} - (p_1 + q_1)E[B_1] \left\{ \lambda_1 \theta_1 E[V_1] \lambda_2 \alpha bE[I] - \lambda_1 \theta_1 \left\{ E[B_1^2](\lambda_1 E[I])^2 + E[B_1] \lambda_1 E[I](I - 1) \right\} \right\} + 2r(\overline{V}_1(\lambda_1) \lambda_2 \alpha bE[I][1 - V_2(\lambda_1 \theta_1)] - V_1(\lambda_1) \overline{V}_2(\lambda_1 \theta_1) \lambda_2 \alpha bE[I]) \lambda_2 \alpha bE[I] - rV_1(\lambda_1)
\]

\[
\left\{ \lambda_2 \alpha bE[I](I - 1)[1 - V_2(\lambda_1 \theta_1)] \right\} \right\}
\]

\[
\left\{ 1 - E[B_2] \lambda_2 \alpha bE[I] \right\} + 2d\lambda_1 \theta_1 (p_1 + q_1)E[B_1] E[B_2] \lambda_2 \alpha bE[I] + 2\theta
\]

\[
\left\{ 1 - B_2(\lambda_1)(p_1 + q_1)E[B_1] \left\{ \theta_1 \lambda_1 E[V_1] \lambda_2 \alpha bE[I] + rV_1(\lambda_1) \lambda_2 \alpha bE[I]\right\} \right\}
\]

\[
[1 - V_2(\lambda_1 \theta_1)] \right\}
\]

\[
q_0(0, 1) \left\{ 3d\lambda_1 \theta_1 \left\{ \lambda_2 \alpha bE[I] \right\}^2 + E[B_2] \lambda_1 \lambda_1 E[I](I - 1) \right\} \right\} + E[B_1] \lambda_2 \alpha bE[I] - \left\{ 1 - E[B_2] \lambda_2 \alpha bE[I] \right\} \left\{ E[B_2^2](\lambda_1 E[I])^2
\]

\[
+ E[B_1] \lambda_1 E[I](I - 1) \right\} \right\} - 6\left\{ 1 - E[B_2] \lambda_2 \alpha bE[I] \right\} E[B_1] \lambda_2 \alpha bE[I]\left\{ d' \lambda_1 \theta_1 - d \lambda_2 \alpha bE[I] \right\] + 6\{d' \lambda_1 \theta_1 - d \lambda_2 \alpha bE[I]\} E[B_2] \lambda_2 \alpha bE[I](p_1 + q_1)E[B_1]
\]

\[
\lambda_2 \alpha bE[I] + 3d\lambda_1 \theta_1 \left\{ (p_1 + q_1) \left\{ E[B_2^2](\lambda_1 E[I])^2 + E[B_1] \lambda_1 E[I](I - 1) \right\} \right\} E[B_2] \lambda_2 \alpha bE[I] + (p_1 + q_1)E[B_1] \lambda_2 \alpha bE[I] \left\{ E[B_2^2](\lambda_1 E[I])^2 + E[B_2] \lambda_1
\]

\[
E[I](I - 1) \right\} \right\} + 6\theta \left\{ E[B_2] \left\{ \lambda_2 \alpha bE[I] \right\} + \lambda_2 \alpha bE[I] \right\} - \overline{V}_2(\lambda_1) \lambda_2 \alpha bE[I] \left\{ (p_1 + q_1)E[B_1] \lambda_2 \alpha bE[I] \left\{ \theta_1 \lambda_1 E[V_1] \lambda_2 \alpha bE[I] + rV_1(\lambda_1) \lambda_2 \alpha bE[I]
\]

\[
[1 - V_2(\lambda_1 \theta_1)] \right\} + 3\theta [1 - B_2(\lambda_1)] \left\{ (p_1 + q_1) \left\{ E[B_1^2](\lambda_1 E[I])^2 + E[B_1]
\]

\[
\lambda_1 E[I](I - 1) \right\} \right\} \left\{ \theta_1 \lambda_1 E[V_1] \lambda_2 \alpha bE[I] + rV_1(\lambda_1) \lambda_2 \alpha bE[I](1 - V_2(\lambda_1 \theta_1)) \right\}
\]

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\]
\[- (p_1 + q_1)E[B_1] \lambda_2 \alpha b E[I] \{ 2 \lambda_2 \alpha b E[I] E[V_1] \lambda_2 \alpha b E[I] - \lambda_1 \theta_1 \{ E[V_1]^2 \} \\
(\lambda_1 E[I])^2 + E[V_1] \lambda_1 E[I(I - 1)] \} + 2 r \lambda_2 \alpha b E[I] \{ V_1'(\lambda_1) \lambda_2 \alpha b E[I] \\
[1 - V_2(\lambda_1 \theta_1)] - V_1(\lambda_1) V_2'(\lambda_1 \theta_1) \lambda_2 \alpha b E[I] - r V_1(\lambda_1) \lambda_2 \alpha b E[I(I - 1)] \\
[1 - V_2(\lambda_1 \theta_1)] \}\right\},

\[D_{r_2}'(1, 1) = 2 d \lambda_1 \theta_1 \lambda_2 \alpha b E[I] (p_1 + q_1) E[B_1] \lambda_2 \alpha b E[I],\]

\[D_{r_2}''(1, 1) = 3 \lambda_2 \alpha b E[I] \{ d \lambda_1 \theta_1 \{ (p_1 + q_1) [E[I] \{ \lambda_1 E[I] \}^2 + E[B_1] \lambda_1 E[I(I - 1)] \\
+ \lambda_2 \alpha b E[I(I - 1)] (p_1 + q_1) E[B_1] \} + 2 (p_1 + q_1) E[B_1] \lambda_2 \alpha b E[I] \\
\{ d' \lambda_1 \theta_1 - d \lambda_2 \alpha b E[I] \}\right\},

\[I_0(0, 1) = \frac{h_1(1)}{g_1(1)},\]

\[q_0(0, 1) = \frac{h_2(1)}{g_1(1)},\]

\[I'_0(0, 1) = \frac{h_1'(1) g_1(1) - h_1(1) g_1'(1)}{(g_1(1))^2},\]

\[q'_0(0, 1) = \frac{g_1(1) h_2'(1) - g_1'(1) h_2(1)}{(g_1(1))^2},\]

\[d(1) = 1 - \theta + \theta V_1(\lambda_1) \{ 1 + r V_2(\lambda_1) - r \theta_1 [1 - V_2(\lambda_1)] - r V_2(\lambda_1 \theta_1) \},\]

\[d'(1) = - \theta V_1'(\lambda_1) \lambda_2 \alpha b E[I] \{ 1 + r V_2(\lambda_1) - r \theta_1 [1 - V_2(\lambda_1)] - r V_2(\lambda_1 \theta_1) \} + \theta V_1(\lambda_1) \\
\{ - r V_2'(\lambda_1) \lambda_2 \alpha b E[I] - r \theta_1 E[I] E[I_1] \{ 1 - V_2(\lambda_1) \} + \frac{r \theta_1 \lambda_2 \alpha b E[I]}{\lambda_1} \{ V_2'(\lambda_1) \\
\lambda_1 + 1 - V_2(\lambda_1) + r V_2'(\lambda_1) \{ \lambda_1 (1 - \theta_1) E[I] E[I_1] + \lambda_2 \alpha b E[I] \} + \theta E[V_1] \\
\{ \lambda_1 E[I] E[I_1] + \lambda_2 \alpha b E[I] \},\]

\[h_1(1) = \lambda_2 \alpha \left\{ \{ 1 - \theta + (1 - r) \theta V_1(\lambda_1) + r \theta V_1(\lambda_1) V_2(\lambda_1) \} + B_2(\lambda_1) \theta \{ V_1(\lambda_1) \\
- r \theta_1 V_1(\lambda_1) [1 - V_2(\lambda_1)] - r V_1(\lambda_1) V_2(\lambda_1 \theta_1) \}\right\} - (\lambda_1 + \lambda_2 \alpha) \{ 1 - \theta \\
+ \theta V_1(\lambda_1) \{ 1 + r V_2(\lambda_1) - r \theta_1 [1 - V_2(\lambda_1)] - r V_2(\lambda_1 \theta_1) \} \},\]

\[h'_1(1) = \lambda_2 \alpha E[I] \left\{ \{ 1 - \theta + (1 - r) \theta V_1(\lambda_1) + r \theta V_1(\lambda_1) V_2(\lambda_1) \} + B_2(\lambda_1) \theta \{ V_1(\lambda_1) \\
- r \theta_1 V_1(\lambda_1) [1 - V_2(\lambda_1)] - r V_1(\lambda_1) V_2(\lambda_1 \theta_1) \}\right\} + \lambda_2 \alpha \left\{ E[B_2] \{ \lambda_1 E[I] E[I_1] \\
+ \lambda_2 \alpha b E[I] \} \{ 1 - \theta + (1 - r) \theta V_1(\lambda_1) + r \theta V_1(\lambda_1) V_2(\lambda_1) \} \} + \{ - (1 - r) \\
\theta V_1'(\lambda_1) \lambda_2 \alpha E[I] - r \theta \{ V_1'(\lambda_1) \lambda_2 \alpha E[I] V_2(\lambda_1) + V_1(\lambda_1) \lambda_2 \alpha E[I] V_2'(\lambda_1) \}\right\}\right\} \]
\[-B_2'(\lambda_1)\lambda_2\alpha E[I]E[I_1] - B_2(\lambda_1)\lambda_2\alpha E[I] \{ r \{ V_1(\lambda_1) - \theta V_1(\lambda_1)[1 - V_2(\lambda_1)] - r V_1(\lambda_1) V_2(\lambda_1) \} \}

+ \theta B_2(\lambda_1) \{ -r V_1(\lambda_1) \lambda_2\alpha E[I] + r \lambda_1 \theta V_1(\lambda_1) \lambda_2\alpha E[I] + V_1(\lambda_1) E[I] E[I_1] [1 - V_2(\lambda_1)] - \frac{r \theta V_1(\lambda_1)}{\lambda_1} \lambda_2\alpha E[I] \{ \lambda_2 V_2(\lambda_1) + [1 - V_2(\lambda_1)] \} - E[V_1] \{ \lambda_1 E[I] E[I_1] + \lambda_2\alpha bE[I] \} \} + r \{ V_1(\lambda_1) \lambda_2\alpha E[I] V_2(\lambda_1) + V_1(\lambda_1) V_2(\lambda_1) \} \}

\{ \lambda_1 (1 - \theta) E[I] E[I_1] + \lambda_2\alpha bE[I] \} \} - (\lambda_1 + \lambda_2\alpha) \left\{ -\theta V_1'(\lambda_1)

\lambda_2\alpha bE[I] \{ 1 + r V_2(\lambda_1) - r \theta_1 [1 - V_2(\lambda_1)] - r V_2(\lambda_1) \} \right\} + \theta V_1(\lambda_1) \{ -r V_2(\lambda_1) \lambda_2\alpha bE[I] + r V_2'(\lambda_1) \lambda_2\alpha bE[I] + B_2(\lambda_1) \} + \theta_1 E[I] E[I_1] [1 - V_2(\lambda_1)] - \frac{r \theta_1}{\lambda_1} \lambda_2\alpha bE[I] \{ \lambda_2 V_2(\lambda_1) + [1 - V_2(\lambda_1)] \} - \theta E[V_1] \{ \lambda_1 E[I] E[I_1] + \lambda_2\alpha bE[I] \} \}

\begin{align*}
g_1(1) &= r \theta \{ V_1(\lambda_1) - \theta V_1(\lambda_1)[1 - V_2(\lambda_1)] - V_1(\lambda_1) V_2(\lambda_1) \} \{ 1 - B_2(\lambda_1) \} \\
&= \{ M(\lambda_1 + \lambda_2\alpha) + \lambda_2\alpha \left[ 1 - \frac{M(\lambda_1 + \lambda_2\alpha)}{\lambda_1 + \lambda_2\alpha} \right] \} \}

\begin{align*}
g'_1(1) &= 1 - \theta + \theta V_1(\lambda_1) \{ 1 + r V_2(\lambda_1) - r \theta_1 [1 - V_2(\lambda_1)] - r V_2(\lambda_1) \} - \lambda_1 \\
&= \left[ 1 - \frac{M(\lambda_1 + \lambda_2\alpha)}{\lambda_1 + \lambda_2\alpha} \right] \{ 1 - \theta \} - \theta V_1(\lambda_1) \lambda_2\alpha bE[I] \{ 1 + r V_2(\lambda_1) - r \theta_1 [1 - V_2(\lambda_1)] - r V_2(\lambda_1) \} + \theta V_1(\lambda_1) r \\
&= \left[ 1 - \frac{M(\lambda_1 + \lambda_2\alpha)}{\lambda_1 + \lambda_2\alpha} \right] \{ 1 - \theta \} + r \theta V_1(\lambda_1) \lambda_2\alpha bE[I] E[I_1] [1 - V_2(\lambda_1)] - \frac{\theta V_1(\lambda_1)}{\lambda_1} \lambda_2\alpha bE[I] \{ \lambda_2 V_2(\lambda_1) + [1 - V_2(\lambda_1)] \} + \theta E[V_1] \}

\{ \lambda_1 E[I] E[I_1] + \lambda_2\alpha bE[I] \} - \lambda_1 E[I] E[I_1] \left[ 1 - \frac{M(\lambda_1 + \lambda_2\alpha)}{\lambda_1 + \lambda_2\alpha} \right] \{ 1 - \theta \}

+(1 - r) \theta V_1(\lambda_1) + \theta V_1(\lambda_1) V_2(\lambda_1) \} - \lambda_1 \left[ 1 - \frac{M(\lambda_1 + \lambda_2\alpha)}{\lambda_1 + \lambda_2\alpha} \right] \{ (1 - r) \theta V_1(\lambda_1) + r \theta V_1(\lambda_1) \} \}

\begin{align*}
&= \theta V_1'(\lambda_1) \lambda_2\alpha bE[I] - \theta \{ V_1(\lambda_1) \lambda_2\alpha bE[I] V_2(\lambda_1) + V_1(\lambda_1) \lambda_2\alpha bE[I] V_2(\lambda_1) \} \\
&= \lambda_2\alpha bE[I] \left[ 1 - \frac{M(\lambda_1 + \lambda_2\alpha)}{\lambda_1 + \lambda_2\alpha} \right] \{ 1 - \theta \} + (1 - r) \theta V_1(\lambda_1) + r \theta V_1(\lambda_1) V_2(\lambda_1) \}

+ \theta V_1'(\lambda_1) \{ 1 + r V_1(\lambda_1) - r \theta V_1(\lambda_1) \} \lambda_1 \left[ 1 - \frac{M(\lambda_1 + \lambda_2\alpha)}{\lambda_1 + \lambda_2\alpha} \right] - 1 - r \theta V_1(\lambda_1) \}

\begin{align*}
&= \theta V_2(\lambda_1) \} - \{ M(\lambda_1 + \lambda_2\alpha) + \lambda_2\alpha \left[ 1 - \frac{M(\lambda_1 + \lambda_2\alpha)}{\lambda_1 + \lambda_2\alpha} \right] \} \}

E[I_1] + \lambda_2\alpha bE[I] \{ 1 - \theta \} + (1 - r) \theta V_1(\lambda_1) + r \theta V_1(\lambda_1) V_2(\lambda_1) \}

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\[ h_2(1) = \lambda_2 \alpha \left\{ 1 - \theta V_1(\lambda_1) \{ 1 + r V_2(\lambda_1) - r \theta [1 - V_2(\lambda_1)] - r V_2(\lambda_1, \theta_1) \} \right\} \]

\[ \left\{ M(\lambda_1 + \lambda_2 \alpha) + \lambda_2 \alpha \left[ \frac{1 - M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha} \right] \right\} - \{ 1 - \theta + (1 - r) \theta V_1(\lambda_1) \right\} \]

\[ + r \theta V_1(\lambda_1) V_2(\lambda_1) \left\{ \lambda_1 \left[ \frac{1 - M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha} \right] \right\}. \]

\[ h_2'(1) = \lambda_2 \alpha \left\{ 1 - \theta V_1(\lambda_1) \{ 1 + r V_2(\lambda_1) - r \theta [1 - V_2(\lambda_1)] - r V_2(\lambda_1, \theta_1) \} \right\} \]

\[ \left\{ M(\lambda_1 + \lambda_2 \alpha) + \lambda_2 \alpha \left[ \frac{1 - M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha} \right] \right\} - \{ 1 - \theta + (1 - r) \theta V_1(\lambda_1) \right\} \]

\[ + r \theta V_1(\lambda_1) V_2(\lambda_1) \left\{ \lambda_1 \left[ \frac{1 - M(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha} \right] \right\}. \]
8.6 The average waiting time in the queue:

Average waiting time of a customer in the high priority queue and the low priority queue(orbit) is

\[ W_{q1} = \frac{L_{q1}}{\lambda_1}, \quad (8.6.1) \]
\[ W_{q2} = \frac{L_{q2}}{\lambda_2}, \quad (8.6.2) \]

where \( L_{q1} \) and \( L_{q2} \) are given in equations (8.5.3) and (8.5.4).

8.7 Particular cases:

Case: I

If there is no vacation, no feedback and no balking that is \( \theta = 0, p_1 = 0 \) and \( b = 1 \), then our model reduces to \( M^{[x_1]}, M^{[x_2]}/G_1, G_2/1 \) retrial queueing system with priority service.

\[ I_0(z) = \left\{ \frac{\lambda_2 \alpha C(z)z_0 I_0 B_2(A(z)) - z_2(\lambda_1 + \lambda_2 \alpha)z_0 I_0}{(\alpha + \lambda_2 \alpha)} \right\} (1 - \bar{M}(\lambda_1 + \lambda_2 \alpha)) \]

\[ P(z) = \frac{-A(z)I_0(0, z_2)\left[1 - \frac{\bar{M}(\lambda_1 + \lambda_2 \alpha)}{\lambda_1 + \lambda_2 \alpha}\right] + q_0(z_2)\{z_2 - B_2(A(z))\}}{1 - B_1(A(z))}, \]

\[ q(z) = \left\{ \frac{\alpha \lambda_2 C(z)z_0 I_0}{A(z)\{z_1 - B_1(A(z))\}} \right\} \]

In this case if there is no retrial that is \( \bar{M}(\lambda_1 + \lambda_2 \alpha) = 1 \) then our model is reduces to \( M^{[x_1]}, M^{[x_2]}/G_1, G_2/1 \) with priority services it is given by Chaudhry, M. L. and Temleton, J. G. C. (1983).
Case: II
If there is no priority arrival, all the arriving customers allow to the system, no customers balking the orbit, no feedback and no vacation that is \( \lambda_1 = 0, \theta = 0, \alpha = 1, p_1 = 0, b = 1 \), then our model reduces to \( M^{[x]} / G / 1 \) retrial queueing system.

\[
I_0(z) = I_{0,0}\left\{ C(z_2)B_2(A(z)) - z_2 \right\} \frac{1 - M(\lambda_2 \alpha)}{\{M(\lambda_2 \alpha) + C(z_2)[1 - M(\lambda_2)]\}B_2(A(z)) - z_2}
\]

\[
q(z) = I_{0,0}\frac{1 - B_2(A(z))}{B_2(A(z)) - z_2}
\]

In this case if \( C(z) = z \), then this model is coincide with Comez corral (1999).

Case: III
If there is no priority arrival, no retrial, no balking and no vacation and all the arriving customers allow to the system that is \( \lambda_1 = 0, \theta = 0, \alpha = 1, b = 1 \) and \( \bar{M}(\lambda_2) = 1 \), then our model reduces to \( M^{[x]} / G / 1 \) queueing system.

\[
q(z) = I_{0,0}\frac{1 - B_2(A(z))}{B_2(A(z)) - z_2}
\]

The above result is coincide with Medhi-[1994].

8.8 Numerical Results

The above queueing model is analysed numerically with the following assumptions.

We consider the single arrival and service time for both high-priority and low-priority customers follows exponential that is, \( C(z_1) = z_1, C(z_2) = z_2 \), \( E(I_1) = \frac{\lambda_2}{\mu - \lambda_1} \), \( E(I_1[I_1 - 1]) = \left( \frac{2\lambda_2\mu}{(\mu - \lambda_1)^3} \right) \). We assume arbitrary values to the parameters such that the stability condition is satisfied. MATLAB software has been used to illustrate the results numerically.

In table 8.1 shows that increasing the arrival rate of high priority customers decrease the idle time and also increase the busy period and queue lengths for the values of \( \lambda_1 = 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 3.0 \), \( \lambda_2 = 3, \mu = 15, \alpha = 0.2, \gamma_1 = 10, \gamma_2 = 15, r = 0.2 \theta = 0.5, \theta_1 = 0.4, v = 0.5, b = 0.4, p_1 = 0.3, q_1 = 0.7 \).

In table 8.2 shows that increasing the non-priority arrival rate decrease the idle time and
queue lengths of high priority and also increase the busy period and queue lengths of low-priority customers for the values of $\lambda_1 = 4$, $\lambda_2 = 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0$, $\mu = 14, \alpha = 0.5, \gamma_1 = 10, \gamma_2 = 15, \rho = 0.5 \theta_1 = 0.5, \nu = 0.5, b = 0.5, p_1 = 0.5, q_1 = 0.5$

All the trends shown by this tables and the graphs are as expected.

Results are presented for the values of $\lambda_1$ and $\lambda_2$ in the following tables with their corresponding graphical representations of the system performance measures.

Table 8.1: Effect of $\lambda_1$ on various queue characteristics

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$I_{0,0}$</th>
<th>$\rho$</th>
<th>$L_{q1}$</th>
<th>$L_{q2}$</th>
<th>$W_{q1}$</th>
<th>$W_{q2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>0.0385</td>
<td>0.9615</td>
<td>3.8921</td>
<td>1.6175</td>
<td>1.9460</td>
<td>0.5392</td>
</tr>
<tr>
<td>2.1</td>
<td>0.0370</td>
<td>0.9630</td>
<td>4.1230</td>
<td>2.0128</td>
<td>1.9634</td>
<td>0.6709</td>
</tr>
<tr>
<td>2.2</td>
<td>0.0356</td>
<td>0.9644</td>
<td>4.3519</td>
<td>2.3992</td>
<td>1.9781</td>
<td>0.7997</td>
</tr>
<tr>
<td>2.3</td>
<td>0.0343</td>
<td>0.9657</td>
<td>4.5780</td>
<td>2.7746</td>
<td>1.9904</td>
<td>0.9249</td>
</tr>
<tr>
<td>2.4</td>
<td>0.0330</td>
<td>0.9670</td>
<td>4.8004</td>
<td>3.1382</td>
<td>2.0002</td>
<td>1.0461</td>
</tr>
<tr>
<td>2.5</td>
<td>0.0319</td>
<td>0.9681</td>
<td>5.0184</td>
<td>3.4893</td>
<td>2.0074</td>
<td>1.1631</td>
</tr>
<tr>
<td>2.6</td>
<td>0.0307</td>
<td>0.9693</td>
<td>5.2313</td>
<td>3.8280</td>
<td>2.0120</td>
<td>1.2760</td>
</tr>
<tr>
<td>2.7</td>
<td>0.0297</td>
<td>0.9703</td>
<td>5.4383</td>
<td>4.1545</td>
<td>2.0142</td>
<td>1.3848</td>
</tr>
<tr>
<td>2.8</td>
<td>0.0287</td>
<td>0.9713</td>
<td>5.6385</td>
<td>4.4691</td>
<td>2.0137</td>
<td>1.4897</td>
</tr>
<tr>
<td>2.9</td>
<td>0.0277</td>
<td>0.9723</td>
<td>5.8312</td>
<td>4.7722</td>
<td>2.0108</td>
<td>1.5907</td>
</tr>
<tr>
<td>3.0</td>
<td>0.0268</td>
<td>0.9732</td>
<td>6.0156</td>
<td>5.0644</td>
<td>2.0052</td>
<td>1.6881</td>
</tr>
</tbody>
</table>

Figure 8.1: Average queue sizes verses priority arrival rate $\lambda_1$
Table 8.2: Effect of $\lambda_2$ on various queue characteristics

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$I_{0,0}$</th>
<th>$\rho$</th>
<th>$L_{q_1}$</th>
<th>$L_{q_2}$</th>
<th>$W_{q_1}$</th>
<th>$W_{q_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0323</td>
<td>0.9677</td>
<td>0.5197</td>
<td>0.3105</td>
<td>0.1299</td>
<td>0.3105</td>
</tr>
<tr>
<td>1.1</td>
<td>0.0322</td>
<td>0.9678</td>
<td>0.5133</td>
<td>0.4095</td>
<td>0.1283</td>
<td>0.3723</td>
</tr>
<tr>
<td>1.2</td>
<td>0.0320</td>
<td>0.9680</td>
<td>0.5071</td>
<td>0.5034</td>
<td>0.1268</td>
<td>0.4195</td>
</tr>
<tr>
<td>1.3</td>
<td>0.0319</td>
<td>0.9681</td>
<td>0.5009</td>
<td>0.5922</td>
<td>0.1252</td>
<td>0.4556</td>
</tr>
<tr>
<td>1.4</td>
<td>0.0318</td>
<td>0.9682</td>
<td>0.4948</td>
<td>0.6762</td>
<td>0.1237</td>
<td>0.4830</td>
</tr>
<tr>
<td>1.5</td>
<td>0.0317</td>
<td>0.9683</td>
<td>0.4889</td>
<td>0.7554</td>
<td>0.1222</td>
<td>0.5036</td>
</tr>
<tr>
<td>1.6</td>
<td>0.0316</td>
<td>0.9684</td>
<td>0.4830</td>
<td>0.8300</td>
<td>0.1207</td>
<td>0.5188</td>
</tr>
<tr>
<td>1.7</td>
<td>0.0316</td>
<td>0.9684</td>
<td>0.4772</td>
<td>0.9000</td>
<td>0.1193</td>
<td>0.5294</td>
</tr>
<tr>
<td>1.8</td>
<td>0.0315</td>
<td>0.9685</td>
<td>0.4714</td>
<td>0.9656</td>
<td>0.1179</td>
<td>0.5365</td>
</tr>
<tr>
<td>1.9</td>
<td>0.0314</td>
<td>0.9686</td>
<td>0.4658</td>
<td>1.0269</td>
<td>0.1164</td>
<td>0.5405</td>
</tr>
<tr>
<td>2.0</td>
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<td>0.9687</td>
<td>0.4602</td>
<td>1.0839</td>
<td>0.1151</td>
<td>0.5419</td>
</tr>
</tbody>
</table>

Figure 8.2: Average queue sizes verses non-priority arrival rate $\lambda_2$