6.0. Introduction

In some cases, before playing the game, the players have the freedom to discuss about the game. So they have some advantage in such cases. They can make some agreements such as sharing of the payoff or keeping the medals they got if they win the game etc. In some games if the payoffs may be money and so it can be shared equally in light of the agreements they have made. But in some other games, the payoffs cannot be transferable. Such a game is called a co-operative game [Jo]. In such cases the payoffs may be a gold medal or a trophy. Then equal division of the payoffs is not possible. So side payments in the form of money can be a solution. One of the player takes the gold and give money to the other players. If the payoff is a trophy, they can make some agreements such as each player can keep the trophy with him for equal periods of time.

In some other cases, the payoff may be in the money form and the players may try to maximize their payoffs. But in some other cases assigning the players may create some problem. For example, two sisters or brothers playing a game, the elder one may accept defeat voluntarily for the younger one. If the payoffs are money, there is another problem that the psychological worth of money may create differences in the attitudes of the players.

In this chapter, we discuss a co-operative game with fuzzy coalition and the payoffs are also fuzzy. We prove some related results of co-operative game with fuzzy coalition. Also we discuss the imputations, strategic and dominance of imputations in co-operative game.
6.1. Co-operative game with fuzzy coalition

**Definition 6.1.1.**

Let \( I = \{1, 2, \ldots, n\} \) be the set of \( n \)-players in a co-operative fuzzy game, then any fuzzy subset \( J \) of \( I \) is called a fuzzy coalition.

**Note:** 1 The set of all fuzzy coalitions is denoted by \( \mathcal{P}(I) \). A one person coalition is denoted by \( \{i\}, i \in I \) and the no-person coalition is denoted by \( \tilde{0} \).

Now we define the fuzzy characteristic function.

**Definition 6.1.2.**

Let \( \mathcal{P}(I) \) be the set of all fuzzy coalition of \( I \) and \( \mathcal{R} \) be the set of all fuzzy numbers on \( R \) in a co-operative fuzzy game, then \( v \) is a function from \( \mathcal{P}(I) \) to \( \mathcal{R} \) such that \( v(I) \) is the maximum joint payoff of the fuzzy coalition \( J \).

We define \( v(0) = \tilde{0} \) and

\[
v(I) = \bigvee_{x \in X_J} \bigwedge_{y \in X_{J^c}} \sum_{\mu_J(i) > 0} P_i(x, y) \mu_J(i),
\]

(6.1.1)

where \( X_J \) and \( X_{J^c} \) denotes the set of co-ordinated mixed strategies for fuzzy coalition \( J \) and \( J^c \) (the complement of \( J \)) respectively.

\( P_i(x, y) \) be the payoff of player \( i \) which is a function from \( X_J \times X_{J^c} \) to \( \mathcal{R} \) and \( \mu_J(i) \) be the grade of membership of player \( i \) in the fuzzy coalition \( J \), i.e., \( \mu_J \) is a function from \( I \) to \([0, 1]\).

**Note:** 2 If \( P_i(x, y) \) is a fuzzy number, \( J \) is a fuzzy coalition of \( I \) and \( \mu_J(i) \), the grade of membership of \( i \) in \( J \), then \( P_i(x, y) \cdot \mu_J(i) \) is a fuzzy number.

Let \( I = \{1, 2, \ldots, n\} \) be the set of \( n \)-players and \( J \) be a fuzzy coalition. Let \( S_{\{i\}} \) be the set of pure strategies of player-\( i \), when his involvement in the fuzzy coalition \( J \), given by the grade of membership of \( \mu_J(i) \). If \( S_J \) denotes the set of co-ordinated pure strategies available
to $\mathcal{J}$, then

$$S_{\mathcal{J}} = \prod_{i} \{ S_{\{i\}} | \mu_{\mathcal{J}}(i) > 0 \}.$$ 

If $X_{\mathcal{J}}$ denotes the set of mixed strategies for $\mathcal{J}$ then $X_{\mathcal{J}} = [S_{\mathcal{J}}]$. Let $X_{\{i\}}$ be the set of mixed strategies of player $i$ in $\mathcal{J}$, then

$$\prod_{i} \{ X_{\{i\}} | \mu_{\mathcal{J}}(i) > 0 \} = \prod_{i} \{ [S_{\{i\}}] | \mu_{\mathcal{J}}(i) > 0 \} \subseteq \prod_{i} \{ S_{\{i\}} | \mu_{\mathcal{J}}(i) > 0 \} = [S_{\mathcal{J}}] = X_{\mathcal{J}}.$$

**Theorem 6.1.3.**

Let $\mathcal{I} = \{1, 2, \ldots, n\}$ be a set of $n$-players in a co-operative fuzzy game. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two fuzzy coalition of $\mathcal{I}$ and $\mathcal{J}_1 \cap \mathcal{J}_2 = \emptyset$ then

$$v(\mathcal{J}_1 \cup \mathcal{J}_2) \geq v(\mathcal{J}_1) + v(\mathcal{J}_2).$$

**Proof.** For the fuzzy coalition $\mathcal{J}_1 \cup \mathcal{J}_2$ and for its complement $(\mathcal{J}_1 \cup \mathcal{J}_2)^c$, let $X_{\mathcal{J}_1 \cup \mathcal{J}_2}$ and $X_{(\mathcal{J}_1 \cup \mathcal{J}_2)^c}$ be the set of co-ordinated mixed strategies.

From (6.1.1), we have

$$v(\mathcal{J}_1 \cup \mathcal{J}_2) = \bigvee_{x \in X_{\mathcal{J}_1 \cup \mathcal{J}_2}} \bigwedge_{y \in X_{(\mathcal{J}_1 \cup \mathcal{J}_2)^c}} \sum_{\mu_{\mathcal{J}_1 \cup \mathcal{J}_2}(i) > 0} P_1(x, y) \mu_{\mathcal{J}_1 \cup \mathcal{J}_2}(i).$$

Let $\sigma \in X_{\mathcal{J}_1}$ and $\tau \in X_{\mathcal{J}_2}$ be independent mixed strategies and the function $P_1(\sigma, \tau, y)$ be a function from $X_{\mathcal{J}_1} \times X_{\mathcal{J}_2} \times X_{(\mathcal{J}_1 \cup \mathcal{J}_2)^c}$ to $\tilde{R}$ denotes the payoff to player $i$ where the mixed strategies $\sigma \in X_{\mathcal{J}_1}$, $\tau \in X_{\mathcal{J}_2}$ and $y \in X_{(\mathcal{J}_1 \cup \mathcal{J}_2)^c}$ are taken.

$$\therefore v(\mathcal{J}_1 \cup \mathcal{J}_2) \geq \bigvee_{\sigma \in X_{\mathcal{J}_1}} \bigvee_{\tau \in X_{\mathcal{J}_2}} \bigwedge_{y \in (\mathcal{J}_1 \cup \mathcal{J}_2)^c} \sum_{\mu_{\mathcal{J}_1 \cup \mathcal{J}_2}(i) > 0} P_1(\sigma, \tau, y) \mu_{\mathcal{J}_1 \cup \mathcal{J}_2}(i).$$

Hence

$$v(\mathcal{J}_1 \cup \mathcal{J}_2) \geq \bigwedge_{y \in X_{(\mathcal{J}_1 \cup \mathcal{J}_2)^c}} \sum_{\mu_{\mathcal{J}_1 \cup \mathcal{J}_2}(i) > 0} P_1(\sigma, \tau, y) \mu_{\mathcal{J}_1 \cup \mathcal{J}_2}(i),$$

for each $\sigma \in X_{\mathcal{J}_1}$ and $\tau \in X_{\mathcal{J}_2}$. 
Where $\mu_{\mathcal{F}_1 \cup \mathcal{F}_2}(i)$ is defined as maximum of $\mu_{\mathcal{F}_1}(i)$ and $\mu_{\mathcal{F}_2}(i)$, i.e.,

$$\mu_{\mathcal{F}_1 \cup \mathcal{F}_2}(i) = \max \left( \mu_{\mathcal{F}_1}(i), \mu_{\mathcal{F}_2}(i) \right).$$

Therefore,

$$v(\mathcal{F}_1 \cup \mathcal{F}_2) \geq \bigwedge_{y \in X(\mathcal{F}_1 \cup \mathcal{F}_2)^c} \sum_{\mu_{\mathcal{F}_1}(i) > 0} P_i(\sigma, \tau, y) \cdot \max \left( \mu_{\mathcal{F}_1}(i), \mu_{\mathcal{F}_2}(i) \right)$$

$$= \bigwedge_{y \in X(\mathcal{F}_1 \cup \mathcal{F}_2)^c} \left[ \sum_{\mu_{\mathcal{F}_1}(i) > 0} P_i(\sigma, \tau, y) \cdot \mu_{\mathcal{F}_1}(i) + \sum_{\mu_{\mathcal{F}_2}(i) > 0} P_i(\sigma, \tau, y) \mu_{\mathcal{F}_2}(i) \right].$$

Since $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$, if $\mu_{\mathcal{F}_1}(i) > 0$ for $i \in \mathcal{F}_1$ then $\mu_{\mathcal{F}_2}(i) = 0$ and if $\mu_{\mathcal{F}_2}(i) > 0$ for $i \in \mathcal{F}_2$ then

$$\mu_{\mathcal{F}_1}(i) = 0,$$

$$v(\mathcal{F}_1 \cup \mathcal{F}_2) = \bigwedge_{y \in X(\mathcal{F}_1 \cup \mathcal{F}_2)^c} \sum_{\mu_{\mathcal{F}_1}(i) > 0} P_i(\sigma, \tau, y) \mu_{\mathcal{F}_1}(i)$$

$$+ \bigwedge_{y \in X(\mathcal{F}_1 \cup \mathcal{F}_2)^c} \sum_{\mu_{\mathcal{F}_2}(i) > 0} P_i(\sigma, \tau, y) \mu_{\mathcal{F}_2}(i).$$

Since $\tau \in X_{\mathcal{F}_2}$ and $y \in X(\mathcal{F}_1 \cup \mathcal{F}_2)^c$

$$(\tau, y) \in X_{\mathcal{F}_2} \times X(\mathcal{F}_1 \cup \mathcal{F}_2)^c$$

and

$$X_{\mathcal{F}_2} \times X(\mathcal{F}_1 \cup \mathcal{F}_2)^c = [S_{\mathcal{F}_2}] \times [S(\mathcal{F}_1 \cup \mathcal{F}_2)^c] \subseteq [S(\mathcal{F}_1 \cup \mathcal{F}_2)^c] = X_{\mathcal{F}_1},$$

then $(\tau, y)$ defines a mixed strategy in $X_{\mathcal{F}_1}$.

Similarly, $X_{\mathcal{F}_1} \times X(\mathcal{F}_1 \cup \mathcal{F}_2)^c \subseteq X_{\mathcal{F}_2}$

$$. v(\mathcal{F}_1 \cup \mathcal{F}_2) \geq \bigwedge_{\gamma \in X_{\mathcal{F}_1} \mu_{\mathcal{F}_1}(i) > 0} \sum P_i(\sigma, \gamma) \mu_{\mathcal{F}_1}(i) + \bigwedge_{\delta \in X_{\mathcal{F}_2} \mu_{\mathcal{F}_2}(i) > 0} \sum P_i(\tau, \delta) \mu_{\mathcal{F}_2}(i)$$

$$= v(\mathcal{F}_1) + v(\mathcal{F}_2). \quad \text{By (6.1.1)}$$

$$. v(\mathcal{F}_1 \cup \mathcal{F}_2) \geq v(\mathcal{F}_1) + v(\mathcal{F}_2).$$
**Definition 6.1.4.** [Jo]

If \( v(\mathcal{J}_1 \cup \mathcal{J}_2) = v(\mathcal{J}_1) + v(\mathcal{J}_2) \), then the characteristic function is to be additive. Then the game is called inessential. Otherwise it is called essential.

**Theorem 6.1.5.**

A necessary and sufficient condition that a finite n-person co-operative fuzzy game is inessential is that \( \sum_{\mu(i) > 0} v(\{i\}) = v(\mathcal{I}) \).

**Proof.** Let the game be essential and \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) are fuzzy coalition of \( \mathcal{I} \) such that \( \mathcal{J}_1 \cap \mathcal{J}_2 = \emptyset \) then \( v(\mathcal{J}_1 \cup \mathcal{J}_2) = v(\mathcal{J}_1) + v(\mathcal{J}_2) \).

So \( v(\mathcal{I}) = \sum_{\mu(i) > 0} v(\{i\}) \).

**Sufficiency.**

Let \( v(\mathcal{I}) = \sum_{\mu(i) > 0} v(\{i\}) \) then by theorem 6.1.3, we have

\[
v(\mathcal{I}) \geq v(\mathcal{I}_1 \cup \mathcal{I}_2) + v((\mathcal{I}_1 \cup \mathcal{I}_2)^c) \tag{6.1.2}\]

\[
v(\mathcal{I}) = \sum_{\mu(i) > 0} v(\{i\}) \geq \sum_{\mu(\mathcal{J}_1(i)) > 0} v(\{i\}) + \sum_{\mu(\mathcal{J}_2(i)) > 0} v(\{i\}) + \sum_{\mu(\mathcal{J}_1 \cup \mathcal{J}_2 \setminus \{i\}) > 0} v(\{i\})
= v(\mathcal{J}_1) + v(\mathcal{J}_2) + v((\mathcal{J}_1 \cup \mathcal{J}_2)^c) \tag{6.1.3}\]

By using the theorem 6.1.3 again

\[
v(\mathcal{J}_1 \cup \mathcal{J}_2) \geq v(\mathcal{J}_1) + v(\mathcal{J}_2).\]

Therefore

\[
v(\mathcal{J}_1 \cup \mathcal{J}_2) + v((\mathcal{J}_1 \cup \mathcal{J}_2)^c) \geq v(\mathcal{J}_1) + v(\mathcal{J}_2) + v((\mathcal{J}_1 \cup \mathcal{J}_2)^c). \tag{6.1.4}\]

So from (6.1.2), (6.1.3) and (6.1.4), we see that

\[
v(\mathcal{I}) = v(\mathcal{J}_1 \cup \mathcal{J}_2) + v((\mathcal{J}_1 \cup \mathcal{J}_2)^c).\]
That is the characteristic function is additive.

So the game is inessential.

**Theorem 6.1.6.**

Let \( \mathcal{I} = \{1, 2, \ldots, n\} \) be a finite set of \( n \)-players in a co-operative fuzzy game with constant sum and \( \mathcal{J} \) be a fuzzy coalition of \( \mathcal{I} \) then

\[
v(\mathcal{J}) + v(\mathcal{J}^c) \leq v(\mathcal{I}).
\]

**Proof.** In a constant sum co-operative fuzzy game

\[
v(\mathcal{I}) = \sum_{\mu(i)>0} v(\{i\}) = \sum_{\mu(\mathcal{J}(i))>0} P_i(x_1, \ldots, x_n)\mu_j(i) = \bar{c},
\]

a constant. Then

\[
v(\mathcal{J}) = \bigvee_{x \in X_\mathcal{J}} \bigwedge_{y \in X_\mathcal{J}^c} \sum_{\mu_\mathcal{J}(i)>0} P_i(x, y)\mu_\mathcal{J}(i)
\]
\[
\leq \bigvee_{x \in X_\mathcal{J}} \bigwedge_{y \in X_\mathcal{J}^c} \left[ \bar{c} - \sum_{\mu_\mathcal{J}(i)>0} P_i(x, y)\mu_\mathcal{J}(i) \right]
\]
\[
= \bar{c} - \bigvee_{x \in X_\mathcal{J}} \bigwedge_{y \in X_\mathcal{J}^c} \sum_{\mu_\mathcal{J}(i)>0} P_i(x, y)\mu_\mathcal{J}(i)
\]
\[
= \bar{c} - \bigvee_{y \in X_\mathcal{J}^c} \bigwedge_{x \in X_\mathcal{J}} \sum_{\mu_\mathcal{J}(i)>0} P_i(x, y)\mu_\mathcal{J}(i)
\]
\[
= \bar{c} - v(\mathcal{J}^c).
\]

Therefore

\[
v(\mathcal{J}) + v(\mathcal{J}^c) \leq \bar{c} = v(\mathcal{I}).
\]
6.2. Imputations in co-operative fuzzy game

6.2.0. Introduction. A possible distribution of available payoff is called an imputation. In a co-operative game let \( \mathcal{I} = \{1, 2, \ldots, n\} \) be the set of \( n \)-players, then \( x = (x_1, \ldots, x_n) \) represents an imputation, where \( x_i \) denotes the payoff received by the player-\( i \). Each member of a coalition will expect an amount better than he receives individually by acting independently of the other players [Jo]. Therefore we need to have the condition that

\[
x_i \geq v(\{i\}) \quad \text{for all } i \in \mathcal{I}.
\]

Also we have the condition that

\[
\sum_{i \in \mathcal{I}} x_i = v(\mathcal{I}).
\]

If the game is a co-operative fuzzy game, we can represent an imputation by \( x = (x_1, \ldots, x_n) \) where \( x_i \) denotes the fuzzy payoff to player-\( i \). In the case of a fuzzy game, we require the condition that

\[
x_i \geq v(\{i\}), \forall i \text{ such that } \mu(i) > 0.
\]  

(6.2.1)

Also we require the condition that

\[
\sum_{\mu(i) > 0} x_i = v(\mathcal{I}).
\]  

(6.2.2)

Since if \( \sum_{\mu(i) > 0} x_i < v(\mathcal{I}) \), then each player can be made to gain without loss to others which is impossible. \( v(\mathcal{I}) \) represents the maximum payoff that player can get from the game by considering the coalition \( \mathcal{I} \). Therefore, \( \sum_{\mu(i) > 0} x_i > v(\mathcal{I}) \) is not possible.

\[\therefore \sum_{\mu(i) > 0} x_i = v(\mathcal{I}).\]

Definition 6.2.1.

Let \( \mathcal{I} = \{1, 2, \ldots, n\} \) be the set of \( n \)-players and \( v \) be the characteristic function from \( \widehat{\mathcal{P}}(\mathcal{I}) \) to \( \widehat{\mathbb{R}} \) such that \( v(0) = \widehat{0} \) where \( \widehat{\mathcal{P}}(\mathcal{I}) \) be the fuzzy power set of \( \mathcal{I} \) and \( \widehat{\mathbb{R}} \) be the set of all fuzzy numbers. Let \( X \) be the set of all imputations which satisfies the conditions that \( x_i \geq v(\{i\}) \) \( \forall i \) such that \( \mu(i) > 0 \) and \( \sum_{\mu(i) > 0} x_i = v(\mathcal{I}) \) then the game is called a classical co-operative fuzzy game and is super additive.
**Theorem 6.2.2.**

Consider a classical $n$-person co-operative fuzzy game in which $\mathcal{I} = \{1, 2, \ldots, n\}$ denotes the set of $n$-persons, then a necessary and sufficient condition that $x = (x_1, \ldots, x_n)$ is an imputation in the game is that there exists fuzzy numbers $a_i, i \in \mathcal{I}$ such that

$$x_i = v(\{i\}) + a_i, \quad (a_i \geq 0)$$

and

$$\sum a_i = v(\mathcal{I}) - \sum_{\mu(i) > 0} v(\{i\}).$$

**Proof.** Let there exists fuzzy numbers $a_i \geq 0, i \in \mathcal{I}$ such that

$$x_i = v(\{i\}) + a_i \quad (6.2.3)$$

and

$$\sum a_i = v(\mathcal{I}) - \sum_{\mu(i) > 0} v(\{i\}). \quad (6.2.4)$$

From (6.2.3), we have,

$$x_i - v(\{i\}) \geq a_i \quad \text{[By 1.2.14]}$$

and since $a_i \geq 0, x_i - v(\{i\}) \geq 0$. Therefore $x_i \geq v(\{i\})$. Condition (6.2.1) is satisfied.

Now, since $a_i \leq x_i - v(\{i\})$,

$$\sum_i a_i \leq \sum_{\mu(i) > 0} [x_i - v(\{i\})] = \sum_{\mu(i) > 0} x_i - \sum_{\mu(i) > 0} v(\{i\}).$$

i.e.,

$$\sum_i a_i \leq \sum_{\mu(i) > 0} x_i - \sum_{\mu(i) > 0} v(\{i\}). \quad (6.2.5)$$

From (6.2.4), we have

$$\sum a_i = v(\mathcal{I}) - \sum_{\mu(i) > 0} v(\{i\}).$$
We have, from (6.2.4) and (6.2.5)

\[ v(\mathcal{I}) - \sum_{\mu(i)>0} v(\{i\}) \leq \sum_{\mu(i)>0} x_i - \sum_{\mu(i)>0} v(\{i\}), \]

i.e., \( v(\mathcal{I}) \leq \sum_{\mu(i)>0} x_i \).

The strict inequality, i.e., \( v(\mathcal{I}) < \sum_{\mu(i)>0} x_i \) is not possible, since \( v(\mathcal{I}) \) represents the maximum payoff that players can get from the game.

\[ : v(\mathcal{I}) = \sum_{\mu(i)>0} x_i. \]

Conversely.

Suppose that \( x \) is an imputation. So \( x \) satisfies the conditions \( x_i \geq v(\{i\}) \) \( \forall i \mu(i) > 0 \) and \( v(\mathcal{I}) = \sum_{\mu(i)>0} x_i \). Consider fuzzy numbers \( a_i \) such that \( a_i = x_i - v(\{i\}) \) since \( x_i \geq v(\{i\}), x_i - v(\{i\}) \geq 0 \) i.e., \( a_i \geq 0 \).

Now

\[ \sum_i a_i = \sum_{\mu(i)>0} x_i - \sum_{\mu(i)>0} v(\{i\}) \]
\[ = v(\mathcal{I}) - \sum_{\mu(i)>0} v(\{i\}), \]

since \( v(\mathcal{I}) = \sum_{\mu(i)>0} x_i \).

Hence the theorem.

**Theorem 6.2.3.**

The imputation \( x = (v(\{1\}), v(\{2\}), \ldots, v(\{n\})) \) is unique in an inessential co-operative fuzzy game or any essential co-operative fuzzy game with at least two players possesses infinitely many imputations.
PROOF. Using the theorem 6.2.2 we can write an imputations \( x = (x_1, \ldots, x_n) \) in the form

\[
x = (v(\{1\}) + a_1, v(\{2\}) + a_2, \ldots, v(\{n\}) + a_n)
\]

where \( x_i = v(\{i\}) + a_i \).

\[
v(\mathcal{I}) = \sum_{\mu(i)>0} v(\{i\})
\]

where \( \mathcal{I} = \{1, 2, \ldots, n\} \) be the set of \( n \)-persons in an inessential game.

By the theorem (6.2.2), \( \sum_i a_i = v(\mathcal{I}) - \sum_{\mu(i)>0} v(\{i\}) \)

\( \therefore \) we have \( \sum_i a_i = 0 \) which implies that \( a_i = 0 \) \( \forall i \in \mathcal{I} \) since \( a_i \geq 0 \).

\[
\therefore x = (v(\{1\}), v(\{2\}), \ldots, v(\{n\}))
\]

for an inessential game.

For an essential game,

\[
v(\mathcal{I}) > \sum_{\mu(i)>0} v(\{i\}).
\]

Therefore, \( \sum_i a_i = v(\mathcal{I}) - \sum_{\mu(i)>0} v(\{i\}) > 0 \) and this fuzzy number can be written in infinitely many ways. \( \square \)

6.3. Strategic equivalence

**DEFINITION 6.3.1.**

Let \( v \) and \( v' \) be the fuzzy characteristic functions of the two \( n \)-person co-operative fuzzy games defined over the same set of players \( \mathcal{I} = \{1, \ldots, n\} \). The two games are said to be strategically equivalent if there exists \( k > 0 \) and fuzzy numbers \( c_i, i \in \mathcal{I} \) such that

\[
v'(\mathcal{I}) = kv(\mathcal{I}) + \sum_{\mu_{\mathcal{I}}(i)>0} c_i \mu_{\mathcal{I}}(i), \quad (6.3.1)
\]

for every fuzzy coalition \( \mathcal{I} \) of \( \mathcal{I} \).

**RESULT 6.3.2.**

The relation, strategic equivalence defined as above is reflexive and transitive but not symmetric.
PROOF. To show that the relation is reflexive, put $k = 1$ and $c_i = \tilde{0}$ for every $i \in \mathcal{I}$ in the relation
\[ v'(\mathcal{J}) = kv(\mathcal{J}) + \sum_{\mu_{\mathcal{J}}(i) > 0} c_i \mu_{\mathcal{J}}(i). \]

Now, suppose $v$ is equivalent to $v'$, then
\[ v'(\mathcal{J}) = kv(\mathcal{J}) + \sum_{\mu_{\mathcal{J}}(i) > 0} c_i \mu_{\mathcal{J}}(i). \]

So by [1.2.14]
\[ v'(\mathcal{I}) - \sum_{\mu_{\mathcal{I}}(i) > 0} c_i \mu_{\mathcal{I}}(i) \geq kv(\mathcal{I}). \]

i.e.,
\[ v(\mathcal{I}) \leq k'v(\mathcal{I}) + \sum_{\mu_{\mathcal{I}}(i) > 0} c'_i \mu_{\mathcal{I}}(i), \]

where $k' = \frac{1}{k} > 0$ and $c'_i = \left(-\frac{1}{k}\right)c_i$.

Therefore the relation is not symmetric.

Now, suppose $v$ is equivalent to $v'$ and $v'$ is equivalent to $v''$. Then for every fuzzy coalition $\mathcal{J}$ of $\mathcal{I}$, there exists $k, k' > 0$ and fuzzy numbers $c_i$ and $c'_i, i \in \mathcal{I}$ such that
\[ v'(\mathcal{J}) = kv(\mathcal{J}) + \sum_{\mu_{\mathcal{J}}(i) > 0} c_i \mu_{\mathcal{J}}(i) \]

and
\[ v''(\mathcal{J}) = k'v'(\mathcal{J}) + \sum_{\mu_{\mathcal{J}}(i) > 0} c'_i \mu_{\mathcal{J}}(i). \]
\[ v''(\mathcal{I}) = k' \left[ kv(\mathcal{I}) + \sum_{\mu_{\mathcal{I}}(i) > 0} c_i \mu_{\mathcal{I}}(i) \right] + \sum_{\mu_{\mathcal{I}}(i) > 0} c_i' \mu_{\mathcal{I}}(i) \]

\[ = kk'v(\mathcal{I}) + \sum_{\mu_{\mathcal{I}}(i) > 0} \left[ c_i' + k'c_i \right] \mu_{\mathcal{I}}(i) \]

\[ = k''v(\mathcal{I}) + \sum_{\mu_{\mathcal{I}}(i) > 0} c_i'' \mu_{\mathcal{I}}(i), \]

where \( k'' = kk' > 0 \) and \( c_i'' = c_i' + kc_i \).

\[ \therefore v \text{ is equivalent to } v''. \]

Hence the relation is transitive.

Thus the fuzzy relation we have defined is not an equivalence relation.

\[ \square \]

**RESULT 6.3.3.**

The relation, strategic equivalence between the games induces an equivalence between imputations.

**Proof.** Let \( x = (x_1, \ldots, x_n) \) be an imputation under the fuzzy characteristic function \( v \). Consider \( x' = (x'_1, \ldots, x'_n) \) where \( x'_i = kx_i + c_i \mu(i) \), \( k > 0 \), \( \mu(i) \) be the grade of membership of the player \( i \) and \( c_i \) for \( i \in \mathcal{I} \) are fuzzy numbers. Now we have to show that \( x' = (x'_1, \ldots, x'_n) \) is an imputation under the characteristic function \( v' \) of a game defined over the same set of players \( \mathcal{I} = \{1, 2, \ldots, n\} \). So we need to prove the two conditions (6.2.1) and (6.2.2) for an imputation.

Since \( x \) is an imputation, it satisfies the condition

\[ x_i \geq v(\{i\}) \text{ for } i \in \mathcal{I}. \]

Therefore

\[ x'_i = kx_i + c_i \mu(i) \geq kv(\{i\}) + c_i v(\{i\}) \]

\[ = v'(\{i\}) \quad \text{by (6.3.1)} \]

\[ \therefore x'_i \geq v'(\{i\}). \]
Now
\[
\sum_{\mu(i) > 0} x'_i = \sum_{\mu(i) > 0} [kx_i + c_i \mu(i)]
\]
\[
= k \sum_{\mu(i) > 0} x_i + \sum_{\mu(i) > 0} c_i \mu(i)
\]
\[
= kv(\mathcal{I}) + \sum_{\mu(i) > 0} c_i \mu(i) \quad \text{since } x \text{ is an imputation}
\]
\[
\sum_{\mu(i) > 0} x_i = v(\mathcal{I})
\]
\[
= v'(\mathcal{I}) \quad \text{by (6.3.1)}
\]

\[\therefore x' \text{ satisfies the required conditions for an imputation. i.e., } x' = (x'_1, \ldots, x'_n) \text{ is an imputation under the characteristic function } v'.\]

Hence the result. \hfill \Box

**DEFINITION 6.3.4.**

Let \( v \) be the characteristic function of a co-operative fuzzy game with a set of \( n \)-players \( \mathcal{I} = \{1, 2, \ldots, n\} \). Then the game is said to be a zero-sum game if the characteristic function is identically zero.

**RESULT 6.3.5.**

An inessential co-operative fuzzy game is not strategically equivalent to a zero game.

**PROOF.** A game is inessential if its characteristic function is additive, i.e., for every fuzzy coalition \( \mathcal{J} \) of \( \mathcal{I} \) the set of \( n \)-players,

\[
v(\mathcal{J}) = \sum_{\mu(i) > 0} v(\{i\}).
\]

Therefore, \( v(\mathcal{J}) - \sum_{\mu(i) > 0} v(\{i\}) \geq 0 \) (By 1.2.14)

If we define

\[
v'(\mathcal{J}) = v(\mathcal{J}) - \sum_{\mu(i) > 0} v(\{i\}) \geq 0,
\]

by taking \( k = 1 \) in (6.3.1), we have \( v \) is equivalent to \( v' \).

But \( v' \geq 0 \). Hence the result. \hfill \Box
6.4. Dominance of imputation

**Definition 6.4.1.**
Consider a classical co-operative fuzzy game with a set of \( n \)-players \( \mathcal{I} = \{1, 2, \ldots, n\} \). Let \( \mathcal{J} \) be a fuzzy coalition of \( \mathcal{I} \) and let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) be two imputations in the game,

if \( \sum_{\mu_{\mathcal{J}}(i) > 0} x_i \neq v(\mathcal{J}) \) and \( x_i > y_i \) for every \( i \in \mathcal{I} \) such that \( \mu_{\mathcal{J}}(i) > 0 \) then we say that \( x \) dominates \( y \) through the fuzzy coalition \( \mathcal{J} \) of \( \mathcal{I} \).

**Note:** For a game when \( \mathcal{J} = \{i\} \), i.e., a 1-person coalition or when \( \mathcal{J} = \mathcal{I} \), i.e., the total coalition, dominance cannot be discussed.

For, if \( \mathcal{J} = \{i\} \) and \( x \) dominates \( y \) then for every \( i, y_i < x_i \leq v(\{i\}) \)
i.e., \( y_i < v(\{i\}) \), therefore the condition \( y_i \geq v(\{i\}) \) for an imputation is not satisfied. Hence \( y \) is not an imputation:

If \( \mathcal{J} = \mathcal{I} \) and \( x \) dominates \( y \) in \( \mathcal{I} \) then \( x_i > y_i \) for every \( i \) such that \( \mu_{\mathcal{J}}(i) > 0 \).

So that \( \sum_{\mu_{\mathcal{J}}(i) > 0} x_i > \sum_{\mu_{\mathcal{J}}(i) > 0} y_i = v(\mathcal{J}) \), i.e., \( \sum_{\mu_{\mathcal{J}}(i) > 0} x_i > v(\mathcal{J}) \).

So \( x \) does not satisfy the condition \( \sum_{\mu_{\mathcal{J}}(i)} x_i = v(\mathcal{J}) \) for an imputation.

\[ \therefore x \text{ is not an imputation.} \]

**Theorem 6.4.2.**
Let \( v \) and \( v' \) be the fuzzy characteristic function of two co-operative game defined over the same set of \( n \)-players \( \mathcal{I} = \{1, 2, \ldots, n\} \). If \( v \) is strategically equivalent to \( v' \) and the imputations \( x' \) and \( y' \) corresponding to the imputations \( x \) and \( y \) under the strategic equivalence then \( x \) dominates \( y \) implies \( x' \) dominates \( y' \).

**Proof.** Since \( v \) is strategically equivalent to \( v' \), then there exists \( k > 0 \) and fuzzy numbers \( c_i, i \in \mathcal{I} \) such that for every fuzzy coalition \( \mathcal{J} \) of \( \mathcal{I} \)
\[ v'(\mathcal{J}) = kv(\mathcal{J}) + \sum_{i, \mu_{\mathcal{J}}(i) > 0} c_i \mu_{\mathcal{J}}(i). \]

Now, since \( x \) dominates \( y \), then \( \sum_{\mu_{\mathcal{J}}(i) > 0} x_i \leq v(\mathcal{J}) \) and \( x_i > y_i \) for every \( i \) such that \( \mu_{\mathcal{J}}(i) > 0 \).

Since \( x' \) and \( y' \) are the imputations corresponding to the imputations \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) under the strategic equivalence, then for \( k > 0 \)

\[ x'_i = kx_i + c_i \mu(i) \]

and \( y'_i = ky_i + c_i \mu(i) \) for every \( i \) such that \( \mu(i) > 0 \).

Hence

\[ \sum_{i, \mu_{\mathcal{J}}(i) > 0} x'_i = k \sum_{\mu_{\mathcal{J}}(i) > 0} x_i + \sum_{\mu_{\mathcal{J}}(i) > 0} c_i \mu_{\mathcal{J}}(i) \leq kv(\mathcal{J}) + \sum_{\mu_{\mathcal{J}}(i) > 0} c_i \mu_{\mathcal{J}}(i) = v'(\mathcal{J}) \]

and

\[ x'_i = kx_i + c_i \mu(i) > ky_i + c_i \mu(i) = y'_i \]

i.e.,

\[ \sum_{\mu_{\mathcal{J}}(i) > 0} x'_i < v'(\mathcal{J}) \]

and \( x'_i > y'_i \) for every \( i \), \( \vdots \) \( x' \) dominates \( y' \).

\[ \square \]

**Definition 6.4.3.**

Let \( X \) and \( X' \) be the set of imputations of two classical co-operative fuzzy game defined over the same set of players \( \mathcal{J} = \{1, 2, \ldots, n\} \). Then the game are said to be isomorphic if there exists a function \( f \) from \( X \) to \( X' \) which is one-to-one and onto such that for every pair of imputations \( x, y \in X \), \( x \) dominates \( y \) if and only if \( f(x) \) dominates \( f(y) \).

**Note:** We have seen that the set of imputations for an inessential game consists of a single imputations but there are infinitely many imputations for an essential game. So an essential game cannot be strategically equivalent to an inessential game.
**Definition 6.4.4.** [Jo]

The set of all undominated imputations in a co-operative game is called the Core.

**Theorem 6.4.5.**

An essential constant sum fuzzy game has an empty core.

**Proof.** Assume that the core is non-empty and let \( x \) an imputation in the core. i.e., \( x \) is an undominated imputation.

Let \( x \) is undominated by some \( y \).

So \( x_i > y_i \) for every \( i \in I \), a fuzzy coalition of \( I \) and \( \sum_{\mu_J(\{i\})>0} x_i \leq v(I) \).

Similarly, for a fuzzy coalition \( J^c \) of \( I \)

\[
x_i > y_i \text{ for every } i \in J^c
\]

and \( \sum_{\mu_{J^c(\{i\})>0}} x_i \leq v(J^c) \).

\( v(I) + v(J^c) = v(I) \) for every fuzzy coalition \( J \) of \( I \)

\[
\therefore v(I) = v(I) + v(J^c) \geq \sum_{\mu_J(\{i\})>0} x_i + \sum_{\mu_{J^c(\{i\})>0}} x_i \\
= \sum_{\mu_J(\{i\})>0} x_i = v(I).
\]

Also, since \( x \) is an imputation, it must satisfy the condition \( x_i \geq v(\{i\}) \) for all \( i \in I \).

So that \( v(I) = \sum_{\mu(i)>0} v(\{i\}) \) i.e., the game is an inessential game.

Hence the theorem. \( \square \)