CHAPTER 4

Games with continuous fuzzy payoff function

4.0. Introduction

A two-person zero-sum game is said to be a continuous game where $X$ and $Y$ are compact metric spaces and the payoff function $P(x, y)$ is continuous in both the variables $x$ and $y$ [Pet,Z]. The sets of probability measures of players 1 and 2 on $\sigma$-algebras $\mathcal{X}$ and $\mathcal{Y}$ are denoted by $\mathcal{X}$ and $\mathcal{Y}$ and $Q(\mu, \nu)$ is the payoff of player-1 in the situation $(\mu, \nu)$ in mixed strategies.

In this chapter, we define a continuous fuzzy game with payoff function $P(x, y)$ from $X \times Y$ to $\tilde{R}$ and keeping the other aspects of the game as crisp. We prove some related results also.

4.1. Continuous fuzzy game

**Definition 4.1.1.**

Let $X$ and $Y$ be compact metric spaces and the payoff function $P$ be a continuous function from $X \times Y$ to $\tilde{R}$, where $\tilde{R}$ is the set of all fuzzy numbers. Then the game is called a continuous fuzzy game.

**Lemma 4.1.2.**

Let $X$ and $Y$ be compact metric spaces and $P$ be a continuous function from $X \times Y$ to $\tilde{R}$ then $Q(\mu, y)$ and $Q(x, \nu)$ are continuous functions of $y$ and $x$ respectively for $\mu \in \mathcal{X}$ and $\nu \in \mathcal{Y}$.
PROOF. The function $Q(\mu, y)$ and $Q(x, \nu)$ are defined as

$$Q(\mu, y) = \int_X P(x, y) d\mu(x)$$

and

$$Q(x, \nu) = \int_Y P(x, y) d\nu(y).$$

Since $P(x, y)$ is a continuous functions from $X \times Y$ to $\mathbb{R}$ with respect to the semi-norm as defined in Chapter-1 and is uniformly continuous, i.e., $\|P(x, y_1) - P(x, y_2)\| < \epsilon$ for given $\epsilon > 0$ then exists $\delta > 0$ such that $\rho(y_1, y_2) < \delta$ for every $y \in Y$.

Therefore,

$$\|Q(\mu_1, y) - Q(\mu, y_2)\| = \left\| \int_X P(x, y_1) d\mu(x) - \int_X P(x, y_2) d\mu(x) \right\|$$

$$\leq \left\| \int_X (P(x, y_1) - P(x, y_2)) d\mu(x) \right\|$$

$$\leq \int_X \|P(x, y_1) - P(x, y_2)\| d\mu(x)$$

$$< \int_X \epsilon d\mu(x) = \epsilon \int d\mu(x).$$

So $\frac{1}{\epsilon} \|Q(\mu_1, y) - Q(\mu_2, y)\| d\mu(x) \rightarrow 0$ as $\epsilon \rightarrow 0$ i.e., $Q(\mu, y)$ is continuous.

Similarly, we can show that $Q(x, \nu)$ is also continuous. \qed

**Lemma 4.1.3.**

Let $X$ and $Y$ be compact metric spaces and the payoff function $P$ be a continuous function from $X \times Y$ to $\mathbb{R}$ in an infinite zero-sum game, then

$$\underline{v} = \bigvee_{\mu \in \mathcal{X}} \bigwedge_{y \in Y} Q(\mu, y)$$

and

$$\overline{v} = \bigwedge_{\nu \in \mathcal{Y}} \bigvee_{x \in X} Q(x, \nu)$$

where $\underline{v}$ and $\overline{v}$ are the lower and upper values of the game fuzzy game.
PROOF. By the previous lemma, since \( P(x, y) \) is continuous, 
\[
Q(\mu, y) = \int_X P(x, y) \, d\mu(x)
\]
is continuous in \( y, y \in Y \) where \( Y \) is a compact metric space.

So \( Q(\mu, y) \) attains a minimum at a particular point \( y \in Y \).

For any \( n \), there exists \( \mu_n \in X \) such that

\[
\bigwedge_y Q(\mu_n, y) \geq v - \frac{1}{n},
\]

Since \( X \) is compact, there exists a subsequence from the sequence \( \{\mu_n\} \), \( \mu_n \in X \). Suppose the sequence \( \{\mu_n\} \) converges to a measure \( \mu_0 \in X \).

\[
\therefore \lim_{n \to \infty} Q(\mu_n, y) = \lim_{n \to \infty} \int_X P(x, y) \, d\mu_n(x) \\
= \int_X P(x, y) \, d\mu_0(x) \\
= Q(\mu_0, y), \ y \in Y.
\]

But \( Q(\mu_0, y) \) is not less than \( v \) for every \( y \in Y \). Hence \( \bigwedge_y Q(\mu_0, y) \geq v \).

So \( Q(\mu_0, y) \) attains a maximum at \( \mu_0 \in X \), i.e.,

\[
v \geq \bigvee_y \bigwedge_y Q(\mu, y).
\]

Similarly,

\[
\overline{v} = \bigvee_x \bigwedge_x Q(x, \nu).
\]

\( \square \)

**Theorem 4.1.4.**

Let \( X \) and \( Y \) be compact metric spaces and the payoff function. \( P \) from \( X \times Y \) to \( \mathbb{R} \) be a continuous function. Then the infinite zero-sum continuous game has a solution.

i.e.,

\[
v = \underline{v} = \overline{v}.
\]
where
\[ v = \bigvee_{\mu} \bigwedge_{y} Q(\mu, y) \quad \text{and} \quad \bar{v} = \bigvee_{x} \bigwedge_{\nu} Q(x, \nu). \]

**PROOF.** Let \( A_n = [\alpha^n_{ij}] \) where \( \alpha^n_{ij} = P(x^n_i, y^n_j) \), \( P \) is the payoff function from \( X \times Y \) to \( \tilde{R} \) for \( x^n_i \in X_n \) and \( y^n_j \in Y_n \), \( X_n \subset X \) and \( Y_n \subset Y \) since \( X \) and \( Y \) are compact metric spaces.

Since the payoff function \( P(x, y) \) is uniformly continuous, for given \( \epsilon > 0 \) and \( x \in X \) there exists \( \delta > 0 \) such that \( \rho(y, y^n_j) < \delta \)
then \( \| P(x, y) - P(x, y^n_j) \| < \epsilon \) where the strategy \( \mu_n \in \bar{X} \) and
\[ Q(\mu_n, y^n_j) = \sum_{j=1}^{r_n} \alpha^n_{ij} \Pi^n_i \geq \tilde{\theta}_n, \]
where \( \tilde{\theta}_n \) is the game value of the matrix \( A_n \) and \((\Pi^n_1, \ldots, \Pi^n_{r_n})\) is the optimal mixed strategy for the player-1.

So for any \( y \in Y \), \( Q(\mu_n, y) > \tilde{\theta}_n - \epsilon \).
By lemma 4.1.3, \( \bigwedge_{y} Q(\mu_n, y) > \tilde{\theta}_n - \epsilon \).
Then
\[ v = \bigvee_{\mu} \bigwedge_{y} Q(\mu, y) > \tilde{\theta}_n - \epsilon. \quad (4.1.1) \]
Similarly, \( Q(x^n_i, \nu_n) \leq \tilde{\theta}_n \), for the mixed strategy \( \nu_n \in \bar{Y} \).
So for any \( x \in X \),
\[ Q(x, \nu_n) < \tilde{\theta}_n + \epsilon. \]
\[ \therefore \bar{v} = \bigvee_{x} \bigwedge_{\nu} Q(x, \nu) < \tilde{\theta}_n + \epsilon. \quad (4.1.2) \]
From (4.1.1) and (4.1.2),
\[ v > \bar{v} - 2\epsilon. \quad (4.1.3) \]
But already we proved that \( v \leq \bar{v} \). So since \( \epsilon > 0 \), we obtain \( v = \bar{v} = v. \) \( \square \)
**Theorem 4.1.5.**

Let $X$ and $Y$ be compact metric spaces and $P(x, y)$ be a continuous function from $X \times Y$ to $\mathbb{R}$ then the infinite two-person zero-sum fuzzy game has $\epsilon$-optimal mixed strategies $\mu_n$ and $\nu_n$ for any $\epsilon > 0$.

**Proof.** Consider a fuzzy matrix game $A_n = [\alpha^n_{ij}]$ ($\alpha^n_{ij}$ are fuzzy numbers) such that $\alpha^n_{ij} = Q(x^n_i, y^n_j)$ where $x^n_i \in X_n$, $y^n_j \in Y_n$ and the mixed strategies of player-1 are $\mu_n \in \overline{X}$ and of player-2 are $\nu_n \in \overline{Y}$.

Then $Q(\mu_n, \nu_n) = \overline{\theta}_n$ be the value of the game where $Q(\mu, \nu)$ is the fuzzy payoff in mixed strategies in the game.

From theorem 4.1.4, we have $Q(\mu_n, y) > \overline{\theta}_n - \epsilon$ and $Q(x, \nu_n) < \overline{\theta}_n + \epsilon$.

$\therefore$ For all $x \in X$ and $y \in Y$

$$Q(x, \nu_n) - \epsilon < \overline{\theta}_n < Q(\mu_n, y) + \epsilon.$$  

Thus the game has $\epsilon$-optimal mixed strategies $\mu_n$ and $\nu_n$ for any $\epsilon > 0$. \qed