CHAPTER 5

INTUITIONISTIC FUZZY OPTIMIZATION FOR MULTIPLE-OBJECTIVE PROGRAMMING WITH INTERDEPENDENCE

5.1. Introduction

Interdependence is an aspect of several economic theories. In most of the modelling approaches in Multiple-Objective Decision Making (MODM) there seems to be an implicit assumption that objectives should be independent and are usually considered independent from each other. But latter C. Carlsson and Robert Fuller and R. Felix [C; RF 94a], [C; RF 95], [C; RF 00] showed that if in a MODM problem all the objectives are conflicting, normally there is no optimal solution which would simultaneously satisfy all the criteria. On the other hand, if we have pair wise supportive objectives, such that attainment of one objective helps us to attain another objective, then we should make use of this property in order to find effective

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optimal solution. We consider Multiple-Objective Programming (MOP) problems with additive interdependence, i.e., when the states of some chosen objective are attained through supportive or inhibitory feedbacks from several other objectives. For example, ‘minimizing cost’ and ‘maximizing the quality of the product’.

C. Carlsson and Robert Fuller introduced interdependence among the objectives of a crisp MOP problem and developed a new method for finding compromise solution for Fuzzy MOP problem [C; RF 94a]. Here we introduce a general method for solving MOP problems with interdependence in the IF environment by considering the degree of satisfaction and rejection of the objectives and apply application functions to both separately. We illustrate an example showing that the degree of satisfaction in this case is higher than that by their crisp or fuzzy analogous.

5.2. Linear Interdependence in MOP

A typical MOP with interdependent objectives in classical theory is as:
Maximize \( x \in X \) \( \{f_1(x), \ldots, f_K(x)\}\) \hspace{1cm} (25)

subject to \( AX \leq B, \; X \geq 0 \)

where \( A, B \) and \( X \) are as defined in Section 1.4.1.

Interdependence among the objectives exists whenever the computed values of an objective function is not equal to its observed value. In this case maximizing/minimizing one of the objectives may depend upon several other objectives. In such case we say that the objectives are linearly interdependent.

**Definition 5.2.1.** [C; RF’94a] Let \( f_i \) and \( f_j \) be two objective functions of (25). We say that

(i) \( f_i \) supports \( f_j \) on \( X \) (denoted by \( f_i \uparrow f_j \)) if \( f_i(x') \geq f_i(x) \) entails \( f_j(x') \geq f_j(x) \), for all \( x', x \in X \).

(ii) \( f_i \) is in conflict with \( f_j \) on \( X \) (denoted by \( f_i \downarrow f_j \)) if

\[
f_i(x') \geq f_i(x) \text{ entails } f_j(x') \leq f_j(x) \text{ for all } x', x \in X.
\]

(iii) \( f_i \) and \( f_j \) are independent on \( X \), otherwise. (See figures 1 and 2)
Definition 5.2.2. [C; RF 94a] The grade of interdependency denoted by $\Delta(f_i)$ of $f_i$ is defined as

$$\Delta(f_i) = \sum_{i \neq j, f_i \uparrow f_j} 1 - \sum_{i \neq j, f_i \downarrow f_j} 1, \quad i = 1, 2, \ldots K \quad (26)$$

Remark 5.2.1. [C; RF 94a] If $\Delta(f_i)$ is positive and large, then $f_i$ supports a majority of the objectives. If $\Delta(f_i)$ is negative and large then
$f_i$ is in conflict with a majority of objectives, if $\Delta(f_i)$ is positive and small then $f_i$ supports more objectives than it hinders, and if $\Delta(f_i)$ is negative and small then $f_i$ hinders more than it supports. Finally if $\Delta(f_i) = 0$, then $f_i$ is independent from the others or supports the same number of objectives as it hinders.

5.3. IFMOP with Interdependence

Let us now consider the Intuitionistic Fuzzy (IF) version of the MOP with independent objectives

$$\max_{x \in X} \{\tilde{f}_1(x), \ldots, \tilde{f}_K(x)\}$$

subject to $AX \leq B$, $X \geq 0$. (27)

where $\tilde{f}_i : \mathbb{R}^n \to \mathcal{F}(\mathbb{R})$ are Intuitionistic Fuzzy functions, $\mathcal{F}(\mathbb{R})$ denotes the family of IF quantities and $X \subseteq \mathbb{R}^n$. We also define the membership and non-membership functions for each $\tilde{f}_i$ as two functions $\mu_i : \mathcal{F}(\mathbb{R}) \to [0, 1]$ and $\nu_i : \mathcal{F}(\mathbb{R}) \to [0, 1]$
where

$$\mu_i(\tilde{f}_i(x)) = \begin{cases} 1 & \text{if } \tilde{f}_i(x) \geq M_i \\ \frac{\tilde{f}_i(x) - m_i}{M_i - m_i} & \text{if } m_i < \tilde{f}_i(x) < M_i \\ 0 & \text{if } \tilde{f}_i(x) \leq m_i \end{cases}$$

as the membership function and the non-membership function as

$$\nu_i(\tilde{f}_i(x)) = \begin{cases} 0 & \text{if } \tilde{f}_i(x) > M_i \\ \frac{M_i - (\tilde{f}_i(x) + l_i)}{M_i - (m_i + l_i)} & \text{if } m_i < \tilde{f}_i(x) \leq M_i \\ 1 & \text{if } \tilde{f}_i(x) \leq m_i \end{cases}$$

for each $i = 1, 2, \ldots K$, where $m_i = \min_{x \in X} \tilde{f}_i(x)$ is the independent minimum and $M_i = \max_{x \in X} \tilde{f}_i(x)$ is the independent maximum.

Also $0 < l_i < M_i - m_i$, so that $m_i + l_i$ is less than or equal to the next minimum value of $f_i(x)$ and $0 \leq \mu_i + \nu_i \leq 1$ for all $i$.

Then using (26) compute $\Delta(\tilde{f}_i)$ for each $i$ and change the shape of $\mu_i$ and $\nu_i$ according to the values of $\Delta(f_i)$ as follows.

(i) If $\Delta(f_i) = 0$ then we do not change the shape.
(ii) If $\Delta(\tilde{f}_i) > 0$, then

$$
\mu_i(\tilde{f}_i(x)) = \begin{cases} 
1 & \text{if } \tilde{f}_i(x) \geq M_i \\
\left(\frac{\tilde{f}_i(x) - m_i}{M_i - m_i}\right)^{\frac{1}{\Delta(\tilde{f}_i)+1}} & \text{if } m_i < \tilde{f}_i(x) < M_i \\
0 & \text{if } \tilde{f}_i(x) \leq m_i
\end{cases}
$$

and

$$
\nu_i(\tilde{f}_i(x)) = \begin{cases} 
0 & \text{if } \tilde{f}_i(x) > M_i \\
\left(\frac{M_i - (\tilde{f}_i(x) + l_i)}{M_i - (m_i + l_i)}\right)^{\frac{1}{\Delta(\tilde{f}_i)+1}} & \text{if } m_i < \tilde{f}_i(x) \leq M_i \\
1 & \text{if } \tilde{f}_i(x) \leq m_i
\end{cases}
$$

(iii) If $\Delta(\tilde{f}_i) < 0$ then

$$
\mu_i(\tilde{f}_i(x)) = \begin{cases} 
1 & \text{if } \tilde{f}_i(x) \geq M_i \\
\left(\frac{\tilde{f}_i(x) - m_i}{M_i - m_i}\right)^{|\Delta(\tilde{f}_i)|+1} & \text{if } m_i < \tilde{f}_i(x) < M_i \\
0 & \text{if } \tilde{f}_i(x) \leq m_i
\end{cases}
$$
\( \nu_i(f_i(x)) = \begin{cases} 
0 & \text{if } \tilde{f}_i(x) > M_i \\
\left( \frac{M_i - (f_i(x) + l_i)}{M_i - (m_i + l_i)} \right)^{|\Delta(f_i)|+1} & \text{if } m_i < \tilde{f}_i(x) \leq M_i \\
1 & \text{if } f_i(x) \leq m_i 
\end{cases} \)

See figures 3, 4 and 5.

Suppose that we have two reference points, denoted by \( m_i \) and \( M_i \) which represent undesired and desired levels for each objective function \( \tilde{f}_i \). Then problem (27) can be stated as, find an \( x^* \in X \) such that \( \tilde{f}_i(x^*) \) is as close as possible to the desired point \( M_i \) and as far as possible to the undesired point \( m_i \), for each \( i = 1, \ldots K \).

We suggest the following family of application functions:

\[
H_i(x) = \min \{ \mu_i(\tilde{f}_i(x)) \} \quad \text{and} \quad G_i(x) = \max \{ \nu_i(\tilde{f}_i(x)) \}.
\]
FIGURE 3. The case of linear membership function $\Delta(f_i) = 0$.

FIGURE 4. $H_i(x)$ and $G_i(x)$ if $\Delta(f_i) > 0$.

FIGURE 5. $H_i(x)$ and $G_i(x)$ if $\Delta(f_i) < 0$.

It is clear that the higher the value of $H_i(x)$ and smaller the value of $G_i(x)$, the closer the value of the $i^{th}$ objective function to the desired level or/and farther from the undesired level, and vice versa, the
smaller the value of $H_i(x)$ and higher the value of $G_i(x)$, the closer its value to the undesired level or/and farther from the desired level.

Thus, similarly to the crisp case, the IFMOP (27) turns to the single-objective problem:

$$\max_{x \in X} \{ \min(H_i(x)), \max(G_i(x)) \}$$

subject to $AX \leq B, \ X \geq 0.$ \hspace{1cm} (28)

or

$$\max \lambda - \mu$$

subject to $H_i(x) \leq \lambda, \ G_i(x) \geq \mu$

$$AX \leq B, \ \lambda + \mu \leq 1,$$ \hspace{1cm} (29)

$\lambda \geq \mu, \ \lambda, X \geq 0.$

5.4. Illustration

Here we illustrate that the consistency and efficiency of solving a simple three objectives optimization problem with interdependence in the IF environment using the proposed functions are higher than that in the crisp or fuzzy case.
Example 5.4.1.

\[
\max \{ \tilde{f}_1(x), \tilde{f}_2(x), \tilde{f}_3(x) \}
\]

subject to \(0 \leq x \leq 1\)

where \(\tilde{f}_1(x) = x, \tilde{f}_2(x) = x + 1\) and \(\tilde{f}_3(x) = 1 - x\).

Suppose that the decision maker has the following reference points \(M_1 = 1, M_2 = 2, M_3 = 0\) and \(m_1 = 0, m_2 = m_3 = 1\).

Even though the model values of the 1st two objective functions are in conflict with the 3rd function, from the choice of the reference points, it is easy to see that the three objectives are supporting objectives. Therefore \(\Delta(f_i) = 2, i = 1, 2, 3\).

\[
\therefore H_1(x) = H_2(x) = x^{1/3}, \quad H_3(x) = (-x)^{1/3}
\]

and \(G_1(x) = (1 - 2x)^{1/3}, \quad G_2(x) = \left(\frac{1}{3} - \frac{2}{3}x\right)^{1/3}, \quad G_3(x) = \left(1 - \frac{2}{3}x\right)^{1/3}\).

Hence the given problem becomes

\[
\text{Max} \{ \min(H_i(x)), \max(G_i(x)) \}
\]

which has the unique solution \(x^* = 1\) and hence we have achieved the desired level.
5.5. Conclusion

In this chapter we have enhanced the multiple criteria optimization instruments with interdependent objectives in IF environment by introducing a general membership and non-membership function for the objectives considering interdependence into account. Also we introduced a family of application function for both the membership and non-membership functions separately and proved that level of satisfaction is higher than the analogous crisp or fuzzy problem by an example.