CHAPTER 5

Games with convex fuzzy payoff function

5.0. Introduction

In this chapter, we consider fuzzy games with convex payoff function and derive some related results and the value of a convex game. In the crisp case [Pet;Z] a convex game is defined with $X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$, the sets of strategies of players 1 and 2 are compact sets, the set $Y$ is convex and the payoff function $P$ is from $X \times Y$ to $\mathbb{R}$ is continuous and is convex with respect to $y \in Y$ for any fixed value of $x \in X$. Here we define a convex fuzzy game with the payoff function $P$ from $X \times Y$ to $\tilde{\mathbb{R}}$ being convex with respect to $y \in Y$.

5.1. Convex fuzzy game

**DEFINITION 5.1.1.**
Let $X$ and $Y$ be compact sets and $P$ be a function from $X \times Y$ to $\tilde{\mathbb{R}}$, the set of all fuzzy numbers, then $P$ is convex with respect to $y \in Y$ for each fixed $x \in X$, if

$$P(x, \lambda y_1 + (1 - \lambda)y_2) \leq \lambda P(x_1, y_1) + (1 - \lambda)P(x_1, y_2)$$

for $\lambda \in (0, 1)$.

**DEFINITION 5.1.2.**
Let $X$ and $Y$ be compact sets and the payoff $P$, a continuous function from $X \times Y$ to $\tilde{\mathbb{R}}$, be convex with respect to $y \in Y$ for any fixed $x \in X$. Then the game is called a fuzzy game with a convex payoff function or a convex fuzzy game.
**Theorem 5.1.3.**

In a convex fuzzy game, the player-2 has an optimal pure strategy when the value of the game is

\[ v = \bigwedge_{y \in Y} \bigvee_{x \in X} P(x, y). \]

**Proof.** In a convex fuzzy game, \( X \) and \( Y \) are compact matric spaces and \( P \), a continuous function from \( X \times Y \) to \( \mathbb{R} \).

By the Lemma 4.1.3, there exists \( \varpi \) and \( \nu \) such that \( \varpi = \bigwedge_{\nu \in \mathfrak{F}} \bigvee_{x \in X} Q(x, \nu) \) and \( \nu = \bigvee_{\mu \in \mathfrak{M}} \bigwedge_{y \in Y} Q(\mu, y) \). Also by the theorem 4.1.4

\[ \bigwedge_{\nu \in \mathfrak{F}} \bigvee_{x \in X} Q(x, \nu) = v = \bigvee_{\mu \in \mathfrak{M}} \bigwedge_{y \in Y} Q(\mu, y), \]

for optimal mixed strategies \( \mu \) and \( \nu \).

Since \( P \) is convex with respect to \( y \in Y \), for fixed \( x \in X \)

\[ P(x, \lambda y_1 + (1 - \lambda)y_2) \leq \lambda P(x_1, y_1) + (1 - \lambda)P(x_1, y_2). \]

The set of all probability measures with finite support is every where dense in the set of all probability measures on \( Y [\mathbb{P}; \mathbb{R}] \). Therefore there exists a sequence of mixed strategies \( \{\nu^n\} \) where \( \nu^n = \{y_{n1}, \ldots, y_{nk}\} \).

\[ \eta_1^n, \ldots, \eta_k^n \] are the corresponding probabilities. Therefore \( \varpi^n = \sum_{j=1}^{k_n} \eta_j^n y_{nj} \).

\[ P(x, \varpi^n) = P \left( x, \sum_{j=1}^{k_n} \eta_j^n y_{nj} \right) \leq \sum_{j=1}^{k_n} \eta_j^n P(x, y_{nj}), \text{ since } P \text{ is convex} \]

\[ = Q(x, \nu^n). \]

i.e., \( Q(x, \nu^n) \geq P(x, \varpi^n) \).

Then we obtain \( Q(x, \nu^*) \geq P(x, \varpi) \), where \( \varpi = \lim_{n \to \infty} \varpi^n \) and \( \nu^n \) converges to \( \nu^* \) as \( n \to \infty \).
Since $P(x, y)$ is continuous, $Q(x, \nu)$ is also continuous as $Q(x, \nu) = \int_y P(x, y) d\nu(y)$. Therefore
\[
\bigvee_{x \in X} Q(x, \nu) \geq \bigvee_{x \in X} P(x, \overline{\nu}).
\]
If the inequality is strict, i.e., if
\[
\bigvee_{x \in X} Q(x, \nu^*) > \bigvee_{x \in X} P(x, \overline{\nu}),
\]
then
\[
v = \bigvee_x Q(x, \nu^*) > \bigvee_x P(x, \overline{\nu}) = \bigwedge_x P(x, \nu) = v.
\]
which is not possible. Thus
\[
v = \bigvee_x Q(x, \nu^*) = \bigvee_x P(x, \overline{\nu}).
\]
Therefore $\overline{\nu}$ is an optimal strategy for player-2. \qed

**Definition 5.1.4.**

*The function $P$ from $X \times Y$ to $\tilde{R}$ is strictly convex if
\[
P(x, \lambda y_1 + (1 - \lambda)y_2) < \lambda P(x, y_1) + (1 - \lambda)P(x, y_2),
\]
for $y_1, y_2 \in Y \subset R$, a convex set for all $\lambda \in (0, 1)$.*

**Theorem 5.1.5.**

*In a convex fuzzy game with strictly convex payoff function $P$ from $X \times Y$ to $\tilde{R}$, player-2 has a unique optimal strategy.*

**Proof.** Consider a convex fuzzy game with value $v$. Let the optimal strategy of player-1 be $\mu^*$. Let $\phi$ be a function from $Y$ to $\tilde{R}$.

Let $\overline{\nu}$ be an optimal strategy of player-2. Then $Q(\mu^*, \overline{\nu}) = v$. Therefore
\[
Q(\mu^*, \overline{\nu}) = \bigwedge_{y \in Y} Q(\mu^*, y) = v.
\]
Let $\phi(y) = Q(\mu^*, y)$, then

$$
\phi(\lambda y_1 + (1 - \lambda)y_2) = Q(\mu^*, \lambda y_1 + (1 - \lambda)y_2)
$$

$$
= \int_X P(x, \lambda y_1 + (1 - \lambda)y_2)d\mu^*(x)
$$

$$
< \int_X [\lambda P(x, y_1) + (1 - \lambda)P(x_1, y_2)]d\mu^*(x)
$$

since $P$ is strictly convex

$$
= \lambda \int_X P(x_1, y_1)d\mu^*(x) + (1 - \lambda) \int_X P(x_2, y_2)d\mu^*(x)
$$

$$
= \lambda Q(\mu^*, y_1) + (1 - \lambda)Q(\mu^*, y_2)
$$

$$
= \lambda \phi(y_1) + (1 - \lambda)\phi(y_2).
$$

Therefore $\phi(y)$ is strictly convex. So there does not exist two optimal strategies for player-2. But by the theorem 5.1.3 there exists an optimal strategy $\overline{y}$ of the function $\phi(y)$.

Hence the theorem. \qed

**Definition 5.1.6.**

Let $X$ and $Y$ are compact sets and $P$ be a function from $X \times Y$ to $\mathbb{R}$, then $P$ is said to be convex with respect to $x \in X$ for fixed $y \in Y$ if

$$
P(\lambda x_1 + (1 - \lambda)x_2, y) \geq \lambda P(x_1, y) + (1 - \lambda)P(x_2, y),
$$

for $\lambda \in (0, 1)$.

**Definition 5.1.7.**

If in a two-person fuzzy game, $X$ and $Y$ are compact sets and the payoff function $P$ from $X \times Y$ to $\mathbb{R}$ is concave with respect to $x \in X$, for each fixed $y \in Y$, then the game is said to be a fuzzy with concave payoff function or a concave fuzzy game.

Now we only state the theorem based on Concave fuzzy game.
**Theorem 5.1.8.**
In a concave fuzzy game, the value of the game is \( v = \bigvee_x \bigwedge_y P(x, y) \), then player-1 has an optimal pure strategy.

**Theorem 5.1.9.**
If the payoff function \( P(x, y) \) is strictly concave with respect to \( x \in X \),

\[ i.e., \quad P(\lambda x_1 + (1 - \lambda)x_2, y) > \lambda P(x_1, y) + (1 - \lambda)P(x_2, y), \]

then player-1 has a unique optimal strategy.

**Theorem 5.1.10.**
In a concave (or respectively convex) fuzzy game the value of the game is

\[ v = \bigwedge_y \bigvee_x P(x, y) = \bigvee_x \bigwedge_y P(x, y). \]  \hspace{1cm} (5.1.1)

And there exists a saddle point \((x^*, y^*)\) where \( x^* \in X \subset \mathbb{R}^m \) and \( y^* \in Y \subset \mathbb{R}^n \) are the pure strategies of players 1 and 2.

**Note-1** In this case, of the function \( P(x, y) \) is strictly concave (respectively convex) with respect to \( x \) (respectively \( y \)) for any fixed \( y \in Y \) (respectively \( x \in X \)) then player-1 (respectively player-2) has a unique optimal strategy.

**Note-2** We recall the famous Helley’s theorem of convex sets (crisp set) without proof. [Rockfellar (1970), Davidov]

**Theorem 5.1.11** (Helley’s Theorem). [Roc] and [Da]
Let \( K \) be a family of atleast \( n + 1 \) convex sets in \( \mathbb{R}^n \), each set from \( K \) being compact, then if each \( n + 1 \) of the sets of the family, there exists a point common to all the sets of the family \( K \).

**Lemma 5.1.12.**
Let \( P(x, y) \) be a continuous function from \( X \times Y \) to \( \mathbb{R} \), \( X \) and \( Y \) be compact sets and \( X^r = X \times X \times X \cdots \times X \) be the Cartesian product of \( r \) copies of \( X \). We define the function \( \phi \)
from \(X^r \times Y\) to \(\mathbb{R}\) as

\[
\phi(x_1, \ldots, x_r, y) = \bigvee_{1 \leq i \leq r} P(x_i, y),
\]

then the function \(\phi\) is continuous on \(X^r \times Y\).

**Proof.** Since the function \(P(x, y)\) is continuous with respect to the semi-norm defined by (Def 1.2.15) on the compact set \(X \times Y\) and hence is uniformly continuous. Then for any \(\delta > 0, \rho_1(x, \overline{x}) < \delta, \rho_2(y, \overline{y}) < \delta\), there exists \(\epsilon > 0\) such that \(\|P(x, y_1) - P(x, y_2)\| < \epsilon\).

Therefore

\[
\|\phi(x_1, \ldots, x_r, y_1) - \phi(x_1, \ldots, x_r, y_2)\| = \bigvee_{1 \leq i \leq r} P(x_i, y_1) - \bigvee_{1 \leq i \leq r} P(x_i, y_2)
\]

\[
= \|P(x_i_1, y_1) - P(x_i_2, y_2)\| < \epsilon,
\]

where \(\bigvee_{1 \leq i \leq r} P(x_i, y_1) = P(x_i_1, y_1)\) and \(\bigvee_{1 \leq i \leq r} P(x_i, y_2) = P(x_i_2, y_2)\).

Hence the theorem. \(\square\)

**Lemma 5.1.13.**

In a convex fuzzy game, the value of the game is

\[
v = \bigwedge_y \bigvee_x P(x, y) = \bigwedge_{1 \leq i \leq n+1} \bigvee_{1 \leq i \leq n+1} P(x_i, y),
\]

(5.1.2)

where \(y \in Y \cdot x_i \in X\) for \(i = 1, \ldots, n + 1\).

**Proof.** Since \(v = \bigwedge_y \bigvee_x P(x, y)\) for each point \((x_1, \ldots, x_{n+1}) \in X^{n+1}\),

\[
\bigwedge_{1 \leq i \leq n+1} \bigvee_{1 \leq i \leq n+1} P(x_i, y) \leq \bigwedge_{y} \bigvee_{x} P(x, y) = v
\]
we have

$$\bigvee_{1 \leq i \leq n+1} P(x_i, \overline{y}) \geq P(x_i, \overline{y}), \quad i = 1, \ldots, n+1$$

and

$$\bigwedge_{y} \bigvee_{1 \leq i \leq n+1} P(x_i, y) = \bigvee_{1 \leq i \leq n+1} P(x_i, \overline{y}) \geq P(x_i, \overline{y}).$$

For a fixed $x$, consider the set

$$D_x \equiv \{ y \mid P(x, y) \leq \Theta \},$$

where $\Theta \equiv \bigwedge_{y} \bigvee_{1 \leq i \leq n+1} P(x_i, y)$. $D_x$ is closed since $P(x, y)$ is continuous in $y$.

For $\lambda \in (0, 1)$,

$$D_{\lambda x_1 + (1-\lambda) x_2} = \{ y \mid P(\lambda x_1 + (1 - \lambda) x_2, y) \leq \Theta \}.$$

Now $y \in D_{\lambda x_1 + (1-\lambda) x_2}$ if and only if

$$P(\lambda x_1 + (1 - \lambda) x_2, y) \leq \Theta.$$

Since $P(x, y)$ is convex, we have

$$\lambda P(x_1, y) + (1 - \lambda) P(x_2, y) \leq P(\lambda x_1 + (1 - \lambda) x_2, y) \leq \Theta.$$

Therefore

$$\lambda P(x_1, y) + (1 - \lambda) P(x_2, y) \leq \Theta.$$

i.e., $y \in \lambda D_{x_1} + (1 - \lambda) D_{x_2}$.

Therefore

$$D_{\lambda x_1 + (1-\lambda) x_2} \subseteq \lambda D_{x_1} + (1 - \lambda) D_{x_2}.$$

Therefore $D_x$ is convex with respect to $x$.

Since $P(x, y)$ is convex and continuous in $y$ and $D_x$ is closed and convex, the sets $\{ D_x \}$ forms a system of convex compact sets on $\mathbb{R}^n$. 
Since \( P(x_i, y) \leq \Theta, y \in Y, i = 1, \ldots, n + 1 \) always has a solution. Therefore there exists a point \( y_0 \in Y \) such that \( P(x, y_0) \leq \Theta \) for any \( x \in X \) by Helley’s theorem.

Suppose that \( \Theta \neq v \), then \( \Theta < v = \bigwedge_{y} \bigvee_{x} P(x, y) \leq \bigvee_{x} P(x, y_0) \leq \Theta \), which is a contradiction.

Hence the theorem. \( \square \)

**THEOREM 5.1.14.**

*In a convex fuzzy game, player-1 has an optimal mixed strategy \( \mu^* \) with the finite spectrum.*

**Note:** Proof of the theorem is based on Helley’s theorem of convex sets.

**PROOF.** From the lemma 5.1.13 we have

\[
v = \bigwedge_{y} \bigvee_{1 \leq i \leq n + 1} P(x_i, y), \quad x_i \in X, i = 1, \ldots, n + 1.
\]

Consider the function

\[
Q(q, y) = \sum_{i=1}^{n+1} P(x_i, y)q_i, \quad y \in Y \subset \mathbb{R}^n;
\]

\( q = (q_1, \ldots, q_{n+1}) \in \mathbb{R}^{n+1}, q_i \geq 0 \) and \( \sum_{i=1}^{n+1} q_i = 1 \). The function \( Q(q, y) \) is continuous in \( q \) and \( y \).

Also \( Q(q, y) \) is convex in \( y \) and concave in \( q \).

Therefore, we have

\[
v = \bigwedge_{y} \bigvee_{q} \sum_{i=1}^{n+1} P(x_i, y)q_i
\]

\[
= \bigvee_{q} \bigwedge_{y} \sum_{i=1}^{n+1} P(x_i, y)q_i. \tag{5.1.3}
\]

So from (5.1.2) and (5.1.3) it follows that there exists \( q^* \in X \subset \mathbb{R}^{n+1} \) and \( y^* \in Y \subset \mathbb{R}^n \) such that for all \( x \in X \) and \( y \in Y \)

\[
P(x, y^*) \leq v \leq \sum_{i=1}^{n+1} H(x_i, y)q^*_i.
\]

Hence the theorem. \( \square \)