7.1 Introduction

A single server deals with the two classes of batch arrival retrial queueing system with general retrial times, priority service, balking, second optional service, Bernoulli vacation, extended vacation, breakdown, delayed repair and stand by server is studied in this chapter. The blocked customers either with probability $p$ join high priority queue or with complementary probability $q$ enter into the retrial group (called orbit). During the interruptions of the main server the stand-by-server serve the customers. The arriving low priority customers may balk the queue(orbit). First service is essential for each customer and second service is optional. After completion of the essential service the

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customer have a option for second service. The main server has four interruptions, there are Bernoulli vacation, extended vacation, delay time and repair time. The retrial time, service time for both high priority and low priority customers, vacation time, extended vacation time, delay time and repair time are all follows general(arbitrary) distribution, breakdown time and service time of stand by server follows exponential distribution. Finally, we derived the probability generating function of service time, vacation time, extended vacation time, delay time and repair time and some important performance measures of the model.

Madan (2000b) has first introduced the concept of second optional service in an M/G/1 queueing system in which he has analyzed the time-dependent as well as the steady state behavior of the model by using supplementary variable technique. Krishnakumar and Arivudainambi (2001) studied M/G/1/1 feedback queue with regular and optional services. Choudhury (2003a) and Medhi (2001) studied an M/G/1 queueing model with second optional channel who developed the explicit expressions for the mean queue length and mean waiting time. Choudhury (2003b) analyzed some aspects of M/G/1 queueing system with second optional service and obtained the steady state queue size distribution at the stationary point of time for general second optional service. Kalyanaraman et al. (2008) studied additional optional batch service with vacation for single server queue.

Madan and Baklizi (2002a) and Choudhury and Paul (2005) studied queue with second optional service by considering feed back also. Krishna Kumar et al. (2002b) investigated an M/G/1 queue with second optional service and server breakdowns such that breakdowns may encounter with a fixed rate while providing service in either phase and also customer departs the system when server breakdown. Wang (2004) studied both transient and steady behavior of an M/G/1 queueing system with second optional service and server breakdowns based on supplementary variable technique. Atencia, I. Moreno, P. (2005) analyzed a single-server retrial queue with general retrial times and Bernoulli schedule, Chesoong Kim et al. (2016) studied a priority tandem queueing system with retrials and reservation of channels as a model of call center, Yang, T. Templeton, J.G.C. (1987) discussed a survey on retrial queue, Yang, T.
al. (1994) have studied an approximation method for the $M/G/1$ retrial queue with
general retrial times, Al-Jaraha,J and Madan.K. (2003) discussed an $M/G/1$ queue
with second optional service with general service time distribution. Ayyappan et al.
(2010) have studied a single server retrial queueing system with stand by server under
pre-emptive priority service, Vishwanath Maurya (2013) analyzed the maximum entropy
of batch arrival two phase retrial queueing system with second phase optional service
and Bernoulli vacations. Ayyappan et al. (2014) discussed an $M^{[X]}/G/1$ retrial queueing
system with second optional service, random breakdown, set up time and Bernoulli
vacation.

The rest of the chapter is organized as follows: Mathematical description of our model
in section (7.2). Equations governing our model and the time dependent solution have
been obtained in section (7.3). The corresponding steady state results have been derived
explicitly in section (7.4). Average queue size and the average waiting time are computed
in section (7.5) and (7.6). Some particular cases have been discussed in section (7.7).

7.2 Mathematical description of our model

(i) Customers arrive at the system in batches of variable size in a compound Pois-
son process. Let $\lambda c_i \, dt (i = 1, 2, 3, \ldots)$ be the first order probability that a batch
of $i$ customers arrives at the system during a short interval of time $(t, t + dt)$,
where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$, and $\lambda > 0$, is the average arrival rate. If the server is
idle upon an arrival, service of the arriving customers commences immediately.
Otherwise, the arriving customers either with probability $p$ joins the priority queue,
where they waits to be served or with complementary probability $q$ joins the retrial
group (called orbit).

(ii) Assume that only the customer at the head of the orbit is allowed to access the
server. If the server is busy upon retrial, the customer join the orbit again. Such
a process is repeated until the retrial customer finds the server idle and gets the
requested service at the time of retrial. Also upon arrival, if the customer finds the
server busy or on interruptions, then the customers join the orbit with probability
$(1-b)$ or balk the orbit with probability $b$. Successive inter retrial times of any
customers follows general(arbitrary) distribution function $A(x)$, density function
\( a(x) \) and the conditional completion rate time for retrials is given by \( \eta(x) = \frac{a(x)}{(1-A(x))} \).

(iii) Each customers are served, by a single server under non-preemptive priority service rule on first come-first served basis. As soon as the customers completes the essential service they have an option for second service with probability \( r_1 \) or may leave the system with probability \((1 - r_1)\). The service time for both essential and second optional service follows general(arbitrary) distributions with distribution functions \( B_i(s) \) and the density functions \( b_i(s) \). \( i = 1, 2 \).

(iv) Let \( \mu_i(x)dx \) be the conditional probability of completion of essential and second optional service(for both high priority and low priority customers) during the interval \((x, x + dx] \), given that the elapsed service time is \( x \), so that

\[
\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)},
\]

and therefore,

\[
b_i(s) = \mu_i(s)e^{-\int_0^s \mu_i(x)dx}.
\]

(v) We further assume that as soon as the completion of each service the server has the option to take a vacation of random length with probability \( \theta \), in which case the vacation starts immediately or else with probability \((1 - \theta)\) he may decide to continue serving the next units present in the system, if any. After completion of a vacation the server has the option to extended the vacation with probability \( r \) or to serve the customer with probability \((1 - r)\).

(vi) The vacation time follows general (arbitrary) distribution with distribution function \( V(s) \) and the density function \( v(s) \). Let \( \gamma(x)dx \) be the conditional probability of completion of vacation during the interval \((x, x + dx] \), given that the elapsed vacation time is \( x \), so that

\[
\gamma(x) = \frac{v(x)}{1 - V(x)},
\]

and therefore,

\[
v(s) = \gamma(s)e^{-\int_0^s \gamma(x)dx}
\]
(vii) The extended vacation time follows general (arbitrary) distribution with distribution function $E(s)$ and the density function $e(s)$. Let $\theta(x)dx$ be the conditional probability of completion of extended vacation during the interval $(x, x + dx]$, given that the elapsed extended vacation time is $x$, so that

$$\theta(x) = \frac{e(x)}{1 - E(x)}$$

and therefore,

$$e(s) = \theta(s)e^{-\int_0^s \theta(x)dx}$$

(viii) On returning from vacation or extended vacation, the server instantly starts serving the customer at the head of the queue, if any, or the server stays in the system for being available.

(ix) The system may break down at random and breakdowns are assumed to occur according to Poisson stream with mean breakdown rate $\alpha > 0$. Whenever the system breaks down, its repairs do not start immediately and there is a delay time to start repair.

(x) The delay time to start repair follows general (arbitrary) distribution with distribution function $D(s)$ and the density function $d(s)$. Let $\xi(x)dx$ be the conditional probability of completion of delay time during the interval $(x, x + dx]$, given that the elapsed delay time is $x$, so that

$$\xi(x) = \frac{d(x)}{1 - D(x)}$$

and therefore,

$$d(s) = \xi(s)e^{-\int_0^s \xi(x)dx}.$$ 

(xi) The repair time follows general (arbitrary) distribution with distribution function $R(s)$ and the density function $r(s)$. Let $\beta(x)dx$ be the conditional probability of completion of repair during the interval $(x, x + dx]$, given that the elapsed repair time is $x$, so that

$$\beta(x) = \frac{r(x)}{1 - R(x)}$$
and therefore,
\[ r(s) = \beta(s)e^{-\int_{0}^{s} \beta(x)dx}. \]

(iii) When the main server is on vacation, extended vacation, waiting for repair to start or under repair in all these interruptions there is a stand by server servers the customers until the main server returns. The stand by server’s service time is exponentially distributed with service rate is \( \delta(>0) \) and mean service time is \( \frac{1}{\delta} \).

When the main server rejoins the system after any interruption, the customer being served by the stand by server transferred to the main server to get the remaining service.

### 7.3 Equations governing the system

The forward Kolmogorov differential difference equations for the queue size distributions for the above assumptions:

\[
\frac{d}{dt} P_{0}(t) = -\lambda P_{0}(t) + (1 - r_{1})(1 - \theta) \int_{0}^{\infty} P_{0,0}^{(1)}(x,t)\mu_{1}(x)dx + (1 - \theta) \int_{0}^{\infty} P_{0,0}^{(2)}(x,t)\mu_{2}(x)dx + (1 - r) \int_{0}^{\infty} V_{0,0}(x,t)\gamma(x)dx
\]

\[
+ \int_{0}^{\infty} E_{0,0}(x,t)\theta(x)dx, \quad (7.3.1)
\]

\[
\frac{\partial}{\partial t} P_{0,n}^{(0)}(x,t) + \frac{\partial}{\partial x} P_{0,n}^{(0)}(x,t) = -(\lambda + a(x))P_{0,n}^{(0)}(x,t); n \geq 1, \quad (7.3.2)
\]

\[
\frac{\partial}{\partial t} P_{m,n}^{(1)}(x,t) + \frac{\partial}{\partial x} P_{m,n}^{(1)}(x,t) = -(\lambda + \mu_{1}(x) + \alpha)P_{m,n}^{(1)}(x,t)
+ \int_{0}^{\infty} R_{m,n}(x,y,t)\beta(y)dy + (1 - \delta_{m0})\lambda p \sum_{i=1}^{m} C_{i}P_{m-i,n}^{(1)}(x,t)
+ (1 - \delta_{0n})\lambda qb \sum_{i=1}^{n} C_{i}P_{m,n-i}^{(1)}(x,t) + \lambda q(1 - b)P_{m,n}^{(1)}(x,t); m, n \geq 0, \quad (7.3.3)
\]

\[
\frac{\partial}{\partial t} P_{m,n}^{(2)}(x,t) + \frac{\partial}{\partial x} P_{m,n}^{(2)}(x,t) = -(\lambda + \mu_{2}(x) + \alpha)P_{m,n}^{(2)}(x,t)
+ \int_{0}^{\infty} R_{m,n}(x,y,t)\beta(y)dy + (1 - \delta_{m0})\lambda p \sum_{i=1}^{m} C_{i}P_{m-i,n}^{(2)}(x,t)
+ (1 - \delta_{0n})\lambda qb \sum_{i=1}^{n} C_{i}P_{m,n-i}^{(2)}(x,t) + \lambda q(1 - b)P_{m,n}^{(2)}(x,t); m, n \geq 0, \quad (7.3.4)
\]

\[
\frac{\partial}{\partial t} V_{m,n}(x,t) + \frac{\partial}{\partial x} V_{m,n}(x,t) = -(\lambda + \gamma(x) + \delta) V_{m,n}(x,t) + p \delta V_{m+1,n}(x,t)
\]

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\[ + \delta_{m0} \delta V_{m,n+1}(x,t) + (1 - \delta_{m0}) \lambda p \sum_{i=1}^{m} C_i V_{m-i,n}(x,t) \]
\[ + (1 - \delta_{mn}) \lambda q b \sum_{i=1}^{n} C_i V_{m,n-i}(x,t) + \lambda q (1 - b) V_{m,n}(x,t); m, n \geq 0, \quad (7.3.5) \]
\[ \frac{\partial}{\partial t} E_{m,n}(x,t) + \frac{\partial}{\partial x} E_{m,n}(x,t) = - (\lambda + \theta(x) + \delta) E_{m,n}(x,t) + p \delta E_{m+1,n}(x,t) \]
\[ + \delta_{m0} \delta E_{m,n+1}(x,t) + (1 - \delta_{m0}) \lambda p \sum_{i=1}^{m} C_i E_{m-i,n}(x,t) \]
\[ + (1 - \delta_{mn}) \lambda q b \sum_{i=1}^{n} C_i E_{m,n-i}(x,t) + \lambda q (1 - b) E_{m,n}(x,t); m, n \geq 0, \quad (7.3.6) \]
\[ \frac{\partial}{\partial t} D_{m,n}(x,y,t) + \frac{\partial}{\partial y} D_{m,n}(x,y,t) = - (\lambda + \xi(y) + \delta) D_{m,n}(x,y,t) \]
\[ + p \delta D_{m+1,n}(x,y,t) + \delta_{m0} \delta D_{m,n+1}(x,y,t) \]
\[ + (1 - \delta_{m0}) \lambda p \sum_{i=1}^{m} C_i D_{m-i,n}(x,y,t) + (1 - \delta_{mn}) \lambda q b \sum_{i=1}^{n} C_i D_{m,n-i}(x,y,t) \]
\[ + \lambda q (1 - b) D_{m,n}(x,y,t); m, n \geq 0, \quad (7.3.7) \]
\[ \frac{\partial}{\partial t} R_{m,n}(x,y,t) + \frac{\partial}{\partial y} R_{m,n}(x,y,t) = - (\lambda + \beta(y) + \delta) R_{m,n}(x,y,t) \]
\[ + p \delta R_{m+1,n}(x,y,t) + \delta_{m0} \delta R_{m,n+1}(x,y,t) \]
\[ + (1 - \delta_{m0}) \lambda p \sum_{i=1}^{m} C_i R_{m-i,n}(x,y,t) + (1 - \delta_{mn}) \lambda q b \sum_{i=1}^{n} C_i R_{m,n-i}(x,y,t) \]
\[ + \lambda q (1 - b) R_{m,n}(x,y,t); m, n \geq 0. \quad (7.3.8) \]

The above set of equations are to be solved under the following boundary conditions at initially we take \( x = 0 \).
\[ P_{0,n}^{(0)}(0,t) = (1 - \theta)(1 - r_1) \int_{0}^{\infty} P_{0,n}^{(1)}(x,t) \mu_1(x) dx + (1 - \theta) \int_{0}^{\infty} P_{0,n}^{(2)}(x,t) \mu_2(x) dx \]
\[ + (1 - r) \int_{0}^{\infty} V_{0,n}(x,t) \gamma(x) dx + \int_{0}^{\infty} E_{0,n}(x,t) \theta(x) dx; \quad n \geq 1, \quad (7.3.9) \]
\[ P_{m,n}^{(1)}(0,t) = \delta_{m0} \lambda p \sum_{i=1}^{m} C_i P_{0}(t) + \delta_{m0} \lambda q \sum_{i=1}^{n} C_{i+1} P_{0}(t) \]
\[ + \delta_{m0} (1 - \delta_{mn}) \sum_{i=1}^{n} \lambda C_i \int_{0}^{\infty} P_{0,n+1-i}(x,t) dx + \delta_{m0} \int_{0}^{\infty} P_{0,n+1}(x,t) \alpha(x) dx \]
\[ + (1 - r_1)(1 - \theta) \int_{0}^{\infty} P_{m+1,n}^{(1)}(x,t) \mu_1(x) dx \]
\[ + (1 - \theta) \int_{0}^{\infty} P_{m+1,n}^{(2)}(x,t) \mu_2(x) dx + (1 - r) \int_{0}^{\infty} V_{m+1,n}(x,t) \gamma(x) dx \]
\[ + \int_{0}^{\infty} E_{m+1,n}(x,t) \theta(x) dx; \phantom{1} m, n \geq 0, \quad (7.3.10) \]
\[ P_{m,n}^{(2)}(0,t) = r_1 \int_{0}^{\infty} P_{m,n}^{(1)}(x,t) \mu_1(x) dx; \phantom{1} m, n \geq 0, \quad (7.3.11) \]
\[ V_{m,n}(0,t) = \theta (1-r_1) \int_0^\infty P_{m,n}^{(1)}(x,t)\mu_1(x)dx + \theta \int_0^\infty P_{m,n}^{(2)}(x,t)\mu_2(x)dx; \quad m, n \geq 0, \]  
\[ (s-1)\bar{P}_0(s) = -\lambda \bar{P}_0(s) + (1-r_1)(1-\theta) \int_0^\infty \bar{P}_{0,0}^{(1)}(x,s)\mu_1(x)dx + (1-\theta) \int_0^\infty \bar{P}_{0,0}^{(2)}(x,s)\mu_2(x)dx + (1-r) \int_0^\infty \bar{V}_{0,0}(x,s)\gamma(x)dx + \int_0^\infty \bar{E}_{0,0}(x,s)\theta(x)dx, \]
\[
\frac{\partial}{\partial x} \mathcal{P}^{(0)}_{m,n}(x,s) + (s + \lambda + a(x))\mathcal{P}^{(0)}_{m,n}(x,s) = 0; \quad n \geq 1, \quad (7.3.20)
\]
\[
\frac{\partial}{\partial x} \mathcal{P}^{(1)}_{m,n}(x,s) + (s + \lambda + \mu_1(x) + \alpha)\mathcal{P}^{(1)}_{m,n}(x,s) = \int_0^\infty \mathcal{R}_{m,n}(x,y,s)\beta(y)dy \\
+ (1 - \delta_{m0})\lambda p \sum_{i=1}^m C_i\mathcal{P}^{(1)}_{m-i,n}(x,s) + (1 - \delta_{m0})\lambda q b \sum_{i=1}^n C_i\mathcal{P}^{(1)}_{m-n-i}(x,s) \\
+ \lambda q(1-b)\mathcal{P}^{(1)}_{m,n}(x,s); m, n \geq 0, \quad (7.3.21)
\]
\[
\frac{\partial}{\partial x} \mathcal{P}^{(2)}_{m,n}(x,s) + (s + \lambda + \mu_2(x) + \alpha)\mathcal{P}^{(2)}_{m,n}(x,s) = \int_0^\infty \mathcal{R}_{m,n}(x,y,s)\beta(y)dy \\
+ (1 - \delta_{m0})\lambda p \sum_{i=1}^m C_i\mathcal{P}^{(2)}_{m-i,n}(x,s) + (1 - \delta_{m0})\lambda q b \sum_{i=1}^n C_i\mathcal{P}^{(2)}_{m-n-i}(x,s) \\
+ \lambda q(1-b)\mathcal{P}^{(2)}_{m,n}(x,s); m, n \geq 0, \quad (7.3.22)
\]
\[
\frac{\partial}{\partial x} \mathcal{V}_{m,n}(x,s) + (s + \lambda + \gamma(x) + \delta)\mathcal{V}_{m,n}(x,s) = p\delta\mathcal{V}_{m+1,n}(x,s) \\
+ \delta_{m0}\delta\mathcal{V}_{m,n+1}(x,s) + (1 - \delta_{m0})\lambda p \sum_{i=1}^m C_i\mathcal{V}_{m-i,n}(x,s) \\
+ (1 - \delta_{m0})\lambda q b \sum_{i=1}^n C_i\mathcal{V}_{m-n-i}(x,s) + \lambda q(1-b)\mathcal{V}_{m,n}(x,s); m, n \geq 0, \quad (7.3.23)
\]
\[
\frac{\partial}{\partial x} \mathcal{E}_{m,n}(x,s) + (s + \lambda + \theta(x) + \delta)\mathcal{E}_{m,n}(x,s) = p\delta\mathcal{E}_{m+1,n}(x,s) \\
+ \delta_{m0}\delta\mathcal{E}_{m,n+1}(x,s) + (1 - \delta_{m0})\lambda p \sum_{i=1}^m C_i\mathcal{E}_{m-i,n}(x,s) \\
+ (1 - \delta_{m0})\lambda q b \sum_{i=1}^n C_i\mathcal{E}_{m-n-i}(x,s) + \lambda q(1-b)\mathcal{E}_{m,n}(x,s); m, n \geq 0, \quad (7.3.24)
\]
\[
\frac{\partial}{\partial y} \mathcal{D}_{m,n}(x,y,s) + (s + \lambda + \xi(y) + \delta)\mathcal{D}_{m,n}(x,y,s) = p\delta\mathcal{D}_{m+1,n}(x,y,s) \\
+ \delta_{m0}\delta\mathcal{D}_{m,n+1}(x,y,s) + (1 - \delta_{m0})\lambda p \sum_{i=1}^m C_i\mathcal{D}_{m-i,n}(x,y,s) \\
+ (1 - \delta_{m0})\lambda q b \sum_{i=1}^n C_i\mathcal{D}_{m-n-i}(x,y,s) + \lambda q(1-b)\mathcal{D}_{m,n}(x,y,s); m, n \geq 0, \quad (7.3.25)
\]
\[
\frac{\partial}{\partial y} \mathcal{R}_{m,n}(x,y,s) + (s + \lambda + \beta(y) + \delta)\mathcal{R}_{m,n}(x,y,s) = p\delta\mathcal{R}_{m+1,n}(x,y,s) \\
+ \delta_{m0}\delta\mathcal{R}_{m,n+1}(x,y,s) + (1 - \delta_{m0})\lambda p \sum_{i=1}^m C_i\mathcal{R}_{m-i,n}(x,y,s) \\
+ (1 - \delta_{m0})\lambda q b \sum_{i=1}^n C_i\mathcal{R}_{m-n-i}(x,y,s) + \lambda q(1-b)\mathcal{R}_{m,n}(x,y,s); m, n \geq 0, \quad (7.3.26)
\]
\[ \mathcal{P}^{(0)}_{0,n}(0,s) = (1 - \theta)(1 - r_1) \int_0^\infty \mathcal{P}^{(1)}_{0,n}(x,s) \mu_1(x)dx + (1 - \theta) \int_0^\infty \mathcal{P}^{(2)}_{0,n}(x,s) \mu_2(x)dx \\
+ (1 - r) \int_0^\infty V_{0,n}(x,s)e(x)dx + \int_0^\infty E_{0,n}(x,s)\theta(x)dx; n \geq 1, \quad (7.3.27) \]
\[ \mathcal{P}^{(1)}_{m,n}(0,s) = \delta_m \lambda p \sum_{i=1}^m C_{i+1} \mathcal{P}_0(0,s) + \delta_m q \sum_{i=1}^n C_{i+1} \mathcal{P}_0(s) \\
+ \delta_m (1 - \delta_{m0}) \sum_{i=1}^n \lambda C_i \int_0^\infty \mathcal{P}^{(0)}_{0,n+1-i}(x,s)dx \\
+ \delta_m \int_0^\infty \mathcal{P}^{(0)}_{0,n+1}(x,s)a(x)dx + (1 - r_1)(1 - \theta) \int_0^\infty \mathcal{P}^{(1)}_{m+1,n}(x,s)\mu_1(x)dx \\
+ (1 - \theta) \int_0^\infty \mathcal{P}^{(2)}_{m+1,n}(x,s)\mu_2(x)dx + (1 - r) \int_0^\infty V_{m+1,n}(x,s)\gamma(x)dx \\
+ \int_0^\infty E_{m+1,n}(x,s)\theta(x)dx; m, n \geq 0, \quad (7.3.28) \]
\[ \mathcal{P}^{2}_{m,n}(0,s) = r_1 \int_0^\infty \mathcal{P}^{(1)}_{m,n}(x,s)\mu_1(x)dx; m, n \geq 0, \quad (7.3.29) \]
\[ \mathcal{V}_{m,n}(0,s) = \theta (1 - r_1) \int_0^\infty \mathcal{P}^{(1)}_{m,n}(x,s)\mu_1(x)dx \\
+ \theta \int_0^\infty \mathcal{P}^{(2)}_{m,n}(x,s)\mu_2(x)dx; m, n \geq 0, \quad (7.3.30) \]
\[ \mathcal{E}_{m,n}(0,s) = r \int_0^\infty \mathcal{V}_{m,n}(x,s)\gamma(x)dx; m, n \geq 0, \quad (7.3.31) \]
\[ \mathcal{D}_{m,n}(x,0,s) = \alpha \mathcal{P}^{(1)}_{m,n}(x,s) + \alpha \mathcal{P}^{(2)}_{m,n}(x,s); m, n \geq 0, \quad (7.3.32) \]
\[ \mathcal{R}_{m,n}(x,0,s) = \int_0^\infty \mathcal{D}_{m,n}(x,y,s)\xi(y)dy; m, n \geq 0. \quad (7.3.33) \]

Now we multiply equations (7.3.21) to (7.3.26) by \( z_1^n \) summing over \( m \) from 0 to \( \infty \),

we get
\[ \frac{\partial}{\partial x} \mathcal{P}^{(1)}_n(x,z_1,s) + (s + \lambda + \mu_1(x) + \alpha) \mathcal{P}^{(1)}_n(x,z_1,s) = \int_0^\infty \mathcal{R}_n(x,y,z_1,s)\beta(y)dy \\
+ \lambda pC(z_1) \mathcal{P}^{(1)}_n(x,z_1,s) + (1 - \delta_{m0}) \lambda q \sum_{i=1}^n C_i \mathcal{P}^{(1)}_{n-i}(x,z_1,s) \\
+ \lambda q(1 - b) \mathcal{P}^{(1)}_n(x,z_1,s); n \geq 0, \quad (7.3.34) \]
\[ \frac{\partial}{\partial x} \mathcal{P}^{(2)}_n(x,z_1,s) + (s + \lambda + \mu_2(x) + \alpha) \mathcal{P}^{(2)}_n(x,z_1,s) = \int_0^\infty \mathcal{R}_n(x,y,z_1,s)\beta(y)dy \\
+ \lambda pC(z_1) \mathcal{P}^{(2)}_n(x,z_1,s) + (1 - \delta_{m0}) \lambda q \sum_{i=1}^n C_i \mathcal{P}^{(2)}_{n-i}(x,z_1,s) \\
+ \lambda q(1 - b) \mathcal{P}^{(2)}_n(x,z_1,s); n \geq 0, \quad (7.3.35) \]
\[ \frac{\partial}{\partial x} \mathcal{V}_n(x,z_1,s) + (s + \lambda + \gamma(x) + \delta) \mathcal{V}_n(x,z_1,s) = \frac{1}{z_1} p \delta \mathcal{V}_n(x,z_1,s) \\
+ q \delta \mathcal{V}_{n+1}(x,z_1,s) + \lambda pC(z_1) \mathcal{V}_n(x,z_1,s) \\
+ (1 - \delta_{m0}) \lambda q \sum_{i=1}^n C_i \mathcal{V}_{n-i}(x,z_1,s) + \lambda q(1 - b) \mathcal{V}_n(x,z_1,s); n \geq 0, \quad (7.3.36) \]

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\[ \frac{\partial}{\partial x} E_n(x, z_1, s) + (s + \lambda + \theta(x) + \delta) E_n(x, z_1, s) = \frac{1}{z_1} p \delta E_n(x, z_1, s) \]
\[ + q \delta E_{n+1}(x, z_1, s) + \lambda p C(z_1) \overline{E}_n(x, z_1, s) \]
\[ + (1 - \delta_n) \lambda q b \sum_{i=1}^{n} C_i E_{n-i}(x, z_1, s) + \lambda q (1-b) \overline{E}_n(x, z_1, s); \ n \geq 0, \]
\[ (7.3.37) \]
\[ \frac{\partial}{\partial y} D_n(x, y, z_1, s) + (s + \lambda + \xi(y) + \delta) D_n(x, y, z_1, s) = \frac{1}{z_1} p \delta D_n(x, y, z_1, s) \]
\[ + q \delta D_{n+1}(x, y, z_1, s) + \lambda p C(z_1) \overline{D}_n(x, y, z_1, s) \]
\[ + (1 - \delta_n) \lambda q b \sum_{i=1}^{n} C_i D_{n-i}(x, y, z_1, s) \]
\[ + \lambda q (1-b) \overline{D}_n(x, y, z_1, s); \ n \geq 0, \]
\[ (7.3.38) \]
\[ \frac{\partial}{\partial y} R_n(x, y, z_1, s) + (s + \lambda + \beta(y) + \delta) R_n(x, y, z_1, s) = \frac{1}{z_1} p \delta R_n(x, y, z_1, s) \]
\[ + q \delta R_{n+1}(x, y, z_1, s) + \lambda p C(z_1) \overline{R}_n(x, y, z_1, s) \]
\[ + (1 - \delta_n) \lambda q b \sum_{i=1}^{n} C_i R_{n-i}(x, y, z_1, s) \]
\[ + \lambda q (1-b) \overline{R}_n(x, y, z_1, s); \ n \geq 0. \]
\[ (7.3.39) \]

Now we multiply equations (7.3.20) and (7.3.34) to (7.3.39) by \( z_n^2 \) summing over \( n \)
\[ (\frac{\partial}{\partial x} + s + \lambda + a(x)) \overline{P}_0(x, z_2, s) = 0, \]
\[ (7.3.40) \]
\[ (\frac{\partial}{\partial x} + a[z, s] + \mu_1(x)) P^{(1)}(x, z_1, z_2, s) = \int_{0}^{\infty} \overline{R}(x, y, z_1, z_2, s) \beta(y) \, dy, \]
\[ (7.3.41) \]
\[ (\frac{\partial}{\partial x} + a[z, s] + \mu_2(x)) P^{(2)}(x, z_1, z_2, s) = \int_{0}^{\infty} \overline{R}(x, y, z_1, z_2, s) \beta(y) \, dy, \]
\[ (7.3.42) \]
\[ (\frac{\partial}{\partial x} + b[z, s] + \gamma(x)) \overline{V}(x, z_1, z_2, s) = 0, \]
\[ (7.3.43) \]
\[ (\frac{\partial}{\partial x} + b[z, s] + \theta(x)) \overline{E}(x, z_1, z_2, s) = 0, \]
\[ (7.3.44) \]
\[ (\frac{\partial}{\partial x} + b[z, s] + \xi(y)) \overline{D}(x, y, z_1, z_2, s) = 0, \]
\[ (7.3.45) \]
\[ (\frac{\partial}{\partial y} + b[z, s] + \beta(y)) \overline{R}(x, y, z_1, z_2, s) = 0. \]
\[ (7.3.46) \]

where
\[ a[z, s] = s + \lambda - \lambda p C(z_1) - \lambda q + \lambda q b(1-C(z_2)), \]
\[ b[z, s] = s + \lambda - \lambda p C(z_1) - \lambda q + \lambda q b(1-C(z_2)) + \delta - \delta \frac{p}{z_1} - \delta \frac{q}{z_2}, \]

Now we multiply equations (7.3.28) to (7.3.33) by \( z_n^m \) summing over \( m \) from 0 to \( \infty \),
we get
\[ P_n^{(1)}(0,z_1,s) = \delta_{n0} \lambda \rho \frac{C(z_1)}{z_1} \mathcal{P}_0(s) + \lambda q \sum_{i=1}^{n} C_{i+1} \mathcal{P}_0(s) \\
+ (1 - \delta_{n0}) \sum_{i=1}^{n} \lambda C_i \int_{0}^{\infty} \mathcal{P}_{0,n+1-i}(x,s) dx + \int_{0}^{\infty} \mathcal{P}_{0,n+1}(x,s) a(x) dx \\
+ \frac{1}{z_1} (1 - r_1)(1 - \theta) \int_{0}^{\infty} \mathcal{P}_n^{(1)}(x,z_1,s) \mu_1(x) dx \\
- \frac{1}{z_1} (1 - r_1)(1 - \theta) \int_{0}^{\infty} \mathcal{P}_0(x,s) \mu_1(x) dx \\
+ \frac{1}{z_1} (1 - \theta) \int_{0}^{\infty} \mathcal{P}_n^{(2)}(x,z_1,s) \mu_2(x) dx \\
- \frac{1}{z_1} (1 - \theta) \int_{0}^{\infty} \mathcal{P}_0(x,s) \mu_2(x) dx + \frac{1}{z_1} (1 - r) \int_{0}^{\infty} \mathcal{V}_n(x,z_1,s) \gamma(x) dx \\
- \frac{1}{z_1} (1 - r) \int_{0}^{\infty} \mathcal{V}_0(x,s) \gamma(x) dx + \frac{1}{z_1} \int_{0}^{\infty} \mathcal{E}_n(x,z_1,s) \theta(x) dx \\
- \frac{1}{z_1} \int_{0}^{\infty} \mathcal{E}_0(x,s) \theta(x) dx; \ n \geq 0, \tag{7.3.47} \]

\[ P_n^2(0,z_1,s) = r_1 \int_{0}^{\infty} \mathcal{P}_n^{(1)}(x,z_1,s) \mu_1(x) dx; \ n \geq 0, \tag{7.3.48} \]

\[ \mathcal{V}_n(0,z_1,s) = \theta (1 - r_1) \int_{0}^{\infty} \mathcal{P}_n^{(1)}(x,z_1,s) \mu_1(x) dx \\
+ \theta \int_{0}^{\infty} \mathcal{P}_n^{(2)}(x,z_1,s) \mu_2(x) dx; \ n \geq 0, \tag{7.3.49} \]

\[ \mathcal{E}_n(0,z_1,s) = r \int_{0}^{\infty} \mathcal{V}_n(x,z_1,s) \gamma(x) dx; \ n \geq 0, \tag{7.3.50} \]

\[ \mathcal{D}_n(x,0,z_1,s) = \alpha \mathcal{P}_n^{(1)}(x,z_1,s) + \alpha \mathcal{P}_n^{(2)}(x,z_1,s); \ n \geq 0, \tag{7.3.51} \]

\[ \mathcal{R}_n(x,0,z_1,s) = \int_{0}^{\infty} \mathcal{D}_n(x,y,z_1,s) \xi(y) dy; \ n \geq 0. \tag{7.3.52} \]

Now we multiply equations (7.3.27) by \( z_2^n \) summing over \( n \) from 1 to \( \infty \) and equations (7.3.47) to (7.3.52) by \( z_2^n \) summing over \( n \) from 0 to \( \infty \), we get

\[ P_0^{(0)}(0,z_2,s) = (1 - \theta)(1 - r_1) \int_{0}^{\infty} \mathcal{P}_0^{(1)}(x,z_2,s) \mu_1(x) dx \\
- (1 - \theta)(1 - r_1) \int_{0}^{\infty} \mathcal{P}_0^{(2)}(x,z_2,s) \mu_2(x) dx \\
+ (1 - \theta) \int_{0}^{\infty} \mathcal{P}_0^{(2)}(x,z_2,s) \mu_2(x) dx - (1 - \theta) \int_{0}^{\infty} \mathcal{P}_0^{(2)}(x,s) \mu_2(x) dx \\
+ (1 - r) \int_{0}^{\infty} \mathcal{V}_0(x,z_2,s) \gamma(x) dx - (1 - r) \int_{0}^{\infty} \mathcal{V}_0(x,s) \gamma(x) dx \\
+ \int_{0}^{\infty} \mathcal{E}_0(x,z_2,s) \theta(x) dx - \int_{0}^{\infty} \mathcal{E}_0(x,s) \theta(x) dx, \tag{7.3.53} \]

substitute (7.3.19) into (7.3.53) we get

\[ P_0^{(0)}(0,z_2,s) = (1 - \theta)(1 - r_1) \int_{0}^{\infty} \mathcal{P}_0^{(1)}(x,z_2,s) \mu_1(x) dx \\
+ (1 - \theta) \int_{0}^{\infty} \mathcal{P}_0^{(2)}(x,z_2,s) \mu_2(x) dx + (1 - r) \int_{0}^{\infty} \mathcal{V}_0(x,z_2,s) \gamma(x) dx \]
\[ + \int_{0}^{\infty} E_0(x, z_2, s) \theta(x) dx + 1 - (s + \lambda) P_0(s), \quad (7.3.54) \]

\[ \overline{P}^{(1)}(0, z_1, z_2, s) = \lambda \mu C(z_1) \overline{P}_0(s) + \lambda \mu C(z_2) \overline{P}_0(s) + \lambda \mu C(z_2) \overline{P}_0(s) + \int_{0}^{\infty} \overline{P}^{(0)}_0(x, z_2, s) dx \]

\[ + \frac{1}{z_2} \int_{0}^{\infty} \overline{P}^{(0)}_0(x, z_2, s) a(x) dx \]

\[ + \frac{1}{z_1} (1 - r_1) (1 - \theta) \int_{0}^{\infty} \overline{P}^{(1)}(x, z_1, z_2, s) \mu_1(x) dx \]

\[ - \frac{1}{z_1} (1 - r_1) (1 - \theta) \int_{0}^{\infty} \overline{P}^{(1)}_0(x, z_2, s) \mu_1(x) dx \]

\[ + \frac{1}{z_1} (1 - \theta) \int_{0}^{\infty} \overline{P}^{(2)}(x, z_1, z_2, s) \mu_2(x) dx \]

\[ - \frac{1}{z_1} (1 - \theta) \int_{0}^{\infty} \overline{P}^{(2)}_0(x, z_2, s) \mu_2(x) dx \]

\[ + \frac{1}{z_1} (1 - r) \int_{0}^{\infty} \overline{V}(x, z_1, z_2, s) \gamma(x) dx \]

\[ - \frac{1}{z_1} (1 - r) \int_{0}^{\infty} \overline{V}_0(x, z_2, s) \gamma(x) dx + \frac{1}{z_1} \int_{0}^{\infty} E(x, z_1, z_2, s) \theta(x) dx \]

\[ - \frac{1}{z_1} \int_{0}^{\infty} E_0(x, z_2, s) \theta(x) dx, \quad (7.3.55) \]

\[ \overline{P}^2(0, z_1, z_2, s) = r_1 \int_{0}^{\infty} \overline{P}^{(1)}(x, z_1, z_2, s) \mu_1(x) dx, \quad (7.3.56) \]

\[ \overline{V}(0, z_1, z_2, s) = \theta (1 - r_1) \int_{0}^{\infty} \overline{P}^{(1)}(x, z_1, z_2, s) \mu_1(x) dx + \theta \int_{0}^{\infty} \overline{P}^{(2)}(x, z_1, z_2, s) \mu_2(x) dx, \quad (7.3.57) \]

\[ \overline{E}(0, z_1, z_2, s) = r \int_{0}^{\infty} \overline{V}(x, z_1, z_2, s) \gamma(x) dx, \quad (7.3.58) \]

\[ \overline{D}(x, 0, z_1, z_2, s) = \alpha \overline{P}^{(1)}(x, z_1, z_2, s) + \alpha \overline{P}^{(2)}(x, z_1, z_2, s), \quad (7.3.59) \]

\[ \overline{R}(x, 0, z_1, z_2, s) = \int_{0}^{\infty} \overline{D}(x, y, z_1, z_2, s) \xi(y) dy, \quad (7.3.60) \]

integrate equations (7.3.40) and (7.3.43) to (7.3.46) between 0 and x we get

\[ \overline{P}^{(0)}_0(x, z_2, s) = \overline{P}^{(0)}_0(0, z_2, s) e^{-s + \lambda} \int_{0}^{x} a(t) dt, \quad (7.3.61) \]

\[ \overline{V}(x, z_1, z_2, s) = \overline{V}(0, z_1, z_2, s) e^{-b[z,s]x} \int_{0}^{x} \gamma(t) dt, \quad (7.3.62) \]

\[ \overline{E}(x, z_1, z_2, s) = \overline{E}(0, z_1, z_2, s) e^{-b[z,s]x} \int_{0}^{x} \theta(t) dt, \quad (7.3.63) \]

\[ \overline{D}(x, y, z_1, z_2, s) = \overline{D}(x, 0, z_1, z_2, s) e^{-b[z,s]y} \int_{0}^{y} \xi(t) dt, \quad (7.3.64) \]

\[ \overline{R}(x, y, z_1, z_2, s) = \overline{R}(x, 0, z_1, z_2, s) e^{-b[z,s]y} \int_{0}^{y} \beta(t) dt, \quad (7.3.65) \]

multiplying equations (7.3.61) to (7.3.65) by a(x), \gamma(x), \theta(x), \xi(y) and \beta(y) respectively,
then integrate we get

\[
\int_0^\infty \mathcal{P}^{(0)}_0(x, z_2, s) a(x) \, dx = \mathcal{P}^{(0)}_0(0, z_2, s) \mathcal{M}(s + \lambda),
\]

(7.3.66)

\[
\int_0^\infty \mathcal{V}(x, z_1, z_2, s) \gamma(x) \, dx = \mathcal{V}(0, z_1, z_2, s) \mathcal{V}(b[z, s]),
\]

(7.3.67)

\[
\int_0^\infty \mathcal{E}(x, z_1, z_2, s) \theta(x) \, dx = \mathcal{E}(0, z_1, z_2, s) \mathcal{E}(b[z, s]),
\]

(7.3.68)

\[
\int_0^\infty \mathcal{D}(x, y, z_1, z_2, s) \xi(y) \, dy = \mathcal{D}(x, 0, z_1, z_2, s) \mathcal{D}(b[z, s]),
\]

(7.3.69)

\[
\int_0^\infty \mathcal{R}(x, y, z_1, z_2, s) \beta(y) \, dy = \mathcal{R}(x, 0, z_1, z_2, s) \mathcal{R}(b[z, s]),
\]

(7.3.70)

at \( z_1 = 0 \), equations (7.3.67) and (7.3.68), becomes

\[
\int_0^\infty \mathcal{V}_0(x, z_2, s) \gamma(x) \, dx = \mathcal{V}_0(0, z_2, s) \mathcal{V}(b_1[z, s]),
\]

(7.3.71)

\[
\int_0^\infty \mathcal{E}_0(x, z_2, s) \theta(x) \, dx = \mathcal{E}_0(0, z_2, s) \mathcal{E}(b_1[z, s]),
\]

(7.3.72)

\[ b_1[z, s] = s + \lambda - \lambda q + \lambda q b(1 - C(z_2)) + \delta - \delta \frac{q}{z_2}, \]

we get from equations (7.3.59) and (7.3.60), we get

\[
\mathcal{R}(x, 0, z_1, z_2, s) = \alpha \mathcal{P}^{(j)}(x, z_1, z_2, s) \mathcal{D}(b[z, s]); \quad j = 1, 2,
\]

(7.3.73)

substitute (7.3.73) into (7.3.70), we get

\[
\int_0^\infty \mathcal{R}(x, y, z_1, z_2, s) \beta(y) \, dy = \alpha \mathcal{P}^{(j)}(x, z_1, z_2, s) \mathcal{D}(b[z, s]) \mathcal{R}(b[z, s]),
\]

(7.3.74)

then (7.3.40) and (7.3.41) becomes

\[
\left\{ \frac{\partial}{\partial x} + a[z, s] + \alpha(1 - \mathcal{D}(b[z, s]) \mathcal{R}(b[z, s])) + \mu_j(x) \right\} \mathcal{P}^{(j)}(x, z_1, z_2, s) = 0; \quad j = 1, 2,
\]

(7.3.75)

then solving (7.3.75) we get

\[
\mathcal{P}^{(j)}(z_1, z_2, s) = \mathcal{P}^{(j)}(0, z_1, z_2, s) e^{-\left(\alpha [z, s] + \alpha(1 - \mathcal{D}(b[z, s]) \mathcal{R}(b[z, s]))\right) x} - \int_0^x \mu_j(t) \, dt; \quad j = 1, 2,
\]

(7.3.76)

\[
\int_0^\infty \mathcal{P}^{(j)}(x, z_1, z_2, s) \mu_j(x) \, dx = \mathcal{P}^{(j)}(0, z_1, z_2, s) \mathcal{B}_j(\phi[z, s]); \quad j = 1, 2,
\]

(7.3.77)

at \( z_1 = 0 \)

\[
\int_0^\infty \mathcal{P}^{(j)}_0(x, z_2, s) \mu_j(x) \, dx = \mathcal{P}^{(j)}_0(0, z_2, s) \mathcal{B}_j(\phi_1[z, s]); \quad j = 1, 2,
\]

(7.3.78)

where

\[
\phi[z, s] = a[z, s] + \alpha(1 - \mathcal{D}(b[z, s]) \mathcal{R}(b[z, s])),
\]

\[
\phi_1[z, s] = a_1[z, s] + \alpha(1 - \mathcal{D}(b_1[z, s]) \mathcal{R}(b_1[z, s])),
\]
\[ a_1[z,s] = s + \lambda - \lambda q + \lambda qb(1 - C(z_2)) \] and

equations (7.3.77) and (7.3.78) gives

\[
\begin{align*}
\int_0^\infty \overline{p}^{(1)}(x,z_1,z_2,s)\mu_1(x)dx &= \overline{p}^{(1)}(0,z_1,z_2,s)B_1(\phi[z,s]), \\
\int_0^\infty \overline{p}^{(2)}(x,z_1,z_2,s)\mu_2(x)dx &= \overline{p}^{(2)}(0,z_1,z_2,s)B_2(\phi[z,s]),
\end{align*}
\] (7.3.79) (7.3.80)

\[
\begin{align*}
\int_0^\infty \overline{p}^{(1)}_0(x,z_2,s)\mu_1(x)dx &= \overline{p}^{(1)}_0(0,z_2,s)B_1(\phi_1[z,s]), \\
\int_0^\infty \overline{p}^{(2)}_0(x,z_2,s)\mu_2(x)dx &= \overline{p}^{(2)}_0(0,z_2,s)B_2(\phi_1[z,s]),
\end{align*}
\] (7.3.81) (7.3.82)

using (7.3.66), (7.3.67), (7.3.68), (7.3.71), (7.3.72) and (7.3.79) to (7.82) into

(7.3.54) and (7.3.55) we get

\[
\begin{align*}
\overline{p}^{(0)}_0(0,z_2,s) &= \overline{p}^{(1)}_0(0,z_2,s)\{ (1 - r_1)B_1(\phi_1[z,s]) [1 - \theta + \theta V(b_1[z,s]) \{ 1 - r + rE(b_1[z,s]) \} ] \\
&\quad + rE(b_1[z,s]) \} + r_1B_1(\phi_1[z,s])B_2(\phi_1[z,s]) [1 - \theta + \theta V(b_1[z,s]) ] \\
&\quad \{ 1 - r + rE(b_1[z,s]) \} \} + 1 - (s + \lambda)\overline{p}^{(0)}_0(s),
\end{align*}
\] (7.3.83)

\[
\begin{align*}
\overline{p}^{(1)}(0,z_1,z_2,s)\{ z_1 - B_1(\phi[z,s]) [1 - \theta + \theta V(b[z,s]) \{ 1 - r + rE(b[z,s]) \} ] \\
&\quad \{ 1 - r_1 + r_1 B_2(\phi[z,s]) \} \} = z_1\lambda \overline{p}^{(0)}_0(s) [p\frac{C(z_1)}{z_1} + q\frac{C(z_2)}{z_2}] + z_1 \overline{p}^{(0)}_0(0,z_2,s) \\
&\quad \{ \lambda \frac{C(z_2)}{z_2} \left[ 1 - \overline{M}(s + \lambda) \right] + \frac{1}{z_2} \overline{M}(s + \lambda) \} - \overline{p}^{(1)}_0(0,z_2,s) \{ B_1(\phi_1[z,s]) \\
&\quad \{ 1 - \theta + \theta V(b_1[z,s]) \} \{ 1 - r + rE(b_1[z,s]) \} \} \\
&\quad \{ 1 - r_1 + r_1 B_2(\phi_1[z,s]) \} ,
\end{align*}
\] (7.3.84)

letting \( z_1 = g(z_2) \) in (7.3.84) we get

\[
\begin{align*}
\overline{p}^{(1)}(0,z_2,s)\{ B_1(\phi_1[z,s]) [1 - \theta + \theta V(b_1[z,s]) \{ 1 - r + rE(b_1[z,s]) \} ] \\
&\quad \{ 1 - r_1 + r_1 B_2(\phi_1[z,s]) \} \} = g(z_2)\lambda \overline{p}^{(0)}_0(s) [p\frac{C(g(z_2))}{g(z_2)} + q\frac{C(z_2)}{z_2}] \\
&\quad + g(z_2)\overline{p}^{(0)}_0(0,z_2,s) \{ \lambda \frac{C(z_2)}{z_2} \left[ 1 - \overline{M}(s + \lambda) \right] + \frac{1}{z_2} \overline{M}(s + \lambda) \} ,
\end{align*}
\] (7.3.85)

substitute (7.3.85) in (7.3.83) and (7.3.84) we get

\[
\overline{p}^{(0)}_0(0,z_2,s) = \left\{ \frac{z_2 - z_2(s + \lambda)\overline{p}^{(0)}_0(s) + z_2\lambda \overline{p}^{(0)}_0(s) pC(g(z_2)) + \lambda qC(z_2)g(z_2)\overline{p}^{(0)}_0(s)}{z_2 - g(z_2)\{ C(z_2)\left[ 1 - \overline{M}(s + \lambda) \right] + \overline{M}(s + \lambda) \}} \right\},
\] (7.3.86)
The inequality $\rho < 1$ is a necessary and sufficient condition for the system to be stable, under this condition the marginal PGF of the server’s state queue size distribution are given by

$$
\tilde{P}(1)(0, z_1, z_2, s) = \left\{ \begin{array}{l}
z_2 \lambda \tilde{P}_0(s) pC(z_1) + z_1 \lambda \tilde{P}_0(s) qC(z_2) - z_2 \lambda \tilde{P}_0(s) pC(g(z_2)) \\
-g(z_2) \lambda \tilde{P}_0(s) qC(z_2) + \tilde{P}_0^{(0)}(0, z_2, s) \{ \lambda C(z_2) \\
\left[ \frac{1 - M(s + \lambda)}{(s + \lambda)} \right] + M(s + \lambda) \} \{ z_1 - g(z_2) \}
\end{array} \right.
$$

(7.3.87)

where $\tilde{P}_0^{(0)}(0, z_2, s)$ is given by (7.3.86), the other boundary conditions are

$$
\tilde{P}^{(2)}(0, z_1, z_2, s) = r_1 \tilde{P}(1)(0, z_1, z_2, s) \tilde{B}_1(\phi[z, s]),
$$

(7.3.88)

$$
\nabla(0, z_1, z_2, s) = \theta \tilde{P}(1)(0, z_1, z_2, s) \tilde{B}_1(\phi[z, s]) [1 - r_1 + r_1 \tilde{B}_2(\phi[z, s])],
$$

(7.3.89)

$$
\mathcal{E}(0, z_1, z_2, s) = r \theta \tilde{P}(1)(0, z_1, z_2, s) \mathcal{B}_1(\phi[z, s]) \mathcal{V}(b[z, s]) [1 - r_1 + r_1 \mathcal{B}_2(\phi[z, s])],
$$

(7.3.90)

$$
\mathcal{D}(0, z_1, z_2, s) = \alpha[\tilde{P}(1)(z_1, z_2, s) + \tilde{P}^{(2)}(z_1, z_2, s)],
$$

(7.3.91)

$$
\mathcal{R}(0, z_1, z_2, s) = \alpha[\tilde{P}(1)(z_1, z_2, s) + \tilde{P}^{(2)}(z_1, z_2, s)] \mathcal{D}(b[z, s]).
$$

(7.3.92)

**Theorem**

The inequality $\rho < 1$ is a necessary and sufficient condition for the system to be stable, under this condition the marginal PGF of the server’s state queue size distribution are given by

$$
\tilde{P}_0^{(0)}(z_2, s) = \tilde{P}_0^{(0)}(0, z_2, s) \left[ \frac{1 - M(s + \lambda)}{(s + \lambda)} \right],
$$

(7.3.93)

$$
\tilde{P}(1)(z_1, z_2, s) = \tilde{P}(1)(0, z_1, z_2, s) \left[ \frac{1 - \mathcal{B}_1(\phi[z, s])}{\phi[z, s]} \right],
$$

(7.3.94)

$$
\tilde{P}^{(2)}(z_1, z_2, s) = r_1 \tilde{P}(1)(0, z_1, z_2, s) \mathcal{B}_1(\phi[z, s]) \left[ \frac{1 - \mathcal{B}_2(\phi[z, s])}{\phi[z, s]} \right],
$$

(7.3.95)

$$
\nabla(z_1, z_2, s) = \theta \tilde{P}(1)(0, z_1, z_2, s) \mathcal{B}_1(\phi[z, s]) [1 - r_1 + r_1 \mathcal{B}_2(\phi[z, s])] \left[ \frac{1 - \mathcal{V}(b[z, s])}{b[z, s]} \right],
$$

(7.3.96)

$$
\mathcal{E}(z_1, z_2, s) = r \theta \tilde{P}(1)(0, z_1, z_2, s) \mathcal{B}_1(\phi[z, s]) \mathcal{V}(b[z, s]) [1 - r_1 + r_1 \mathcal{B}_2(\phi[z, s])] \left[ \frac{1 - \mathcal{E}(b[z, s])}{b[z, s]} \right],
$$

(7.3.97)

$$
\mathcal{D}(z_1, z_2, s) = \alpha[\tilde{P}(1)(z_1, z_2, s) + \tilde{P}^{(2)}(z_1, z_2, s)] \left[ \frac{1 - \mathcal{D}(b[z, s])}{b[z, s]} \right],
$$

(7.3.98)

$$
\mathcal{R}(z_1, z_2, s) = \alpha[\tilde{P}(1)(z_1, z_2, s) + \tilde{P}^{(2)}(z_1, z_2, s)] \mathcal{D}(b[z, s]) \left[ \frac{1 - \mathcal{R}(b[z, s])}{b[z, s]} \right].
$$

(7.3.99)
Proof: Integrating equations (7.3.61), (7.3.66), (7.3.62) and (7.3.63) with respect to \( x \) and using the well known result of renewal theory

\[
\int_0^\infty [1 - H(x)] e^{-sx} dx = \frac{1 - \overline{h(s)}}{s},
\]

(7.3.100)

where \( \overline{h(s)} \) is the LST of the distribution function of a random variable \( H(x) \), we get the formulae (7.3.93) to (7.3.97). Similarly, integrating (7.3.64) and (7.3.65) with respect to \( y \), we get

\[
\mathcal{D}_i(x, z) = \int_0^\infty \mathcal{D}_i(x, y, z) dy = \frac{\alpha \overline{P}^{(i)}(x, z)[1 - \overline{D}_i[A(z)]]}{A(z)}; i = 1, 2., \quad (7.3.101)
\]

\[
\mathcal{R}_i(x, z) = \int_0^\infty \mathcal{R}_i(x, y, z) dy = \frac{\alpha \overline{P}^{(i)}(x, z)\overline{D}_i[A(z)][1 - \overline{R}_i[A(z)]]}{A(z)}; i = 1, 2. \quad (7.3.102)
\]

Further integrating (7.3.101) and (7.3.102) with respect to \( x \), we can get the formulae (7.3.98) and (7.3.99) respectively. Thus, we obtained the complete solution for the probability generating function for the states are \( \mathcal{P}^0(z_2, s) \), \( \mathcal{P}^{(1)}(z_1, z_2, s) \), \( \mathcal{P}^{(2)}(z_1, z_2, s) \), \( \mathcal{V}(z_1, z_2, s) \), \( \mathcal{E}(z_1, z_2, s) \), \( \mathcal{D}(z_1, z_2, s) \) and \( \mathcal{R}(z_1, z_2, s) \).

### 7.4 Steady state Analysis: Limiting behaviour

In this section, we derive the steady state probability distribution for our queueing model. By applying the well-known Tauberian property,

\[
\lim_{s \to 0} s \mathcal{P}(s) = \lim_{t \to \infty} f(t).
\]

to the equations (7.3.93) to (7.3.99). In order to determine \( P_0 \) we use the normalizing condition (7.3.16)

\[
P_0 + P_0^0(1) + P^{(1)}(1, 1) + P^{(2)}(1, 1) + V(1, 1) + E(1, 1) + D(1, 1) + R(1, 1) = 1.
\]

The steady state probability for high priority and low priority customers with second optional service, balking, Bernoulli and extended vacation for an unreliable server are

\[
P_0^0(z_2) = P_0^0(0, z_2)\left[1 - \overline{M}(\lambda)\right], \quad (7.4.1)
\]

\[
P^{(1)}(z_1, z_2) = P^{(1)}(0, z_1, z_2)\left[1 - \frac{\overline{B}_1(\phi[z])}{\phi[z]}\right], \quad (7.4.2)
\]

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\[ P^{(2)}(z_1, z_2) = r_1 P^{(1)}(0, z_1, z_2) B_1(\phi[z]) \left[ \frac{1 - B_2(\phi[z])}{\phi[z]} \right], \]  

(7.4.3)

\[ V(z_1, z_2) = \theta P^{(1)}(0, z_1, z_2) B_1(\phi[z]) \left[ 1 - r_1 + r_1 B_2(\phi[z]) \right] \left[ \frac{1 - V(b[z])}{(b[z])} \right], \]  

(7.4.4)

\[ E(z_1, z_2) = r \theta P^{(1)}(0, z_1, z_2) B_1(\phi[z]) \left[ V(b[z]) \right] \left[ 1 - r_1 + r_1 B_2(\phi[z]) \right] \left[ \frac{1 - E(b[z])}{(b[z])} \right], \]  

(7.4.5)

\[ D(z_1, z_2) = \alpha P^{(1)}(z_1, z_2) + P^{(2)}(z_1, z_2) \left[ \frac{1 - D(b[z])}{(b[z])} \right], \]  

(7.4.6)

\[ R(z_1, z_2) = \alpha P^{(1)}(z_1, z_2) + P^{(2)}(z_1, z_2) D(b[z]) \left[ \frac{1 - R(b[z])}{(b[z])} \right], \]  

(7.4.7)

where

\[ P^{(0)}_0(0, z_2) = \frac{-z_2 \lambda P_0 + z_2 \lambda P_0 pC[g(z_2)] + \lambda qC(z_2) g(z_2) P_0}{z_2 - g(z_2) \{ C(z_2)[1 - \bar{M}(\lambda)] + \bar{M}(\lambda) \}}, \]  

(7.4.8)

\[ P^{(1)}(0, z_1, z_2) = \frac{\left\{ z_2 \lambda P_0 pC(z_1) + z_1 \lambda P_0 qC(z_2) - z_2 \lambda P_0 pC(g(z_2)) - g(z_2) \lambda P_0 qC(z_2) \right\}}{z_2 \{ z_1 - f(z) \}}, \]  

(7.4.9)

\[ f(z) = \bar{B}_1(\phi[z]) \left[ 1 - \theta + \theta \bar{V}(b[z]) \{ 1 - r + r \bar{E}(b[z]) \} \right] \left[ 1 - r_1 + r_1 \bar{B}_2(\phi[z]) \right]. \]

Let \( P_q(z_1, z_2) \) be the probability generating function of the queue size irrespective of the state of the system

\[ P_q(z_1, z_2) = P^{(0)}_0(z_2) + P^{(1)}(z_1, z_2) + P^{(2)}(z_1, z_2) + V(z_1, z_2) + E(z_1, z_2) \]

\[ + D(z_1, z_2) + R(z_1, z_2), \]  

(7.4.10)

\[ P_q(z_1, z_2) = \frac{\left\{ \left[ 1 - \bar{B}_1(\phi[z]) \right] + r_1 \bar{B}_1(\phi[z]) \left[ 1 - B_2(\phi[z]) \right] \right\} \{ b(z) \} + \alpha \left[ 1 - D(b[z]) \right] \left\{ \phi(z) \right\}}{D(z)}, \]  

(7.4.11)

where

\[ N_r(z) = \left\{ \left[ 1 - \bar{B}_1(\phi[z]) \right] + r_1 \bar{B}_1(\phi[z]) \left[ 1 - B_2(\phi[z]) \right] \right\} \{ b(z) \} + \alpha \left[ 1 - D(b[z]) \right], \]

\[ \bar{R}(b(z))] + \theta \phi(z) \bar{B}_1(\phi[z]) \left[ 1 - r_1 + r_1 \bar{B}_2(\phi[z]) \right] \left[ 1 - \bar{V}(b[z]) \right] + r \]

\[ \bar{V}(b(z)) \left[ 1 - \bar{E}(b(z)) \right], \]  

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If the system is in steady state condition, then we have

(i) The probability of the system is free

\[ P_0 = \frac{1 - [\rho_1 - \rho_2]}{D_r} \]

(ii) The probability of the system is occupied

\[ P_0^{(0)}(1) + P_1^{(1)}(1,1) + P_2^{(2)}(1,1) = P_0^{(0)}(0,1) \left[ \frac{1 - \overline{M}(\lambda)}{\lambda} \right] \]

\[ \left\{ \begin{array}{l} [E(B_1) + r_1 E(B_2)] \left\{ \lambda P_0[1 + E(I)] - \lambda P_0 \{ p[1 + E(I)] \} \right. \\
\left. + \left\{ g'(1) + q[g'(1) + E(I)] \right\} + P_0^{(0)}(0,1)[1 - g'(1)] \right\} \frac{1}{1 - \rho_1 + \rho_2} \right. \]

(iii) The probability of the server is idle

\[ P_0 + P_0^{(0)}(1) \]
Note that

\[
\begin{align*}
&\left\{ [E(B_1) + r_1 E(B_2)][1 + \alpha(E(D) + E(R))] + \theta[E(V) \\
&+ r E(E_v)] \right\} \times \left\{ \lambda P_0[1 + E(I)] - \lambda P_0 \{ p[1 + E(I)]g'(1) \} \\
&+ q[g'(1) + E(I)] + P_0^{(0)}(0,1)[1 - g'(1)] \right\} \\
= 1 - \frac{1}{[1 - \rho_1 + \rho_2]}.
\end{align*}
\]

(iv) The probability of the server is busy

\[
P^{(1)}(1,1) + P^{(2)}(1,1) = \frac{[E(B_1) + r_1 E(B_2)] \left\{ \lambda P_0[1 + E(I)] - \lambda P_0 \{ p[1 + E(I)] \\
g'(1) + q[g'(1) + E(I)] \} + P_0^{(0)}(0,1)[1 - g'(1)] \right\}}{[1 - \rho_1 + \rho_2]}.
\]

(v) The probability of the server is waiting for repair

\[
D(1,1) = \frac{\alpha E(D)[E(B_1) + r_1 E(B_2)] \left\{ \lambda P_0[1 + E(I)] - \lambda P_0 \{ p[1 + E(I)] \\
g'(1) + q[g'(1) + E(I)] \} + P_0^{(0)}(0,1)[1 - g'(1)] \right\}}{[1 - \rho_1 + \rho_2]}.
\]

(vi) The probability of the server is under repair

\[
R(1,1) = \frac{\alpha E(R)[E(B_1) + r_1 E(B_2)] \left\{ \lambda P_0[1 + E(I)] - \lambda P_0 \{ p[1 + E(I)] \\
g'(1) + q[g'(1) + E(I)] \} + P_0^{(0)}(0,1)[1 - g'(1)] \right\}}{[1 - \rho_1 + \rho_2]}.
\]

(vii) The probability of the server is on vacation

\[
V(1,1) = \frac{\theta E(V) \left\{ \lambda P_0[1 + E(I)] - \lambda P_0 \{ p[1 + E(I)] \\
g'(1) + q[g'(1) + E(I)] \} + P_0^{(0)}(0,1)[1 - g'(1)] \right\}}{[1 - \rho_1 + \rho_2]}.
\]

(viii) The probability of the server is on extended vacation

\[
E(1,1) = \frac{\theta r E(E_v) \left\{ \lambda P_0[1 + E(I)] - \lambda P_0 \{ p[1 + E(I)] \\
g'(1) + q[g'(1) + E(I)] \} + P_0^{(0)}(0,1)[1 - g'(1)] \right\}}{[1 - \rho_1 + \rho_2]}.
\]

**Proof:**

Note that
Proof: By considering the following equations we get the results

\[ P_0(1) + P_1(1, 1) + P_2(1, 1) = \lim_{z_1 \to 1} \lim_{z_2 \to 1} [P_0(0)(z_2) + P_1(z_1, z_2) + P_2(z_1, z_2)], \]

\[ P_0 + P_0(0)(1) = P_0 + \lim_{z_2 \to 1} P_0(0)(z_2), \]

\[ P_1(1, 1) + P_2(1, 1) = \lim_{z_1 \to 1} \lim_{z_2 \to 1} [P_1(1, z_2) + P_2(1, z_2)], \]

\[ D(1, 1) = \lim_{z_1 \to 1} D(z_1, z_2), \quad R(1, 1) = \lim_{z_1 \to 1} R(z_1, z_2), \]

\[ V(1, 1) = \lim_{z_1 \to 1} V(z_1, z_2) \quad \text{and} \quad E(1, 1) = \lim_{z_1 \to 1} E(z_1, z_2) \]

under the steady state condition the availability of the server and failure of the server

are given by

\[ A_v = 1 - \frac{\left\{ \alpha [E(B_1) + r_1 E(B_2)] \left[ E(D) + E(R) \right] + \theta [E(V) + r E(E_v)] \right\} \times \left\{ \lambda P_0[1 + E(I)] - \lambda P_0(p[1 + E(I)g'(1)] + qg'(1) \\
+ E(I))] + P_0(0, 1)[1 - g'(1)] \right\}}{[1 - \rho_1 + \rho_2]} \]

and

\[ M_f = \frac{\left\{ \alpha [E(B_1) + r_1 E(B_2)] \left\{ \lambda P_0[1 + E(I)] - \lambda P_0 \left\{ p[1 + E(I)g'(1)] \\
+ qg'(1) + E(I)] \right\} + P_0(0, 1)[1 - g'(1)] \right\}}{[1 - \rho_1 + \rho_2]} \].

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by direct calculation we get the above formulae.

Proof: By considering the following equations we get the results

\[ A_v = P_0 + \lim_{z_1 \to 1} \lim_{z_2 \to 1} [P_1(z_1, z_2) + P_2(z_1, z_2)] \]

and

\[ M_f = \alpha [P_1(1, 1) + P_2(1, 1)]. \]
7.5 The average queue length:

The mean number of customers in high priority queue and the low priority queue (orbit) under the steady state are

\[
L_{q_1} = \frac{d}{dz_1}P_{q_1}(z_1, 1)|_{z_1=1}, \tag{7.5.1}
\]

\[
L_{q_2} = \frac{d}{dz_2}P_{q_2}(1, z_2)|_{z_2=1}, \tag{7.5.2}
\]

then

\[
L_{q_1} = \frac{D''_1(1)[N''_1(1) + N''_2(1)] - D''_1(1)[N''_1(1) + N''_2(1)]}{4(D''_1(1))^2}, \tag{7.5.3}
\]

\[
L_{q_2} = \frac{D''_2(1)[N''_3(1) + N''_4(1)] - D''_2(1)[N''_3(1) + N''_4(1)]}{4(D''_2(1))^2}, \tag{7.5.4}
\]

where

\[
N''_1(1) = P_0^{(0)}(0, 1)\{\frac{1 - \overline{M}(\lambda)}{\lambda}D''_1(1) + 3N''_1(1)\},
\]

\[
N''_1(1) = P_0^{(0)}(0, 1)\{\frac{1 - \overline{M}(\lambda)}{\lambda}D''_1(1) + 4N''_1(1)\},
\]

\[
N''_2(1) = 3N''_1(1)\lambda P_b[pE(I) + q],
\]

\[
N''_2(1) = 4N''_1(1)\lambda P_b[pE(I) + q] + 6N''_1(1)\lambda P_0 pE(I[l - 1]),
\]

\[
D''_1(1) = 6\{\lambda pE(I) - \delta p\}\{\lambda pE(I)[1 + \alpha[E(D) + E(R)]] - \alpha\delta p[E(D) + E(R)]\}
\]

\[
\{1 - [\rho_1 - \rho_2]\},
\]

\[
D''_1(1) = -12\{\lambda pE(I) - \delta p\}\{\lambda pE(I)[1 + \alpha[E(D) + E(R)]] - \alpha\delta p[E(D) + E(R)]\}
\]

\[
+E(R)]\}\{\{\lambda pE(I)[1 + \alpha[E(D) + E(R)]] - \alpha\delta p[E(D) + E(R)]\}\}^2
\]

\[
[E(B_1^2) + 2r_1E(B_1)E(B_2) + r_1E(B_2^2)] + \lambda pE(I) + \alpha[\lambda pE(I) - \delta p]^2
\]

\[
[E(D^2) + 2E(D)E(R) + E(R^2)] + [\lambda pE(I[l - 1]) + 2\delta p][E(D)
\]

\[
+E(R)]\}\{E(B_1) + r_1E(B_2)) + \theta[\lambda pE(I) - \delta p]^2|E(V^2) + 2rE(V)
\]

\[
E(E_v) + rE(E_v^2) + \theta[\lambda pE(I[l - 1]) + 2\delta p][E(V) + rE(E_v)] + 2\theta
\]

\[
\{\lambda pE(I) - \delta p\}{\lambda pE(I)[1 + \alpha[E(D) + E(R)]] - \alpha\delta p[E(D) + E(R)]\}
\]

\[
|E(V) + rE(E_v)|E(B_1) + r_1E(B_2)]\} + 12\{\{\lambda pE(I) - \delta p\}\{\lambda p
\]

\[
E(I[l - 1]) + \alpha[\lambda pE(I) - \delta p]^2[E(D^2) + 2E(D)E(R) + E(R^2)] + [\lambda pE
\]

\[
(I[l - 1]) + 2\delta p][E(D) + E(R)]\} + \{\lambda pE(I)[1 + \alpha[E(D) + E(R)]] - \alpha
\]

\[
\delta p[E(D) + E(R)]\} [\lambda pE(I[l - 1]) + 2\delta p]\} \{1 - [\rho_1 - \rho_2]\},
\]
$N''_1(1) = 2\{\lambda pE(I) - \delta p\}\{\lambda pE(I)[1 + \alpha[E(D) + E(R)] - \alpha\delta p[E(D) + E(R)])
\{E(B_1) + r_1E(B_2)]\{1 + \alpha[E(D) + E(R)] + \theta[E(V) + rE(E_v)]},

N''_1(1) = 3\{\lambda pE(I) - \delta p\}\{\lambda pE(I)[1 + \alpha[E(D) + E(R)] - \alpha\delta p[E(D) + E(R)])^2
\{E(B_1^2) + 2r_1E(B_1)E(B_2) + r_1E(B_2]\} + \{\lambda pE(I[I - 1]) + \alpha
\[(\lambda pE(I) - \delta p)^2[E(D^2) + 2E(D)E(R) + E(R^2)] + [\lambda pE(I[I - 1]) + 2\delta
p][E(D) + E(R)]\}(E(B_1) + r_1E(B_2)[1 + \alpha[E(D) + E(R)])] + 3\{\lambda p
E(I)[1 + \alpha[E(D) + E(R)] - \alpha\delta p[E(D) + E(R)])\}(E(B_1) + r_1E(B_2))
\{(\lambda pE(I[I - 1]) + 2\delta p) + \alpha[(\lambda pE(I) - \delta p)^2[E(D^2) + 2E(D)E(R)
+ E(R^2)] + [\lambda pE(I[I - 1]) + 2\delta p][E(D) + E(R)])\}(E(V) + rE(E_v)] + 6\theta\{\lambda p
E(I) - \delta p\}\{\lambda pE(I)[1 + \alpha[E(D) + E(R)] - \alpha\delta p[E(D) + E(R)])\]^2
\{E(V) + rE(E_v)](E(B_1) + r_1E(B_2)) + 3\theta\{\lambda pE(I)[1 + \alpha[E(D)
+ E(R)] - \alpha\delta p[E(D) + E(R)])\}\{(\lambda pE(I) - \delta p)^2[E(V^2) + 2rE(V)
E(E_v) + rE(E_v)]\} + \{\lambda pE(I[I - 1]) + 2\delta p][E(V) + rE(E_v)]\},

N''_3(1) = P^{(0)}_0(0, 1)[\frac{1 - \bar{M}(\lambda)}{\lambda}]D''_2(1) - 3g''(1)N''_2(1),

N''_3(1) = 4P^{(0)}_0(0, 1)[\frac{1 - \bar{M}(\lambda)}{\lambda}]D''_2(1) - 3g''(1)N''_2(1) + P^{(0)}_0(0, 1)[\frac{1 - \bar{M}(\lambda)}{\lambda}
D''_2(1) - 12E[I][1 - \bar{M}(\lambda)]g''(1)N''_2(1) - 6g''(1)N''_2(1) - 4g''(1)N''_2(1),

N''_4(1) = 3N''_2(1)\{\lambda P_0[p + qE(I)] - \lambda P_0[p + pE(I)g'(1) + qg'(1) + qE(I)],

N''_4(1) = 4N''_2(1)\lambda P_0[p + qE(I)] - \lambda P_0[p + pE(I)g'(1) + qg'(1) + qE(I)] + 6
N''_2(1)\{\lambda P_0[qE(I[I - 1]) - \lambda P_0[2pE(I)g'(1) + p(E[I[I - 1])]g'(1)^2
+ E(I)g''(1)] + g''(1)q + 2qg'(1)E(I) + qE(I[I - 1])],

D''_2(1) = -6\{\lambda qE(I) - \delta q\}\{\lambda qE(I)[1 + \alpha[E(D) + E(R)] - \alpha\delta q[E(D) + E(R)]}\}
\{\rho_1 - \rho_2z,

D''_2(1) = -12\{\lambda qE(I) - \delta q\}\{\lambda qE(I)[1 + \alpha[E(D) + E(R)] - \alpha\delta q[E(D) + E(R)]\}
\{2[\rho_1 - \rho_2] + \{\lambda qE(I)[1 + \alpha[E(D) + E(R)] - \alpha\delta q[E(D) + E(R)]\]^2
\{E(B_1^2) + 2r_1E(B_1)E(B_2) + r_1E(B_2]\} + \{\lambda qE(I) + \alpha[(\lambda qE(I) - \delta q)^2
\{E(D^2) + 2E(D)E(R) + E(R^2)] + [\lambda qE(I[I - 1]) + 2\delta q][E(D) + E(R)]\}
\{\rho_1 - \rho_2z,
\( (E(B_1) + r_1E(B_2)) + \theta(\lambda qbE(I) - \delta q)^2[E(V^2) + 2rE(V)E(E_v) + rE(E_v^2)] \\
+ \theta[\lambda qbE(I[I - 1]) + 2\delta q][E(V) + rE(E_v)] + 2\theta(\lambda qbE(I) - \delta q)\{\lambda qbE(I) \\
[1 + \alpha[E(D) + E(R)]] \} - \alpha\delta q[E(D) + E(R)]\} [E(V) + rE(E_v)] (E(B_1) \\
+ r_1E(B_2)) \} - 12\{\lambda qbE(I) - \delta q]\{\lambda qbE(I[I - 1]) + \alpha[(\lambda qbE(I) - \delta q)^2 \\
[E(D^2) + 2E(D)E(R) + E(R^2)] + [\lambda qbE(I[I - 1]) + 2\delta q][E(D) + E(R)]] \} \\
+ \{\lambda qbE(I)[1 + \alpha[E(D) + E(R)]] - \alpha\delta q[E(D) + E(R)]\} [\lambda qbE(I[I - 1]) \\
+ 2\delta q]\} \} \} \} \}

\[ N''_{12}(1) = 2\{\lambda qbE(I) - \delta q\} \{\lambda qbE(I)[1 + \alpha[E(D) + E(R)]] - \alpha\delta q[E(D) + E(R)]\} \\
\{(E(B_1) + r_1E(B_2))[1 + \alpha[E(D) + E(R)]] + \theta[E(V) + rE(E_v)]\}, \]

\[ N''_{12}(1) = 3\{\lambda qbE(I) - \delta q\} \{\lambda qbE(I)[1 + \alpha[E(D) + E(R)]] - \alpha\delta q[E(D) + E(R)]\}^2 \\
[E(B_1) + 2r_1E(B_1)E(B_2) + r_1E(B_2^2)] + \{\lambda qbE(I[I - 1]) + \alpha[(\lambda qbE(I) - \delta q)^2 \\
[E(D^2) + 2E(D)E(R) + E(R^2)] + [\lambda qbE(I[I - 1]) + 2\delta q][E(D) + E(R)]] \} \\
(E(B_1) + r_1E(B_2))[1 + \alpha[E(D) + E(R)]]\} + 3\{\lambda qbE(I)[1 + \alpha[E(D) + E(R)]] \\
- \alpha\delta q[E(D) + E(R)]\} [\lambda qbE(I[I - 1]) + 2\delta q] \\
+ \alpha[(\lambda qbE(I) - \delta q)^2[E(D^2) + 2E(D)E(R) + E(R^2)] + [\lambda qbE(I[I - 1]) \\
+ 2\delta q][E(D) + E(R)]] \} + 3\theta \{\lambda qbE(I) - \delta q\} \{\lambda qbE(I[I - 1]) + \alpha \\
[(\lambda qE(I) - \delta q)^2[E(D^2) + 2E(D)E(R) + E(R^2)] + [\lambda qbE(I[I - 1]) \\
+ 2\delta q][E(D) + E(R)]]\} [E(V) + rE(E_v)] + 6\theta \{\lambda qbE(I) - \delta q\} \{\lambda qbE(I) \\
[1 + \alpha[E(D) + E(R)] - \alpha\delta q[E(D) + E(R)]\}^2[E(V) + rE(E_v)] (E(B_1) + r_1 \\
E(B_2)) + 3\theta \{\lambda qbE(I)[1 + \alpha[E(D) + E(R)] - \alpha\delta q[E(D) + E(R)]\} \\
\{\lambda qbE(I) - \delta q)^2[E(V^2) + 2rE(V)E(E_v) + rE(E_v^2)] + [\lambda qbE(I[I - 1]) \\
+ 2\delta q][E(V) + rE(E_v)]\}). \]

7.6 The average waiting time in the queue:

Average waiting time of a customer in the high priority queue is

\[ W_{q_1} = \frac{L_{q_1}}{\lambda p}. \] (7.6.1)
Average waiting time of a customer in the low priority queue (orbit) is

\[
W_{q_2} = \frac{L_{q_2}}{\lambda q b}, \quad (7.6.2)
\]

where \( L_{q_1} \) and \( L_{q_2} \) are given in equations (7.5.3) and (7.5.4).

### 7.7 Particular cases

**Case: I**

If we let no high priority arrival, no balking, no stand by server, no second optional service and no extended vacation, i.e. \( q = 1, z_2 = z, \ p = 0, b = 1, \delta = 0, r_1 = 0, r = 0 \) then our model reduces to \( M^{[X]} / G / 1 \) retrial queueing system with general retrial times under Bernoulli vacation for unreliable server and delaying repair. The probability generating functions for this model are

\[
P^{(0)}(z) = \frac{P_0 \{ z - C(z)B_1(\phi_1[z]) \{ 1 - \theta + \theta V(b_1[z]) \} \} \{ 1 - \bar{M}(\lambda) \}}{\bar{B}_1(\phi_1[z]) \{ 1 - \theta + \theta V(b_1[z]) \} \{ C(z) \{ 1 - \bar{M}(\lambda) \} + \bar{M}(\lambda) \} - z},
\]

\[
P^{(1)}(z) = \frac{\lambda P_0 \bar{M}(\lambda) \{ 1 - C(z) \} \{ 1 - \bar{B}_1(\phi_1[z]) \}}{\bar{B}_1(\phi_1[z]) \{ 1 - \theta + \theta V(b_1[z]) \} \{ C(z) \{ 1 - \bar{M}(\lambda) \} + \bar{M}(\lambda) \} - z},
\]

\[
V(z) = \frac{\theta \lambda P_0 \bar{M}(\lambda) \{ 1 - C(z) \} \{ 1 - \bar{B}_1(\phi_1[z]) \}}{\bar{B}_1(\phi_1[z]) \{ 1 - \theta + \theta V(b_1[z]) \} \{ C(z) \{ 1 - \bar{M}(\lambda) \} + \bar{M}(\lambda) \} - z},
\]

\[
D(z) = \frac{\alpha P_0 \bar{M}(\lambda) \{ 1 - \bar{B}_1(\phi_1[z]) \} \{ 1 - \bar{D}(b_1[z]) \}}{\bar{B}_1(\phi_1[z]) \{ 1 - \theta + \theta V(b_1[z]) \} \{ C(z) \{ 1 - \bar{M}(\lambda) \} + \bar{M}(\lambda) \} - z},
\]

\[
R(z) = \frac{\alpha P_0 \bar{M}(\lambda) \{ 1 - \bar{B}_1(\phi_1[z]) \} \bar{D}(\phi_1[z]) \{ 1 - \bar{R}(b_1[z]) \}}{\bar{B}_1(\phi_1[z]) \{ 1 - \theta + \theta V(b_1[z]) \} \{ C(z) \{ 1 - \bar{M}(\lambda) \} + \bar{M}(\lambda) \} - z}.
\]

This result is coincide with Gautam Choudhury, Jau-Chaun Ke. (2012).

**Case: II**

If we let no high priority arrival, no balking, no stand by server, no second optional service, no breakdown and \( \bar{M}(\lambda) \rightarrow 1 \) i.e. \( q = 1, z_2 = z, \ p = 0, b = 1, \delta = 0, r = 0, \alpha = 0 \) then our model reduces to \( M^{[X]} / G / 1 \) queueing system with second optional service and Bernoulli vacation. The probability generating functions for this model are

\[
P^{(1)}(z) = \frac{P_0 \{ 1 - \bar{B}_1(a_1[z]) \}}{\bar{B}_1(a_1[z]) \{ 1 - \theta + \theta V(a_1[z]) \} - z},
\]
\[ p^{(z)}(z) = \frac{r_1 P_0 B_1(a_1[z])[1 - \overline{B}_2(a_1[z])]}{\overline{B}_1(a_1[z])[1 - \theta + \theta \overline{V}(a_1[z]) - z]}, \]

\[ V(z) = \frac{\theta P_0 B_1(a_1[z])[1 - r_1 + r_1 \overline{B}_2(a_1[z])][1 - \overline{V}(a_1(z))]}{\overline{B}_1(a_1[z])[1 - \theta + \theta \overline{V}(a_1[z]) - z]}, \]

In this case if \( C(z) = z, \theta = 0 \) then this model is coincide with Al-Jaraha and Madan (2003).

**Case: III**

If we let no high priority arrival, no balking, no stand by server, no second optional service and no extended vacation, no breakdown and no vacation i.e. \( q = 1, z_2 = z, p = 0, b = 1, \delta = 0, r_1 = 0, r = 0, \alpha = 0 \) and \( \theta = 0 \) then our model reduces to \( M^{[X]} / G / 1 \) retrial queueing system. The probability generating functions for this model are

\[ p^{(0)}(z) = \frac{P_0 \{ z - C(z) \overline{B}_1(a_1[z]) \} [1 - \overline{M}(\lambda)]}{\overline{B}_1(a_1[z]) \{ C(z) [1 - \overline{M}(\lambda)] + \overline{M}(\lambda) \} - z}, \]

\[ p^{(1)}(z) = \frac{P_0 \overline{M}(\lambda) \{ 1 - \overline{B}_1(a_1[z]) \}}{\overline{B}_1(a_1[z]) \{ C(z) [1 - \overline{M}(\lambda)] + \overline{M}(\lambda) \} - z}. \]

In this case if \( C(z) = z \), then this model is coincide with Comez corral (1999).

### 7.8 Numerical Analysis

The above queueing model is analysed numerically with the following assumptions.

(i) Service time distribution for essential service follows exponential distribution the mean service rate is \( \mu_1 = 2 \).

(ii) service time for the second optional service follows exponential distribution with \( r_1 = 0.2 \) and \( \mu_2 = 2 \)

(iii) The service time of stand by server follows exponential distribution with parameter \( \delta = 0.3 \)

(iv) Retrial time follows exponential distribution with parameter \( \nu = 0.6 \) and \( \overline{M}(\lambda) = \left[ \frac{\nu}{\nu + \lambda} \right] \).

(v) If the arriving customers join the priority queue with probability \( p = 0.4 \)

(vi) If the arriving customers join the non-priority queue(orbit) with probability \( q = 0 \)
(vii) $g'(1) = \left[ \frac{\lambda q b}{\mu_1 - \lambda p} \right], \quad g''(1) = \left[ \frac{2\lambda q b \mu_1}{(\mu_1 - \lambda p)^2} \right].$

(viii) Vacation time follows exponential distribution with $\theta = 0.20$ and $\gamma = 4$

(ix) Extended vacation time follows exponential distribution with $r = 0.2$ and $\Theta_v = 4$

(x) Random Breakdown follows exponential distribution with parameter $\alpha = 0.5$

(xi) Delay time to start repair follows exponential distribution with $\xi = 0.5$

(xii) Repair time follows exponential distribution with $\beta = 3$

(xiii) Arriving low priority customers may balk the orbit with probability $b = 0.2$.

(xiv) Arrivals are single and the average arrival rate ranging from $\lambda = 0.5$ to 1.5.

In this case increasing the arrival rate the idle time decreases and increase the queue size and waiting time of high priority queue in table 7.1 and graph 1.

Results are presented for the values of $\lambda$ in the following table with their corresponding graphical representation.
Table 7.1: Effect of $\lambda$ on various queue characteristics

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$P_0$</th>
<th>$L_{q_1}$</th>
<th>$W_{q_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.5844</td>
<td>0.1533</td>
<td>0.7666</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5249</td>
<td>0.2366</td>
<td>0.9857</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4709</td>
<td>0.3423</td>
<td>1.2225</td>
</tr>
<tr>
<td>0.8</td>
<td>0.4215</td>
<td>0.4743</td>
<td>1.4822</td>
</tr>
<tr>
<td>0.9</td>
<td>0.3763</td>
<td>0.6376</td>
<td>1.7711</td>
</tr>
<tr>
<td>1.0</td>
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<td>0.8390</td>
<td>2.0974</td>
</tr>
<tr>
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<td>1.0876</td>
<td>2.4718</td>
</tr>
<tr>
<td>1.2</td>
<td>0.2606</td>
<td>1.3963</td>
<td>2.9090</td>
</tr>
<tr>
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<td>3.4295</td>
</tr>
<tr>
<td>1.4</td>
<td>0.1968</td>
<td>2.2756</td>
<td>4.0635</td>
</tr>
<tr>
<td>1.5</td>
<td>0.1680</td>
<td>2.9141</td>
<td>4.8568</td>
</tr>
</tbody>
</table>

Figure 7.1: Average queue length, waiting time and idle time of high priority customers verses arrival rate $\lambda$