Chapter 3

Estimation of Augmented Gamma Strength Reliability for Non-Identical Stress-Strength set up

3.1 Introduction

In real life situations, it is noticed that the choice of identical distributions for stress and strength random variables may not be feasible and it is attempted due to mathematical simplicity. The stress imposed on the system can be controlled and may be identical to the strength, when the system is being tested or working in controlled and sophisticated laboratory. But in real life situation, many forces viz. temperature, pressure, voltage etc. are applied and which are not in under human control. Keeping this view in mind, this chapter presents the augmented strength reliability and its classical and Bayesian estimation by considering that the original strength (X) of the equipment and the common stress (Y) imposed on it are independent but non-identically distributed as two parameter gamma distribution.

In this chapter, it is assumed that the strength (X) and stress (Y) of the equipment are independently but non-identically distributed as gamma distribution with parameters $(\alpha_1, \lambda_1)$ and $(\alpha_2, \lambda_2)$ respectively. The probability density functions (PDF) of random variables $X$ (or $Y$) are given as

$$f_X(x; \alpha_p, \lambda_p) = \frac{\alpha_p^\lambda_p}{\Gamma(\lambda_p)} x^{\lambda_p-1} \exp(-\alpha_p x); \quad p = 1, 2; \quad x > 0, \alpha_p > 0, \lambda_p > 0$$

(3.1)
where \( \alpha_p \) and \( \lambda_p \) \((p=1,2)\) are the scale and shape parameters of strength \((p=1)\) and stress \((p=2)\) gamma random variables. The cumulative distribution function and survival function are defined as in equations (2.2) and (2.5) for real valued shape parameters and in equations (2.4) and (2.6) for integer valued shape parameters respectively of Chapter 2.

Constantine et al. (1986) firstly considered the two parameter gamma distribution as stress-strength model and derived the ML and uniformly minimum variance unbiased estimators (UMVUE) of system reliability assuming that both the shape parameters are known integers. In similar fashion Constantine et al. (1989 & 1990) attempted bootstrap study to compare different methods of constructing confidence intervals of system reliability under the same assumption. Recently, Huang et al (2012) developed ML and UMVUE of stress-strength reliability and compared through Monte-Carlo simulation assuming that the two shape parameters are any positive real numbers.

In this study, the augmented strength reliability expressions under the generalized form of ASP are derived assuming that both the shape parameters \( \lambda_1 \) and \( \lambda_2 \) are any positive real numbers. The ML and Bayes estimators of augmented strength reliability are computed by assuming that the shape parameters \( \lambda_1 \) and \( \lambda_2 \) and scale parameters \( \alpha_1 \) and \( \alpha_2 \) are unknown constants. The Bayes estimators are calculated for informative (gamma and inverse gamma) and non-informative (uniform and Jeffrey’s) types of priors under symmetric (SELF) and asymmetric (LLF) loss functions. The proposed estimators are compared through simulated and real data sets on the basis of their mean square errors and absolute biases.

Rest of the chapter is organized as follow. Section 3.2 presents the generalized augmented strength reliability under ASP for real and integer valued shape parameters. In section 3.3, the ML estimators of augmented strength reliability for unknown shape and scale parameters. Bayes estimators of augmented strength reliability parameters for informative types of priors (gamma and inverted gamma priors) and non-informative priors (uniform and Jeffrey’s priors) under SELF and LLF are considered in section 3.4. A numerical illustration of proposed estimators of augmented strength reliability through real
data sets along with the simulation study are reported in section 3.5. Finally, the conclusion of the article is given in section 3.6.

### 3.2 Generalized Augmented Strength Reliability

In this section, the generalized augmented strength reliability under ASP, which consists of three cases, is derived. The generalized case of ASP to enhance the strength reliability of a system can be stated as “The strength of the equipment is increased by adding ‘n’ identical components each having increased strength ($X'_i$) is set to face the common stress of the equipment ($Y$). Thus the augmented strength $Z_k = \sum_{i=1}^{n} X'_i$ follow Gamma distribution with parameters ($\alpha_i/m , n\lambda_i$), where, $X'_i (i = 1, 2, 3, \ldots, n)$ is ‘m’ times of initial stress of the component, which follows gamma ($\alpha_i/m , \lambda_i$) independently.” The probability density function (PDF) of augmented strength ($Z_k; k = 1, 2, 3$) under generalized case of ASP is given as

$$f_{Z_k} (z_k / \alpha_i, \lambda_i) = \frac{1}{\Gamma(n\lambda_i)} \left( \frac{\alpha_i}{m} \right)^{n\lambda_i} \exp \left( -\frac{\alpha_i z_k}{m} \right) z_k^{n\lambda_i-1} ; z_k > 0, \alpha_i, \lambda_i > 0 \quad (3.2)$$

where 'm' is a positive real number and 'n' is positive integer. The PDFs of case-I, case-II and case-III can directly be obtained by substituting ($k = 1, n = 1$), ($k = 2, m = 1$) and ($k = 3$) respectively in equation (3.2). The augmented system reliability parameter $R_k = P(Z_k > Y)$, when the shape parameters are assumed to real valued parameters, is given by

$$R_k = \Pr(Z_k > Y) = P \left( \frac{Z_k}{Y} > 1 \right). \quad (3.3)$$
It may be noted that, if \( Z_k \sim \text{Gamma}(\alpha_1/m, n \lambda_1) \) and \( Y \sim \text{Gamma}(\alpha_2, \lambda_2) \) then, \( \frac{2\alpha_1 Z_k}{m} \sim \chi^2(2n \lambda_1) \) and \( 2\alpha_2 Y \sim \chi^2(2\lambda_2) \) which are independently distributed. Thus the augmented strength reliability becomes

\[
R_k = P \left( \frac{2(\alpha_1/m) Z_k}{2\alpha_2 Y} > \frac{\lambda_2}{n \lambda_1} \frac{1}{\rho} \right) = P \left( \tau > \frac{\lambda_2}{n \lambda_1} \frac{1}{\rho} \right)
\]

\[
= 1 - F \left( \frac{\lambda_2}{\rho n \lambda_1}; 2n \lambda_1, 2\lambda_2 \right)
\]

(3.4)

where \( \rho = m \alpha_2 / \alpha_1 \) and \( F(.; 2n \lambda_1, 2\lambda_2) \) is cumulative distribution of F-distribution with \( 2n \lambda_1 \) and \( 2\lambda_2 \) degrees of freedom.

**Remark: (iv).** Let us suppose that one of the shape parameters (\( \lambda_1 \), say) is an integer. Then the system reliability \( R_k \) can be expressed as a finite sum, given as follows

\[
R_k = \Pr(Z_k > Y) = \int_0^\infty \int_0^y f_{Z_k}(z_k) dz_k \int_0^{\infty} g_Y(y) dy = \int_0^\infty F_{Z_k}(y) g_Y(y) dy
\]

\[
= \int_0^{\infty} \exp(-\alpha_1 y/m) \sum_{i=0}^{n \lambda_1-1} (\alpha_1/m)^i \frac{\alpha_2}{\Gamma(\lambda_2)} y^{\lambda_2-1} \exp(-\alpha_2 y) dy
\]

\[
= \sum_{i=0}^{n \lambda_1-1} \frac{\alpha_1^i \alpha_2 \lambda_2}{m^i i! \Gamma(\lambda_2)} \int_0^{\infty} \left( \frac{\alpha_1/m + \alpha_2}{\Gamma(\lambda_2 + i)} \right)^{\lambda_2+i} y^{\lambda_2+i-1} e^{-(\alpha_1/m + \alpha_2)y} dy
\]

\[
= \sum_{i=0}^{n \lambda_1-1} \frac{\Gamma(\lambda_2 + i)}{m^i i! \Gamma(\lambda_2)} \left( \frac{m}{\alpha_1/m + \alpha_2} \right)^i \left( \frac{\rho}{1 + \rho} \right)^{\lambda_2}
\]

(3.5)

where the terms in the integral part of the expression are the density function of \( \text{Gamma}(\alpha_1/m + \alpha_2, \lambda_2 + i) \) becomes unity and \( \rho = m \alpha_2 / \alpha_1 \).
The augmented strength reliability expressions for case-I, case-II and case-III under ASP for real as well as integer valued shape parameters can be obtained directly by substituting

\( k = 1, n = 1, \quad k = 2, m = 1 \) and \( k = 3 \) in equations (3.4) as well as (3.5) respectively.

### 3.3 Maximum Likelihood Estimation of Generalized Augmented Strength Reliability \( \hat{R}_k \)

This section deals with the maximum likelihood estimation of parameters of augmented strength reliability \( \hat{R}_k \) by considering the model parameters \( \alpha_1, \alpha_2, \lambda_1 \) and \( \lambda_2 \) unknown. Suppose \( Z_k = (z_{k1}, z_{k2}, \ldots, z_{kn_k}) \) and \( Y = (y_1, y_2, \ldots, y_{n_y}) \) be the two independent random samples of sizes \( n_1 \) and \( n_2 \) drawn from the distributions of augmented strength and the common stress respectively. Then the likelihood function based on the strength and stress random observations is given by

\[
L_k = L_k(\alpha_1, \lambda_1, \alpha_2, \lambda_2 / z_k, y) = \prod_{i=1}^{n_1} f_{z_k}(z_{ki}) \prod_{j=1}^{n_2} g_Y(y_j)
\]

\[
= \prod_{i=1}^{n_1} \left[ \frac{1}{\Gamma(n\lambda_1)} \left( \frac{\alpha_1}{m} \right)^{n\lambda_1} \exp\left( -\frac{\alpha_1 z_{ki}}{m} \right) z_{ki}^{n\lambda_1 - 1} \right] \times \prod_{j=1}^{n_2} \left[ \frac{\alpha_2 \lambda_2}{\Gamma(\lambda_2)} y_j^{\lambda_2 - 1} \exp\left( -\lambda_2 y_j \right) \right]
\]

\[
L_k = \alpha_1^{n_1 n_1 \lambda_1} \alpha_2^{n_2 \lambda_2} m^{-n_1 n_1 \lambda_1} \left\{ \Gamma(n\lambda_1) \right\}^{-n_1} \left\{ \Gamma(\lambda_2) \right\}^{-n_2} \exp\left[ -\left( \frac{\alpha_1 n_1 \bar{z}_k + \alpha_2 n_2 \bar{y}}{m} \right) \right]
\]

\[
+ (n\lambda_1 - 1)s_1 + (\lambda_2 - 1)s_2 \tag{3.6}
\]

where \( \bar{z}_k = \frac{1}{n_1} \sum_{i=1}^{n_1} z_{ki} \) and \( \bar{y} = \frac{1}{n_2} \sum_{j=1}^{n_2} y_j \). The product terms \( \prod_{i=1}^{n_1} z_{ki}^{(n\lambda_1 - 1)} \) and \( \prod_{j=1}^{n_2} y_j^{(\lambda_2 - 1)} \) can be expressed as

\[
\prod_{i=1}^{n_1} z_{ki}^{(n\lambda_1 - 1)} \prod_{j=1}^{n_2} y_j^{(\lambda_2 - 1)} = \exp\left[ \log \left\{ \prod_{i=1}^{n_1} z_{ki}^{(n\lambda_1 - 1)} \right\} \right] \exp\left[ \log \left\{ \prod_{j=1}^{n_2} y_j^{(\lambda_2 - 1)} \right\} \right]
\]
\[
\prod_{i=1}^{n_1} z_{ki}^{n_1 \lambda_i - 1} \prod_{j=1}^{n_2} y_j^{n_j \lambda_j - 1} = \exp\left( (n_1 \lambda_1 - 1) s_1 \right) \exp\left( (n_2 \lambda_2 - 1) s_2 \right)
\]

Taking logarithm in both sides of equation (3.6), the log-likelihood function is given by

\[
\log L_k = nn_1 \lambda_1 (\log \alpha_1 - \log m) + n_2 \lambda_2 \log \alpha_2 - n_1 \log \{\Gamma(n_1 \lambda_1)\} - n_2 \log \{\Gamma(n_2 \lambda_2)\} - \frac{\alpha_1 n_1 z_k}{m} - \alpha_2 n_2 y + (n_1 \lambda_1 - 1) s_1 + (n_2 \lambda_2 - 1) s_2.
\]

Assuming that the scale parameters \( \alpha_1 \) and \( \alpha_2 \) as well as the shape parameters \( \lambda_1 \) and \( \lambda_2 \) are unknown constants, the MLEs \( \hat{\alpha}_1, \hat{\alpha}_2, \hat{\lambda}_1 \) and \( \hat{\lambda}_2 \) of \( \alpha_1, \alpha_2, \lambda_1 \) and \( \lambda_2 \) can be obtained as the simultaneous solution of the following partial derivative equations with respect to \( \alpha_1, \alpha_2, \lambda_1 \) and \( \lambda_2 \) respectively, which are given by

\[
\frac{\partial \log L_k}{\partial \alpha_1} = 0 \Rightarrow \hat{\alpha}_1 = \frac{mn \lambda_1}{z_k}
\]

\[
\frac{\partial \log L_k}{\partial \alpha_2} = 0 \Rightarrow \hat{\alpha}_2 = \frac{\lambda_2}{\bar{y}}
\]

\[
\frac{\partial \log L_k}{\partial \lambda_1} = 0 \Rightarrow nn_1 \log (\alpha_1/m) - n_1 \psi(n_1 \lambda_1) + ns_1 = 0
\]

\[
\frac{\partial \log L_k}{\partial \lambda_2} = 0 \Rightarrow n_2 \log \alpha_2 - n_2 \psi(\lambda_2) + s_2 = 0
\]

where \( \psi(.) \) is the digamma function, which is defined as \( \psi(x) = \partial \ln \Gamma(x)/\partial x, \forall x > 0 \). It is noticed from the likelihood equations, simultaneous trivial solutions are not possible. Thus, the maximum likelihood estimators \( \hat{\alpha}_1, \hat{\alpha}_2, \hat{\lambda}_1 \) and \( \hat{\lambda}_2 \) can be obtained by using any numerical iterative technique. The Newton-Raphson technique is used to find out the simultaneous solution of the above likelihood equations. The ML estimator of \( R_k \), denoted
as $\hat{R}_k$ can be easily obtained due to invariance property of MLE by replacing $\lambda_1 = \hat{\lambda}_1, \lambda_2 = \hat{\lambda}_2$ and $\rho = \hat{\rho}$; where $\hat{\rho} = m\hat{\alpha}_2/\hat{\alpha}_1$ in the expression of $R_k$ given in (3.4), we have

$$\hat{R}_k = 1 - F\left(\frac{\hat{\lambda}_2}{\hat{\rho}n\hat{\lambda}_1}; 2n\hat{\lambda}_1, 2\hat{\lambda}_2\right) \tag{3.13}$$

The ML estimates of $R_k$ for known values of shape and scale parameters are also discussed as special cases in the following remarks.

**Remark: (v).** When both the shape parameters $(\lambda_1, \lambda_2)$ are known then the MLEs $\hat{\alpha}_1$ and $\hat{\alpha}_2$ of scale parameters $\alpha_1$ and $\alpha_2$ respectively can easily be found from equations (3.9) and (3.10) by keeping fixed values of $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

**Remark: (vi).** When the scale parameters $(\alpha_1, \alpha_2)$ are known then the MLEs $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of shape parameters $\lambda_1$ and $\lambda_2$ can also be obtained through Newton-Raphson method from equations (3.11) and (3.12) by keeping fixed values of $\hat{\alpha}_1$ and $\hat{\alpha}_2$.

### 3.4 Bayesian Estimation of Generalized Augmented strength Reliability

This section presents the Bayesian estimation of generalized augmented strength reliability $(R_k; k = 1, 2, 3)$ for informative (gamma and inverted gamma priors) as well as non-informative (uniform and Jeffery’s priors) types of priors under two different loss functions symmetric (SELF) and asymmetric (LLF) are considered when each of the scale parameters $(\alpha_1, \alpha_2)$ and shape parameters $(\lambda_1, \lambda_2)$ of stress and strength distributions are unknown.
3.4.1 Assuming gamma prior:

In this subsection, it is assumed that \( \alpha_1, \alpha_2, \lambda_1 \) and \( \lambda_2 \) are independent random variables having gamma prior distributions i.e. \( \alpha_1 \sim \text{Gamma}(a_1, b_1) \), \( \alpha_2 \sim \text{Gamma}(a_2, b_2) \); \( \lambda_1 \sim \text{Gamma}(c_1, d_1) \) and \( \lambda_2 \sim \text{Gamma}(c_2, d_2) \). The hyper-parameters \( a_1, a_2, c_1, c_2 \) are the scale hyper-parameters and \( b_1, b_2, d_1, d_2 \) are the shape hyper-parameters and are assumed to be known and chosen in such a way to reflect the prior belief about the unknown parameters. Thus, their respective prior density function are given as

\[
h_1(\alpha_p) = \frac{\alpha_p^{b_p-1} \exp(-a_p \alpha_p)}{\Gamma(b_p)}; \quad \alpha_p, a_p, b_p > 0; \quad p = 1, 2 \tag{3.14}
\]

\[
h_2(\lambda_p) = \frac{\lambda_p^{d_p-1} \exp(-c_p \lambda_p)}{\Gamma(d_p)}; \quad \lambda_p, c_p, d_p > 0; \quad p = 1, 2. \tag{3.15}
\]

Thus, the joint prior distribution of \( \alpha_1, \alpha_2, \lambda_1 \) and \( \lambda_2 \) is the product of their marginal prior densities, given by

\[
g_1(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = C_1 \alpha_1^{b_1-1} \alpha_2^{b_2-1} \lambda_1^{d_1-1} \lambda_2^{d_2-1} \exp\left\{-(a_1 \alpha_1 + a_2 \alpha_2 + c_1 \lambda_1 + c_2 \lambda_2)\right\} \tag{3.16}
\]

where \( C_1 \) is an arbitrary constant defined as

\[
C_1 = \frac{a_1^{b_1} a_2^{b_2} c_1^{d_1} c_2^{d_2}}{\Gamma(b_1) \Gamma(b_2) \Gamma(d_1) \Gamma(d_2)} \tag{3.17}
\]

The joint posterior probability distribution of random variables \( \alpha_1, \alpha_2, \lambda_1 \) and \( \lambda_2 \) is obtained by combining both likelihood function \( L_k \) given in equation (3.6) and joint prior probability density function \( g_1(\alpha_1, \alpha_2, \lambda_1, \lambda_2) \) given in equation (3.16) via Bayes theorem, given by
\[ \Pi_{k1}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = \Pi_{k1}(\alpha_1, \alpha_2, \lambda_1, \lambda_2 / z_k, y) \]

\[ = K_1 L_k(\alpha_1, \alpha_2, \lambda_1, \lambda_2 / z_k, y) g_1(\alpha_1, \alpha_2, \lambda_1, \lambda_2) \quad (3.18) \]

where \( K_1 \) is the denominator part of the joint posterior distribution also known as the normalizing constant, defined by

\[ K_1^{-1} = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty L_k(\alpha_1, \alpha_2, \lambda_1, \lambda_2 / z_k, y) g_1(\alpha_1, \alpha_2, \lambda_1, \lambda_2) \, d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2 \quad (3.19) \]

Substituting the values of \( L_k(\alpha_1, \alpha_2, \lambda_1, \lambda_2 / z_k, y) \) and \( g_1(\alpha_1, \alpha_2, \lambda_1, \lambda_2) \) from (3.6) and (3.16) respectively in (3.18), some of the constant terms are cancelled from both numerator and denominator. We have the joint posterior density of \( \alpha_1, \alpha_2, \lambda_1 \) and \( \lambda_2 \) as

\[ \Pi_{k1}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = K_1 \frac{\alpha_1^{n_1 \lambda_1 + b_1 - 1} \alpha_2^{n_2 \lambda_2 + b_2 - 1} \lambda_1^{d_1 - 1} \lambda_2^{d_2 - 1}}{m^{n_1 \lambda_1} \{\Gamma(n \lambda_1)\}^{n_1} \{\Gamma(\lambda_2)\}^{d_2}} \exp \left[ -\left( \alpha_1 \left( \frac{n_1 \bar{z}_k}{m} + a_1 \right) + \alpha_2 \left( n_2 \bar{y} + a_2 \right) + \lambda_1 (c_1 - n s_1) + \lambda_2 (c_2 - s_2) \right) \right] \quad (3.20) \]

where the normalizing constant \( K_1 \) is defined as

\[ K_1^{-1} = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{\alpha_1^{n_1 \lambda_1 + b_1 - 1} \alpha_2^{n_2 \lambda_2 + b_2 - 1} \lambda_1^{d_1 - 1} \lambda_2^{d_2 - 1}}{m^{n_1 \lambda_1} \{\Gamma(n \lambda_1)\}^{n_1} \{\Gamma(\lambda_2)\}^{d_2}} \exp \left[ -\left( \alpha_1 \left( \frac{n_1 \bar{z}_k}{m} + a_1 \right) + \alpha_2 \left( n_2 \bar{y} + a_2 \right) + \lambda_1 (c_1 - n s_1) + \lambda_2 (c_2 - s_2) \right) \right] \, d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2 \quad (3.21) \]

Thus, the joint posterior density given in (3.20) can be written as

\[
\Pi_{k1}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) \propto \left[ \alpha_1^{n_1 \lambda_1 + b_1 - 1} \exp \left( -\alpha_1 \left( \frac{n_1 \bar{z}_k}{m} + a_1 \right) \right) \right] \times \left[ \alpha_2^{n_2 \lambda_2 + b_2 - 1} e^{-\alpha_2 (n_2 \bar{y} + a_2)} \right] \\
\times \left[ \frac{\lambda_1^{d_1 - 1}}{m^{n_1 \lambda_1} \{\Gamma(n \lambda_1)\}^{n_1}} e^{-\lambda_1 (c_1 - n s_1)} \right] \times \left[ \frac{\lambda_2^{d_2 - 1}}{\{\Gamma(\lambda_2)\}^{d_2}} e^{-\lambda_2 (c_2 - s_2)} \right]
\]
\[
\propto G_{a_1} \left( \frac{n \bar{x}_k}{m} + a_1, (n \lambda_1 + b_1) \right) \times G_{a_2} \left( n \bar{y} + a_2, (n \lambda_2 + b_2) \right) 
\]

\[
\times \frac{\Gamma(n \lambda_1 + b_1) \lambda_1^{d_1-1} e^{-\lambda_1(c_1 - m_1)}}{\left( \frac{n \bar{x}_k}{m} + a_1 \right)^{(m \lambda_1 + b_1)}} m^{n \lambda_1} \{ \Gamma(n \lambda_1) \}^{n_1} 
\]

\[
\times \frac{\Gamma(n \lambda_2 + b_2) \lambda_2^{d_2-1} e^{-\lambda_2(c_2 - m_2)}}{(n \bar{y} + a_2)^{(n \lambda_2 + b_2)}} \{ \Gamma(n \lambda_2) \}^{n_2} 
\]

(3.22)

with the proportionality constant \( K_1 \), defined in (3.21) and \( G_{a_1}(...) \) and \( G_{a_2}(...) \) being the gamma functions. Thus the conditional marginal posterior densities of \( \alpha_1, \alpha_2, \lambda_1 \) and \( \lambda_2 \) to generate the random observations through M-H algorithm are given by

\[
\pi_{k11}(\alpha_1 / \bar{x}_k, \lambda_1) \propto \text{Gamma}\left[ \left( \frac{n \bar{x}_k}{m} + a_1, (n \lambda_1 + b_1) \right) \right] 
\]

(3.23)

\[
\pi_{k12}(\alpha_2 / \bar{y}, \lambda_2) \propto \text{Gamma}\left[ \left( n \bar{y} + a_2, (n \lambda_2 + b_2) \right) \right] 
\]

(3.24)

\[
\pi_{k13}(\lambda_1 / \bar{z}_k) \propto \frac{\Gamma(n \lambda_1 + b_1) \lambda_1^{d_1-1} e^{-\lambda_1(c_1 - m_1)}}{\left( \frac{n \bar{x}_k}{m} + a_1 \right)^{(m \lambda_1 + b_1)}} m^{n \lambda_1} \{ \Gamma(n \lambda_1) \}^{n_1} 
\]

(3.25)

\[
\pi_{k14}(\lambda_2 / y) \propto \frac{\Gamma(n \lambda_2 + b_2) \lambda_2^{d_2-1} e^{-\lambda_2(c_2 - m_2)}}{(n \bar{y} + a_2)^{(n \lambda_2 + b_2)}} \{ \Gamma(n \lambda_2) \}^{n_2} 
\]

(3.26)

**Bayes Estimator \( \hat{R}_{k1} \) under SELF**

Under the squared error loss function, the Bayes estimator \( \hat{R}_{k1} \) of augmented strength reliability \( R_k; k = 1, 2, 3 \) is defined by its posterior expectation and is given by

\[
\hat{R}_{k1} = E \left( R_k / z_k, y \right) = \int_0^\infty \int_0^\infty \int_0^\infty R_k \prod_{k11} (\alpha_1, \alpha_2, \lambda_1, \lambda_2) d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2
\]

(3.27)
Substituting the expressions of $R_k$ and $\Pi_{k_1}(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ from equations (3.4) and (3.20) respectively in equation (3.27), we have

$$g \hat{R}_k^{\text{self}} = K_1 \int \int \int \int R_k \left[ \alpha_1^{n_1, \lambda_1 + b_1 - 1} \exp \left\{ -\alpha_1 \left( \frac{n_1 z_k}{m} + a_1 \right) \right\} \times \alpha_2^{n_2, \lambda_2 + b_2 - 1} e^{-\alpha_2 (\lambda_2 \gamma + a_2)} \right]$$

$$\left\{ \frac{\hat{\lambda}_1^{d_1 - 1} m^{n_1 \lambda_1} \left\{ \Gamma(n \lambda_1) \right\}^{n_1}}{\Gamma(n \lambda_1) \Gamma(n \lambda_1)} \right\} \times \left\{ \frac{\hat{\lambda}_2^{d_2 - 1} e^{-\lambda_2 (\epsilon_2 - x_2)}}{\Gamma(n \lambda_2) \Gamma(n \lambda_2)} \right\} d\alpha_1 d\lambda_1 d\lambda_2 d\lambda_2$$

where $K_1$ is the denominator part, defined in (3.21), of the joint posterior density and equation (3.28) is the required Bayes estimator of $R_k$ under SELF.

**Bayes Estimator** $(g \hat{R}_k^{\text{llf}})$ **under LLF**

Under line loss function, the Bayes estimator $(g \hat{R}_k^{\text{llf}})$ of augmented strength reliability model defined by

$$g \hat{R}_k^{\text{llf}} = \frac{-1}{p} \ln \left[ E \left( e^{-p R_k} \mid z_k, y \right) \right] = \frac{-1}{p} \ln \left[ \int \int \int \int e^{-p R_k} \Pi_{k_1}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2 \right]$$

Substituting the expressions of $R_k$ and $\Pi_{k_1}(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ from equations (3.4) and (3.20) respectively in equation (3.29), we have

$$g \hat{R}_k^{\text{llf}} = \frac{-1}{p} \ln \left[ K_1 \int \int \int \int e^{-p R_k} \left\{ \alpha_1^{n_1, \lambda_1 + b_1 - 1} e^{-\alpha_1 \left( \frac{n_1 z_k}{m} + a_1 \right)} \right\} \times \alpha_2^{n_2, \lambda_2 + b_2 - 1} e^{-\alpha_2 (\lambda_2 \gamma + a_2)} \right]$$

$$\left\{ \frac{\hat{\lambda}_1^{d_1 - 1} m^{n_1 \lambda_1} \left\{ \Gamma(n \lambda_1) \right\}^{n_1}}{\Gamma(n \lambda_1) \Gamma(n \lambda_1)} \right\} \times \left\{ \frac{\hat{\lambda}_2^{d_2 - 1} e^{-\lambda_2 (\epsilon_2 - x_2)}}{\Gamma(n \lambda_2) \Gamma(n \lambda_2)} \right\} d\alpha_1 d\lambda_1 d\lambda_2 d\lambda_2$$

where $p$ is the shape parameter of LLF, which determine the amount of overestimation and underestimation. It is to be noted that the expressions of Bayes estimators of augmented strength reliability under each of the loss functions, SELF and LLF given in (3.28) and (3.30) respectively consist multiple integrals and therefore a close form solution is not
possible. Thus, the numerical approximation method known as MCMC technique via Metropolis-Hasting algorithm is used.

### 3.4.2 Assuming Inverted Gamma prior:

Assuming that $\alpha_1, \alpha_2, \lambda_1, \text{and } \lambda_2$ are independent random variables having prior distribution as inverted gamma distribution with their respective scale hyper-parameters as $a_1, a_2, c_1, c_2$ and the shape hyper-parameters as $b_1, b_2, d_1, d_2$ i.e. $\alpha_1 \sim \text{Inv.Gamma}(a_1, b_1)$, $\alpha_2 \sim \text{Inv.Gamma}(a_2, b_2)$; $\lambda_1 \sim \text{Inv.Gamma}(c_1, d_1)$ and $\lambda_2 \sim \text{Inv.Gamma}(c_2, d_2)$. Their respective prior density function are given as

\[
h_{21}(\alpha_p) = \frac{a_p^{b_p}}{\Gamma(b_p)}\alpha_p^{-b_p-1} \exp\left(-\frac{a_p}{\alpha_p}\right); \quad \alpha_p, a_p, b_p > 0; \quad p = 1, 2 \tag{3.31}
\]

\[
h_{22}(\lambda_p) = \frac{c_p^{d_p}}{\Gamma(d_p)}\lambda_p^{-d_p-1} \exp\left(-\frac{c_p}{\lambda_p}\right); \quad \lambda_p, c_p, d_p > 0; \quad p = 1, 2. \tag{3.32}
\]

Thus, the joint prior distribution of $\alpha_1, \alpha_2, \lambda_1 \text{ and } \lambda_2$ is the product of their marginal prior densities, given by

\[
g_2(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = C_1 \alpha_1^{-b_1-1} \alpha_2^{-b_2-1} \lambda_1^{-d_1-1} \lambda_2^{-d_2-1} \exp\left\{-\left(\frac{a_1}{\alpha_1} + \frac{a_2}{\alpha_2} + \frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}\right)\right\} \tag{3.33}
\]

where $C_1$ is an arbitrary constant, defined in (3.17). Thus, the joint posterior probability distribution of random variables $\alpha_1, \alpha_2, \lambda_1 \text{ and } \lambda_2$ for inverted gamma prior is obtained by combining both likelihood function $L_k(\alpha_1, \lambda_1, \alpha_2, \lambda_2 / z_k, y)$ and joint prior probability density function $g_2(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ via Bayes theorem as in (3.18). Further, substituting the values of $L_k(\alpha_1, \lambda_1, \alpha_2, \lambda_2 / z_k, y)$ and $g_2(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ given in equations (3.6) and (3.33) respectively, some of the constant terms get cancelled, we have
\[ \Pi_{k2}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = K_2 \frac{\alpha_1^{n_{k1}} \lambda_1^{-b_1-1}}{m_{n_{k1}} \{\Gamma(n)\}^{n_{k1}}} \frac{\alpha_2^{n_{k2}} \lambda_2^{-b_2-1}}{m_{n_{k2}} \{\Gamma(n)\}^{n_{k2}}} \exp \left[ - \left( \frac{\alpha_1 n_k \bar{z}_k + a_1}{m} + \frac{\alpha_2}{\alpha_1} \right) \right] 
+ \left( \alpha_2 n_2 \bar{y} + \frac{a_2}{\alpha_2} \right) + \left( \frac{c_1}{\lambda_1} - \lambda_1 n_s_1 \right) + \left( \frac{c_2}{\lambda_2} - \lambda_2 n_s_2 \right) \right] 
\] (3.34)

where the normalizing constant \( K_2 \) is defined as

\[ K_2^{-1} = \iiint_{0000} \frac{\alpha_1^{n_{k1}} \lambda_1^{-b_1-1}}{m_{n_{k1}} \{\Gamma(n)\}^{n_{k1}}} \frac{\alpha_2^{n_{k2}} \lambda_2^{-b_2-1}}{m_{n_{k2}} \{\Gamma(n)\}^{n_{k2}}} \exp \left[ - \left( \frac{\alpha_1 n_k \bar{z}_k + a_1}{m} + \frac{\alpha_2}{\alpha_1} \right) \right] 
+ \left( \alpha_2 n_2 \bar{y} + \frac{a_2}{\alpha_2} \right) + \left( \frac{c_1}{\lambda_1} - \lambda_1 n_s_1 \right) + \left( \frac{c_2}{\lambda_2} - \lambda_2 n_s_2 \right) \right] d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2 \] (3.35)

Thus, the joint posterior density of \( \alpha_1, \alpha_2, \lambda_1 \) and \( \lambda_2 \) given in (3.34) with proportionality constant \( K_2 \) can be represented as

\[ \Pi_{k2}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) \propto \left[ \alpha_1^{n_{k1} \lambda_1^{-b_1-1}} \exp \left\{- \left( \frac{\alpha_1 n_k \bar{z}_k + a_1}{m} \right) \right\} \right] 
\times \left[ \alpha_2^{n_{k2} \lambda_2^{-b_2-1}} \exp \left\{- \left( \frac{\alpha_2 n_2 \bar{y} + a_2}{\alpha_2} \right) \right\} \right] 
\times \left[ \frac{\lambda_1^{-d_1-1}}{m_{n_{k1}} \{\Gamma(n)\}^{n_{k1}}} \exp \left\{- \left( \frac{c_1}{\lambda_1} - n_s_1 \lambda_1 \right) \right\} \right] 
\times \left[ \frac{\lambda_2^{-d_2-1}}{\{\Gamma(n)\}^{n_{k2}}} \exp \left\{- \left( \frac{c_2}{\lambda_2} - n_s_2 \lambda_2 \right) \right\} \right] \] (3.36)

To generate the random observations through M-H algorithm, the conditional marginal posterior densities of \( \alpha_1, \alpha_2, \lambda_1 \) and \( \lambda_2 \) are respectively given by

\[ \pi_{k21}(\alpha_1 / z_k, \lambda_1) \propto \alpha_1^{n_{k1} \lambda_1^{-b_1-1}} \exp \left\{- \left( \frac{\alpha_1 n_k \bar{z}_k + a_1}{m} \right) \right\} \] (3.37)
\[ \pi_{k22}(\alpha_2 / y, \lambda_2) \propto \alpha_2^{n_2}\lambda_2^{-b_2-1} \exp \left\{ -\left( n_2 \bar{y} \alpha_2 + \frac{a_2}{\alpha_2} \right) \right\} \] (3.38)

\[ \pi_{k23}(\lambda_1 / z_k) \propto \frac{\lambda_1^{-d_1-1}}{m^{n_1\lambda_1} \{ \Gamma(n\lambda_1) \}^{n_1}} \exp \left\{ -\left( \frac{c_1}{\lambda_1} - ns_1\lambda_1 \right) \right\} \] (3.39)

\[ \pi_{k24}(\lambda_2 / y) \propto \frac{\lambda_2^{-d_2-1}}{\Gamma(\lambda_2)} \exp \left\{ -\left( \frac{c_2}{\lambda_2} - s_2\lambda_2 \right) \right\} \] (3.40)

**Bayes Estimator \( (\hat{R}_k) \) under SELF**

The Bayes estimator \( (\hat{R}_k) \) of augmented strength reliability \( R_k; k = 1, 2, 3 \) under the squared error loss function is defined by the posterior expectation of \( R_k \), given by

\[
_{IG} \hat{R}_k^{self} = E(R_k / z_k, y) = \int \int \int \int R_k \Pi_{k2} (\alpha_1, \alpha_2, \lambda_1, \lambda_2) d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2
\] (3.41)

Substituting the expressions of \( R_k \) and \( \Pi_{k2} (\alpha_1, \alpha_2, \lambda_1, \lambda_2) \) from equations (3.4) and (3.36) respectively in equation (3.41), we have

\[
_{IG} \hat{R}_k^{self} = K_2 \int \int \int \int R_k \left[ \alpha_1^{n_2}\lambda_1^{-b_1-1} \exp \left\{ -\left( \frac{\alpha_1 n_1 \bar{z}_k}{m} + \frac{a_1}{\alpha_1} \right) \right\} \right] \\
\times \left[ \alpha_2^{n_2}\lambda_2^{-b_2-1} \exp \left\{ -\left( \alpha_2 n_2 \bar{y} + \frac{a_2}{\alpha_2} \right) \right\} \right] \\
\times \frac{\lambda_1^{-d_1-1}}{m^{n_1\lambda_1} \{ \Gamma(n\lambda_1) \}^{n_1}} \exp \left\{ -\left( \frac{c_1}{\lambda_1} - ns_1\lambda_1 \right) \right\} \\
\times \frac{\lambda_2^{-d_2-1}}{\Gamma(\lambda_2)} \exp \left\{ -\left( \frac{c_2}{\lambda_2} - s_2\lambda_2 \right) \right\} \partial\alpha_1 \partial\alpha_2 \partial\lambda_1 \partial\lambda_2
\] (3.42)

where \( K_2 \) is the denominator part of the joint posterior density which is defined in (3.35) and the expression in (3.42) is the required Bayes estimator of \( R_k \) under SELF.
Bayes Estimator \(\hat{R}_k^{llf}\) under LLF

The Bayes estimator \(\hat{R}_k^{llf}\) of augmented strength reliability model \((R_k)\) under LINEX loss function for inverted gamma prior is defined by

\[
\hat{R}_k^{llf} = -\frac{1}{p} \ln \left[ E \left( e^{-\rho R_k} / z_k, y \right) \right] = -\frac{1}{p} \ln \left[ \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-\rho R_k} \Pi_{k2} (\alpha_1, \alpha_2, \lambda_1, \lambda_2) d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2 \right] (3.43)
\]

Substituting the expressions of \(R_k\) and \(\Pi_{k2} (\alpha_1, \alpha_2, \lambda_1, \lambda_2)\) from equations (3.4) and (3.36) respectively in equation (3.43), we get

\[
\hat{R}_k^{llf} = -\frac{1}{p} \ln \left[ K_2 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-\rho R_k} \alpha_1^{n_1} \lambda_1^{-b_1-1} \exp \left\{ - \left( \frac{\alpha_1 n_1 \bar{z}_k}{m} + \frac{a_1}{\alpha_1} \right) \right\} \right.
\]

\[
\times \left[ \alpha_2^{n_2} \lambda_2^{b_2-1} \exp \left\{ - \left( \frac{\alpha_2 n_2 \bar{y}}{\alpha_2} \right) \right\} \right]
\]

\[
\times \left[ \frac{\lambda_1^{d_1-1}}{m^{n_1} \lambda_1 \Gamma (n_1)} \exp \left\{ - \left( \frac{c_1}{\lambda_1} - n_1 \lambda_1 \right) \right\} \right]
\]

\[
\times \left[ \frac{\lambda_2^{d_2-1}}{\Gamma (n_2)} \exp \left\{ - \left( \frac{c_2}{\lambda_2} - s_2 \lambda_2 \right) \right\} \partial \lambda_1 \partial \lambda_2 \right] (3.44)
\]

where \(K_2\) is the denominator part of the joint posterior density which is defined in (3.35).

The Bayes estimators of augmented strength reliability \((R_k; k = 1,2,3)\) obtained for inverted gamma prior under SELF as well as LLF given in equations (3.42) and (3.44) respectively are not in explicit form and cannot be evaluated analytically. Hence alternatively, a numerical method viz., Markov Chain Monte Carlo (MCMC) sampling procedure via M-H algorithm is used to approximate the integrals.

### 3.4.3 Assuming Uniform Prior:

In this subsection, the model parameters \(\alpha_1, \alpha_2, \lambda_1\) and \(\lambda_2\) are assumed to be independent random variables having prior distribution as uniformly distributed. To find
out the Bayes estimators, we assume the shape \((\lambda_1, \lambda_2)\) and scale \((\alpha_1, \alpha_2)\) parameters of stress and strength are unknown. The prior densities of \(\alpha_1, \alpha_2, \lambda_1\) and \(\lambda_2\) are given by

\[
h_{31}(\alpha_p) = \frac{1}{\alpha_p^p}; \quad \alpha_p > 0, \quad p = 1, 2
\]  
(3.45)

\[
h_{32}(\lambda_p) = \frac{1}{\lambda_p^p}; \quad \lambda_p > 0, \quad p = 1, 2.
\]  
(3.46)

The joint prior density of random variables \(\alpha_1, \alpha_2, \lambda_1, \lambda_2\) can be obtained by taking product of their respective marginal prior densities, given as follows

\[
g_3(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = \frac{1}{\alpha_1 \alpha_2 \lambda_1 \lambda_2}; 0 < \alpha_1, \alpha_2, \lambda_1, \lambda_2 < \infty
\]  
(3.47)

Thus, the joint posterior probability density is obtained by combining both likelihood function \(L_k(\alpha_1, \alpha_2, \lambda_1, \lambda_2 / z, y)\) given in equation (3.6) and joint prior probability density function \(g_3(\alpha_1, \alpha_2, \lambda_1, \lambda_2)\) given in (3.47) via Bayes theorem as in (3.17), the constant terms from both the numerator and denominator get cancelled, given by

\[
\Pi_{k3}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = K_3 \frac{\alpha_1^{n_1 \lambda_1 - 1} \alpha_2^{n_2 \lambda_2 - 1} \lambda_1^{-1} \lambda_2^{-1}}{m^{n_1 \lambda_1} \left\{ \Gamma(n_1) \right\}^{n_1} \left\{ \Gamma(n_2) \right\}^{n_2}} \exp \left[ - \left\{ \alpha_1 m_1 \bar{z} m^{-1} + \alpha_2 n_2 \bar{y}_2 \\ - n \lambda_1 - s \lambda_2 \right\} \right]
\]  
(3.48)

where \(K_3\) is the normalizing constant, defined as

\[
K_3^{-1} = C_3 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{\alpha_1^{n_1 \lambda_1 - 1} \alpha_2^{n_2 \lambda_2 - 1} \lambda_1^{-1} \lambda_2^{-1}}{m^{n_1 \lambda_1} \left\{ \Gamma(n_1) \right\}^{n_1} \left\{ \Gamma(n_2) \right\}^{n_2}} \exp \left[ - \left\{ \alpha_1 m_1 \bar{z} m^{-1} + \alpha_2 n_2 \bar{y}_2 \\ - n \lambda_1 - s \lambda_2 \right\} \right] d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2
\]  
(3.49)

Thus, the joint posterior density given in (3.48) with proportionality constant \(K_3\) can be represented as
\[
\Pi_{k,3}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) \propto \left[ \alpha_1^{n_1, \lambda_1^{-1}} \exp\left\{-\alpha_1 \left( \frac{n_i \xi_k}{m} \right) \right\} \right] \times \left[ \alpha_2^{n_2, \lambda_2^{-1}} \exp\left\{-\alpha_2 \left( n_2 \bar{y} \right) \right\} \right] \\
\times \left[ \frac{\lambda_1^{-1} \exp\left( \lambda_1 n s_1 \right)}{m^{n_1, \lambda_1^1} \{ \Gamma(n \lambda_1)^{n_1} \}} \right] \times \frac{\lambda_2^{-1} \exp\left( \lambda_2 s_2 \right)}{\{ \Gamma(\lambda_2)^{n_2} \}} 
\]

\[
\propto G_{\alpha_1} \left( \frac{n_i \xi_k}{m}, n_1, \lambda_1 \right) \times G_{\alpha_2} \left( n_2 \bar{y}, n_2, \lambda_2 \right) \\
\times \left\{ \frac{\lambda_1^{-1} \Gamma(n \lambda_1) \exp(\lambda_1 n s_1)}{\left( \frac{n_i \xi_k}{m} \right)^{n_1, \lambda_1^1} m^{n_1, \lambda_1^1} \{ \Gamma(n \lambda_1)^{n_1} \}} \right\} \\
\times \left\{ \frac{\lambda_2^{-1} \Gamma(n_2 \lambda_2 + b_2) \exp(\lambda_2 s_2)}{(n_2 \bar{y})^{n_2, \lambda_2^2} \{ \Gamma(\lambda_2)^{n_2} \}} \right\} \tag{3.50}
\]

with the proportionality constant \( K_3 \), defined in (3.49). Thus the conditional marginal posterior densities of \( \alpha_1, \alpha_2, \lambda_1 \) and \( \lambda_2 \) to draw the random observations through M-H algorithm are given by

\[
\pi_{k,33}(\alpha_1 / \xi_k, \lambda_1) \propto \text{Gamma} \left( \frac{n_i \xi_k}{m}, n_1, \lambda_1 \right) \tag{3.51}
\]

\[
\pi_{k,32}(\alpha_2 / \bar{y}, \lambda_2) \propto \text{Gamma} \left( n_2 \bar{y}, n_2, \lambda_2 \right) \tag{3.52}
\]

\[
\pi_{k,33}(\lambda_1 / \xi_k) \propto \frac{\lambda_1^{-1} \Gamma(n \lambda_1) \exp(\lambda_1 n s_1)}{\left( \frac{n_i \xi_k}{m} \right)^{n_1, \lambda_1^1} m^{n_1, \lambda_1^1} \{ \Gamma(n \lambda_1)^{n_1} \}} \tag{3.53}
\]

\[
\pi_{k,34}(\lambda_2 / y) \propto \frac{\lambda_2^{-1} \Gamma(n_2 \lambda_2 + b_2) \exp(\lambda_2 s_2)}{(n_2 \bar{y})^{n_2, \lambda_2^2} \{ \Gamma(\lambda_2)^{n_2} \}} \tag{3.54}
\]
Bayes Estimator \( \hat{U}_k \) under SELF

Under the squared error loss function, the Bayes estimator \( \hat{U}_k \) of augmented strength reliability \( (R_k; k = 1, 2, 3) \) is defined by its posterior expectation and is given by

\[
\hat{U}_k = E(R_k / z_k, y) = \int \int \int R_k \Pi_{k^3} (\alpha_1, \alpha_2, \lambda_1, \lambda_2) \, d\alpha_1 \, d\alpha_2 \, d\lambda_1 \, d\lambda_2
\]  

(3.55)

Substituting the expressions of \( R_k \) and \( \Pi_{k^3} (\alpha_1, \alpha_2, \lambda_1, \lambda_2) \) from equations (3.4) and (3.48) respectively in equation (3.55), we have

\[
\hat{U}_k = K_3 \int \int \int R_k \left[ \alpha_1^{n_1} \lambda_1^{-1} \exp \left( -\alpha_1 \left( \frac{n_1 \lambda_1}{m} \right) \right) \right] \times \left[ \alpha_2^{n_2} \lambda_2^{-1} \exp \left( -\alpha_2 \left( n_2 \bar{y} \right) \right) \right] \times \left[ \frac{\lambda_1^{-1} \exp (\lambda_1 n_1)}{m \Gamma (n_1)} \right] \times \left[ \frac{\lambda_2^{-1} \exp (\lambda_2 n_2)}{\Gamma (\lambda_2)} \right] \, d\alpha_1 \, d\alpha_2 \, d\lambda_1 \, d\lambda_2
\]  

(3.56)

where \( K_3 \) is the normalizing constant defined in (3.49) and equation (3.56) is the required Bayes estimator of \( R_k \) under SELF.

Bayes Estimator \( \hat{U}_k \) under LLF

Under LINEX loss function, the Bayes estimate \( \hat{U}_k \) of augmented strength reliability is defined by

\[
\hat{U}_k = -\frac{1}{p} \ln \left[ E(e^{-pR_k} / z_k, y) \right] = -\frac{1}{p} \ln \left[ \int \int \int e^{-pR_k} \Pi_{k^3} (\alpha_1, \alpha_2, \lambda_1, \lambda_2) \, d\alpha_1 \, d\alpha_2 \, d\lambda_1 \, d\lambda_2 \right]
\]  

(3.57)

Substituting the expressions of \( R_k \) and \( \Pi_{k^3} (\alpha_1, \alpha_2, \lambda_1, \lambda_2) \) from equations (3.4) and (3.48) respectively in equation (3.57), we get
As the Bayes estimators of augmented strength reliability for uniform prior under SELF and LLF given in equations (3.56) and (3.58) consist multiple integrals and cannot be evaluated analytically. Thus, the numerical approximation technique viz. M-H algorithm is applied to obtain the Bayes estimators for various parameter values.

### 3.4.4 Assuming Jeffrey’s Prior:

In this subsection, assuming $\alpha_1, \alpha_2, \lambda_1$ and $\lambda_2$ as independent random variables having a type of non-informative prior known as Jeffreys prior, which is proposed by Jeffrey (1998), defined as

$$J(\alpha_1, \alpha_2, \lambda_1, \lambda_2) \propto \sqrt{\det I(\alpha_1, \alpha_2, \lambda_1, \lambda_2)}$$  \hspace{1cm} (3.59)

where, $I(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ is Fisher information matrix, defined as follows

$$I(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = -E \begin{bmatrix}
\frac{\partial^2 \log L}{\partial \alpha_1^2} & \frac{\partial^2 \log L}{\partial \alpha_1 \partial \alpha_2} & \frac{\partial^2 \log L}{\partial \alpha_1 \partial \lambda_1} & \frac{\partial^2 \log L}{\partial \alpha_1 \partial \lambda_2} \\
\frac{\partial^2 \log L}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 \log L}{\partial \alpha_2^2} & \frac{\partial^2 \log L}{\partial \alpha_2 \partial \lambda_1} & \frac{\partial^2 \log L}{\partial \alpha_2 \partial \lambda_2} \\
\frac{\partial^2 \log L}{\partial \lambda_1 \partial \alpha_1} & \frac{\partial^2 \log L}{\partial \lambda_1 \partial \alpha_2} & \frac{\partial^2 \log L}{\partial \lambda_1^2} & \frac{\partial^2 \log L}{\partial \lambda_1 \partial \lambda_2} \\
\frac{\partial^2 \log L}{\partial \lambda_2 \partial \alpha_1} & \frac{\partial^2 \log L}{\partial \lambda_2 \partial \alpha_2} & \frac{\partial^2 \log L}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 \log L}{\partial \lambda_2^2}
\end{bmatrix} \hspace{1cm} (3.60)$$

To find out the Bayes estimators under Jeffreys prior, we assume the shape $(\lambda_1, \lambda_2)$ and scale $(\alpha_1, \alpha_2)$ parameters of stress and strength are unknown. The elements of Fisher Information matrix in (3.60) are obtained by taking second derivatives of log-likelihood function given in equation (3.8), given by
\[
\frac{\partial^2 \log L_k}{\partial \alpha_i^2} = -\frac{nn_i \lambda_i}{\alpha_i^2}; \quad \frac{\partial^2 \log L_k}{\partial \alpha_2^2} = -\frac{n_2 \lambda_2}{\alpha_2^2},
\]
(3.61)

\[
\frac{\partial^2 \log L_k}{\partial \lambda_1^2} = -n_i \psi'(n_1); \quad \frac{\partial^2 \log L_k}{\partial \lambda_2^2} = -n_2 \psi'(\lambda_2)
\]
(3.62)

\[
\frac{\partial^3 \log L_k}{\partial \alpha_i \partial \alpha_2} = \frac{\partial^3 \log L_k}{\partial \alpha_2 \partial \alpha_i} = \frac{\partial^3 \log L_k}{\partial \lambda_1 \partial \lambda_2} = \frac{\partial^3 \log L_k}{\partial \lambda_2 \partial \lambda_1} = 0
\]
(3.63)

\[
\frac{\partial^3 \log L_k}{\partial \alpha_i \partial \lambda_1} = \frac{\partial^3 \log L_k}{\partial \lambda_1 \partial \alpha_i} = \frac{nn_i}{\alpha_i}
\]
(3.64)

\[
\frac{\partial^3 \log L_k}{\partial \alpha_2 \partial \lambda_2} = \frac{\partial^3 \log L_k}{\partial \lambda_2 \partial \alpha_2} = \frac{n_2}{\alpha_2}
\]
(3.65)

\[
\frac{\partial^3 \log L_k}{\partial \alpha_i \partial \lambda_2} = \frac{\partial^3 \log L_k}{\partial \lambda_2 \partial \alpha_i} = \frac{\partial^3 \log L_k}{\partial \alpha_2 \partial \lambda_i} = \frac{\partial^3 \log L_k}{\partial \lambda_i \partial \alpha_2} = 0
\]
(3.66)

The Fisher information matrix defined in (3.66) can be given as

\[
I(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = \begin{bmatrix}
\frac{nn_i \lambda_i}{\alpha_i^2} & 0 & -\frac{nn_i}{\alpha_i} & 0 \\
0 & \frac{n_2 \lambda_2}{\alpha_2} & 0 & -\frac{n_2}{\alpha_2} \\
-\frac{nn_i}{\alpha_i} & 0 & n_i \psi'(n_1 \lambda_i) & 0 \\
0 & -\frac{n_2}{\alpha_2} & 0 & n_2 \psi'(\lambda_2)
\end{bmatrix}
\]
(3.67)

where \( \psi'(.) \) is tri-gamma function, defined as \( \psi'(x) = \frac{\partial^2 \ln \Gamma(x)}{\partial x^2}, \forall x > 0 \). Then the determinant of the matrix is given by

\[
\det[I(\alpha_1, \alpha_2, \lambda_1, \lambda_2)] = n \left( \frac{nn_i \lambda_i}{\alpha_i \alpha_2} \right)^2 \left[ \lambda_1 \lambda_2 \alpha_2 \psi'(n_1 \lambda_i) \psi'(\lambda_2) - \lambda_1 \psi'(n_1 \lambda_i) - n_2 \psi'(\lambda_2) + n \right]
\]
(3.68)

Thus, the Jeffreys prior of \( \alpha_1, \alpha_2, \lambda_1 \) and \( \lambda_2 \) can be defined as
\[ J(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = \sqrt{n} \left( \frac{n \cdot n}{\alpha_1 \cdot \alpha_2} \right) \left[ \lambda_1 \lambda_2 \alpha_2 \psi'(n\lambda_1) \psi'(\lambda_2) - \lambda_1 \psi'(n\lambda_1) - n\lambda_2 \psi'(\lambda_2) + n \right]^{1/2} \] (3.69)

The joint posterior density \( \Pi_{k_4}(\alpha_1, \alpha_2, \lambda_1, \lambda_2 / z_k, y) \) can be defined by combining both likelihood function \( L_k(\alpha_1, \alpha_2, \lambda_1, \lambda_2 / z_k, y) \) given in (3.8) and joint Jeffreys prior density function \( J(\alpha_1, \alpha_2, \lambda_1, \lambda_2) \) given in (3.69) via Bayes theorem as in (3.17), the constant terms from both numerator and denominator of joint posterior are cancelled, we have

\[
\Pi_{k_4}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = K_4 \left\{ \frac{\alpha_1^{m_1 n_1 \lambda_1 - 1} \alpha_2^{n_2 \lambda_2 - 1} e^{(m_1 \lambda_1 + n_2 \lambda_2)}}{m^{n_1 \lambda_1} \Gamma(n\lambda_1)^{n_1} \Gamma(\lambda_2)^{n_2}} \exp \left[ - \left( \frac{\alpha_1 n_1 \bar{z}_k}{m} + \alpha_2 n_2 \bar{y} \right) \right] \right\}
\times \left\{ \lambda_1 \lambda_2 \alpha_2^2 \psi'(n\lambda_1) \psi'(\lambda_2) - \lambda_1 \psi'(n\lambda_1) - n\lambda_2 \psi'(\lambda_2) + n \right\}^{1/2}
\] (3.70)

where \( K_4 \) is the normalizing constant given by

\[
K_4^{-1} = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \alpha_1^{m_1 n_1 \lambda_1 - 1} \alpha_2^{n_2 \lambda_2 - 1} e^{(m_1 \lambda_1 + n_2 \lambda_2)} \exp \left[ - \left( \frac{\alpha_1 n_1 \bar{z}_k}{m} + \alpha_2 n_2 \bar{y} \right) \right] \psi'(n\lambda_1) \psi'(\lambda_2) - \lambda_1 \psi'(n\lambda_1) - n\lambda_2 \psi'(\lambda_2) + n \right\}^{1/2} d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2
\] (3.71)

\textbf{Bayes Estimator (} \( \hat{R}_k^{\text{self}} \)) \textit{under SELF}

Under the squared error loss function, the Bayes estimators \( \left( \hat{R}_k^{\text{self}} \right) \) of augmented strength reliability \( (R_k; k = 1, 2, 3) \) is defined by the posterior expectation of \( R_k \), given by

\[
\hat{R}_k^{\text{self}} = E(R_k / z_k, y) = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} R_k \Pi_{k_4}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2.
\] (3.72)

Substituting the values of \( R_k \) from (3.4) and \( \Pi_{k_4}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) \) from (3.70) in equation (3.72), we get

\[
\hat{R}_k^{\text{self}} = K_4 \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} R_k \frac{\alpha_1^{m_1 n_1 \lambda_1 - 1} \alpha_2^{n_2 \lambda_2 - 1} e^{(m_1 \lambda_1 + n_2 \lambda_2)}}{m^{n_1 \lambda_1} \Gamma(n\lambda_1)^{n_1} \Gamma(\lambda_2)^{n_2}} \exp \left[ - \left( \frac{\alpha_1 n_1 \bar{z}_k}{m} + \alpha_2 n_2 \bar{y} \right) \right] \psi'(n\lambda_1) \psi'(\lambda_2) - \lambda_1 \psi'(n\lambda_1) - n\lambda_2 \psi'(\lambda_2) + n \right\}^{1/2} d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2
\] (3.73)
where $K_4$ being the normalizing constant, defined in (3.71) and $\hat{R}^{\text{self}}_k$ is the Bayes estimator of augmented strength reliability under Jeffreys prior.

**Bayes Estimator $\left( u \hat{R}^{\text{llf}}_k \right)$ under LLF**

Under LINEX loss function, the Bayes estimator $\left( u \hat{R}^{\text{llf}}_k \right)$ of augmented strength reliability $R_k$ is defined by

$$
\hat{R}^{\text{llf}}_k = -\frac{1}{p} \ln \left[ E \left( e^{-\nu_k} / z_k, y \right) \right] = -\frac{1}{p} \ln \left[ \int_{0}^{\infty} \int_{0}^{\infty} \frac{e^{-\nu_k}}{n_{\lambda_1} \Gamma(n_{\lambda_2})} \exp \left\{ -\left( \frac{\alpha_1 n_{\lambda_k} \bar{z}_k + \alpha_2 n_{\lambda_2} \bar{y}}{m} \right) \right\} d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2 \right] \quad (3.74)
$$

Substituting the expressions of $R_k$ and $\Pi_{k4} \left( \alpha_1, \alpha_2, \lambda_1, \lambda_2 \right)$ from equations (3.4) and (3.70) respectively in equation (3.74), we get

$$
\hat{R}^{\text{llf}}_k = -\frac{1}{p} \ln \left[ \int_{0}^{\infty} \int_{0}^{\infty} \frac{e^{-\nu_k}}{n_{\lambda_1} \Gamma(n_{\lambda_2})} \exp \left\{ -\left( \frac{\alpha_1 n_{\lambda_k} \bar{z}_k + \alpha_2 n_{\lambda_2} \bar{y}}{m} \right) \right\} \times \left\{ \lambda_1 \bar{z}_k \psi'(n_{\lambda_1}) \psi'(\lambda_2) - \lambda_1 \psi'(n_{\lambda_1}) - n_{\lambda_2} \psi'(\lambda_2) + n \right\}^{1/2} d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2 \right] \quad (3.75)
$$

The Bayes estimators for Jeffreys prior under SELF and LLF given in equations (3.73) and (3.75) respectively consist multiple integrals and therefore a closed form solution is not possible. Thus, to approximate the Bayes estimators of augmented strength reliability for Jeffreys’s prior under SELF and LLF, the M-H approximation technique is used.

**3.5 Numerical Illustrations**

In this section, the proposed ML estimator $\hat{R}_k$ of $R_k$ in (3.13) as well as Bayes estimators for various priors under SELF and LLF, $g \hat{R}^{\text{self}}_k$ and $g \hat{R}^{\text{llf}}_k$ given in (3.28) and (3.30), $\hat{R}^{\text{self}}_k$ and $\hat{R}^{\text{llf}}_k$ in (3.42) and (3.44), $u \hat{R}^{\text{self}}_k$ and $u \hat{R}^{\text{llf}}_k$ in (3.56) and (3.58) and $\hat{R}^{\text{self}}_k$ and $\hat{R}^{\text{llf}}_k$ in equations (3.73) and (3.75) of generalized augmented strength reliability under ASP are numerically illustrated by considering simulated as well as real data sets. The real strength data sets are fitted and analyzed for two-parameter gamma distribution.
The MCMC technique is proposed to use for obtaining the Bayes estimation of $R_k$. To generate a posterior random sample of size $N$ (say) from the above marginal posterior distributions, we follow the following steps:

Step 1: Start with $q=1$.

Step 2: Choose initial value $\lambda_1^{(0)}$, $\lambda_2^{(0)}$.

Step 3: Using M-H algorithm, generate $\lambda_{1,q} \sim \pi_{kp3}(\cdot/ z_k)$; $p = 1,3$

Step 4: Using M-H algorithm, generate $\lambda_{2,q} \sim \pi_{kp4}(\cdot/ y)$; $p = 1,3$

Step 5: Generate $\alpha_{1,q} \sim \pi_{kp1}(\cdot/ z_k)$; $p = 1,3$

Step 6: Generate $\alpha_{2,q} \sim \pi_{kp2}(\cdot/ y)$; $p = 1,3$

Step 7: Repeat steps 3 to 6 for $q=1,2,...,N$.

The above procedure of generating the random observations can be employed in gamma prior and uniform prior cases. Whereas, in case of inverted gamma prior and Jeffreys prior, the conditional marginal posteriors are not any distributional form, thus the random samples can be drawn directly from their respective joint posteriors through M-H algorithm. The steps of basic Metropolis-Hasting algorithm are given in Section 2.5 of Chapter 2. Thus, the Bayes estimates of $R_k$ under SELF and LLF can be approximated respectively through M-H algorithm as follows

$$G\hat{R}_{k}^{\text{self}} = \frac{1}{N} \sum_{q=1}^{N} \left( R_k \right)_{a_1=\alpha_{1,q}, a_2=\alpha_{2,q}, \lambda_1=\lambda_{1,q}, \lambda_2=\lambda_{2,q}}$$

$$G\hat{R}_{k}^{\text{llf}} = \frac{-1}{p} \ln \left[ \exp \left\{ -p \frac{1}{N} \sum_{q=1}^{N} \left( R_k \right)_{a_1=\alpha_{1,q}, a_2=\alpha_{2,q}, \lambda_1=\lambda_{1,q}, \lambda_2=\lambda_{2,q}} \right\} \right]$$

where $\alpha_{1,q}, \alpha_{2,q}$, $\lambda_{1,q}$ and $\lambda_{2,q}; q=1,2,...,N$ are the four independent random samples drawn from their respective marginal posteriors through M-H algorithm.

### 3.5.1 Data Analysis

In this subsection the simulated as well as real data sets are analyzed to illustrate the proposed ML and Bayes estimators of augmented strength reliability
3.5.1.1 Example 1: Simulated Data

Here, the simulated data analysis of the augmented strength reliability is considered. Two samples of sizes 40 and 30 are drawn from the distributions of augmented strength and the common stress i.e. from $\text{gamma}(\alpha_1/m, n\lambda_1)$ and $\text{gamma}(\alpha_2, \lambda_2)$ respectively by fixing the parameters $\alpha_1=\alpha_2=1.50, \lambda_1=\lambda_2=2.50$ and $m=n=2$. The calculated true value of augmented strength reliability for generalized case of ASP is given as $R_3 = 0.970325$ with its ML estimate $\hat{R}_3 = 0.998448$. The Bayes estimates for all the considered priors under squared error and LINEX loss are computed by generating fifteen thousand samples through Metropolis-Hasting algorithm. Thus the Bayes estimate of augmented strength reliability for gamma, inverted gamma, uniform and Jeffreys priors under SELF (LLF) are $0.977789$ ($0.977647$), $0.942335$ ($0.942324$), $0.972275$ ($0.972189$) and $0.953223$ ($0.95416$) respectively. It is seen that the Bayes estimators under gamma and uniform priors are quite close to the true value of augmented strength reliability. The ML estimator is over estimated, whereas the Bayes estimators for inverted gamma and Jeffreys priors are underestimated.

3.5.1.2 Example 2: Real Data Sets

In this example, fitting and analysis of breaking strength data of carbon fiber tows, reported in Badar and Priest (1982), are carried out. The detailed discussion about the data sets are discussed in example 2 of subsection 2.5.1(b) in Chapter 2. Here, two parameters gamma distribution is considered to analyze the strength data sets. The goodness of fit of the following data for two parameter gamma distribution using Kolmogorov-Smirnov (KS) statistic, Akaike information criterion (AIC) and Bayesian information criterion (BIC) are tested. The fitting summary of gamma distribution is given in the following Table 3.7, which clearly shows that the gamma distribution fits well as compared to log-normal and exponential distributions. The graphs of fitted and empirical distribution functions for both the data sets x and y for two-parameter gamma, log-normal and exponential distributions are given in Figure 3.1, which clearly show that the underlying datasets give best fit to the two-parameter gamma distribution.
The true value of augmented strength reliability for generalized case of ASP is given by $R_3 = 0.970325$ and the maximum likelihood estimate based on augmented strength data set is $\hat{R}_3 = 0.992728$. The Bayes estimates of augmented strength reliability under square error and LINEX loss functions for all the underlying priors have been carried out through Metropolis-Hasting algorithm. Fifteen thousand random samples of $\alpha_1, \alpha_2, \lambda_1$ and $\lambda_2$ are drawn from their respective marginal posterior distributions by considering Normal density as candidate density to evaluate the Bayes estimates of augmented strength reliability for different priors. Thus the Bayes estimate of augmented strength reliability under SELF (LLF) for gamma, inverted gamma, uniform and Jeffreys priors are given by $0.9132396 (0.9122107), 0.986994 (0.986986)$, $0.987012 (0.986999)$ and $0.961924 (0.968371)$ respectively. For the real data sets the Bayes estimates under inverted gamma, uniform and Jeffreys priors gives closer result to true value of augmented strength reliability.

### 3.5.2 Simulation Study and Discussions

In this subsection, a comparison among the proposed estimators of augmented strength reliability is carried out for different combinations of sample sizes $(n_1, n_2)$ and for varying values of stress-strength reliability parameters $\alpha_1, \alpha_2, \lambda_1, \lambda_2, m, n$ for 1000 replications of MCMC samples. Here, the behavior of generalized augmented strength reliability parameters is studied under the proposed augmentation strategy plan through simulated samples for real valued shape parameter. The performance of proposed maximum likelihood estimator and Bayes estimators of generalized augmented strength reliability for informative (gamma and inverted gamma) and non-informative (uniform and Jeffreys) types of priors under two different loss functions (i.e. SELF and LLF) were compared through their mean square errors (MSEs) and absolute biases.

It is noticed that the maximum likelihood equations of $\alpha_1, \alpha_2, \lambda_1$ and $\lambda_2$ given in (3.9), (3.10), (3.11) and (3.12) respectively are mutually dependent to each other and the likelihood equations of $\lambda_1$ and $\lambda_2$ in (3.11) and (3.12) consist special function viz.
The expressions of Bayes estimators of augmented strength reliability for all the four priors under both the loss functions SELF and LLF given in (3.28), (3.30), (3.42), (3.44), (3.56), (3.58), (3.73) and (3.75) are not in explicit forms and cannot be solved analytically and it is also to be noticed that the marginal posteriors are also not in any standard distributional form. In such a situation, the well-known MCMC technique viz. Metropolis-Hastings algorithm can be recommended for drawing the samples from any arbitrary posterior distribution. For generating a random sample of size $N$ (say) from any arbitrary posterior distribution $\pi(\theta | data); \theta = (\alpha_1, \alpha_2, \lambda_1, \lambda_2)'$, the basic steps of Metropolis-Hastings algorithm are given in Section 2.5 of Chapter 2.

The initial values $\alpha_1^{(0)} = \hat{\alpha}_1$, $\alpha_2^{(0)} = \hat{\alpha}_2$ and $\lambda_1^{(0)} = \hat{\lambda}_1$, $\lambda_2^{(0)} = \hat{\lambda}_2$ are fixed as the ML estimates along with its asymptotic variance covariance matrix to draw the initial random samples from proposal density. To evaluate the Bayes estimate of augmented strength reliability, the Metropolis-Hastings algorithm was carried out with ten thousands of intermediate iterations by considering asymptotic normal distribution as proposal distribution. The first thousand samples were discarded as burn-in period of Markov chain. We also tested the autocorrelation and it is noticed that the chains are highly auto correlated. For reducing the autocorrelation within the chain, we thinned the chain equally spaced at every second simulation. The choice of prior distributions are considered as gamma and inverted-gamma under informative types and uniform and Jeffrey’s priors under non-informative types of priors, the Bayes estimates were calculated under squared error loss function and LINEX loss functions.

The following presented Tables 3.1-3.3 contain the average estimates, mean square errors (MSE) and absolute biases for MLE of generalized augmented strength reliability under ASP as well as Bayes estimates for gamma and inverted gamma priors under informative types of priors under both the loss functions i.e. SELF and LLF, whereas in Tables 3.4-3.6 the results of Bayes estimators for uniform and Jeffrey’s priors under non-informative types of priors under SELF and LLF of augmented strength reliability along
with its ML estimators are presented. The following observations are made from the given tables on the basis of simulation analysis

**Variations of parameters: (For gamma and inverted gamma priors)**

(i) **Variations in model parameters:** Table 3.1 presents the results for variations in model parameters \((\alpha_1, \alpha_2, \lambda_1, \lambda_2)\) by keeping rest of the augmentation parameters and hyper-parameters as fixed constant as \(m=2, n=2\) and \(a_1 = 0.50, b_1 = 0.75, a_2 = 0.60, b_2 = 0.35, c_1 = 0.75, c_2 = 0.63, d_1 = 0.80, d_2 = 0.72,\) and \(p = 2.5.\) Three sets of variations in model parameters are taken as \((\alpha_1 = 0.5, \alpha_2 = 1.5, \lambda_1 = 0.75, \lambda_2 = 1.5), (\alpha_1 = 1.5, \alpha_2 = 2.75, \lambda_1 = 1.75, \lambda_2 = 2.5)\) and \((\alpha_1 = \alpha_2 = 1.5, \lambda_1 = \lambda_2 = 2.5).\)

**Findings:** It is observed from Table 3.1 that the Bayes estimator for gamma prior under both the loss functions gives better results with smaller MSEs and absolute biases as compared to ML estimator. Even for first sets of variations in model parameters \((\alpha_1 = 0.5, \alpha_2 = 1.5, \lambda_1 = 0.75, \lambda_2 = 1.5),\) the Bayes estimator for inverted gamma prior also perform better than ML estimator in terms of MSEs, but in context of absolute biases the ML estimator dominates over the inverted gamma prior. For second set of variation \((\alpha_1 = 1.5, \alpha_2 = 2.75, \lambda_1 = 1.75, \lambda_2 = 2.5),\) the Bayes estimator for gamma prior gives minimum MSEs and absolute biases, whereas the inverted gamma performs better than ML estimators in terms of MSEs and absolute biases except the for the sample sizes (20,40) and (50,50). For third set of variation of model parameters \((\alpha_1 = \alpha_2 = 1.5, \lambda_1 = \lambda_2 = 2.5),\) the Bayes estimators for gamma as well as inverted gamma priors perform better with smaller MSEs and absolute biases than ML estimators. Overall it is observed that the Bayes estimator for gamma prior gives more accurate estimates of augmented strength reliability with smallest MSEs and absolute biases.
(ii) **Variation in augmentation parameter \(m\):** In Table 3.2, the variation of augmentation parameter \(m\) is considered as 1.5 and 3.5 by keeping rest of the model parameters and hyper-parameters as fixed constant as \(a_1 = a_2 = 1.5, \lambda_1 = \lambda_2 = 2.5, n = 2; a_1 = 0.50, b_1 = 0.75, a_2 = 0.60, b_2 = 0.35, c_1 = 0.75, c_2 = 0.63, d_1 = 0.80, d_2 = 0.72, p = 2.5\). The following observations are made from the table.

**Findings:** It is observed from the table that the values of augmented strength reliability gradually increases for increasing values of augmentation parameter \(m\). For \(m=1.5\), it is noticed that the Bayes estimators for gamma and inverted gamma priors under both the loss functions gives smaller MSEs than the ML estimators for all the sample choices of sizes. But, for the choices of sample sizes (20,20) and (50,50) the gamma prior gives higher values of absolute biases. The over performance of Bayes estimator for inverted gamma prior under both SELF and LLF is better with smaller MSEs and absolute biases. For \(m=3.5\), the performance of gamma prior under both the loss function is better than its other counterparts in context of both MSEs and absolute biases. The performance of both the loss function SELF and LLF are similar.

(iii) **Variation in augmentation parameter \(n\):** Table 3.3 presents the average estimates MSEs and absolute biases for ML as well as Bayes estimators for gamma and inverted gamma priors under SELF and LLF for the variations in augmentation parameter \(n\) as 3 and 5 by keeping rest of the parameter and hyper-parameters fixed constants as \(a_1 = a_2 = 1.5, \lambda_1 = \lambda_2 = 2.5, m = 2; a_1 = 0.50, b_1 = 0.75, a_2 = 0.60, b_2 = 0.35, c_1 = 0.75, c_2 = 0.63, d_1 = 0.80, d_2 = 0.72, p = 2.5\). The following observations are made from the table.

**Finding:** It is noticed from the table that the values of augmented strength reliability increase after adding more numbers of components \((n)\). It is observed that the Bayes estimator for inverted gamma prior under both the loss functions (SELF and LLF) give overall better results with minimum MSEs and absolute biases as compared to ML as well as Bayes estimators under gamma prior for both the variations \(n=3\) as well as \(n=5\). Even for \(n=3\), the gamma prior also perform
better than ML estimator in context of MSEs but in terms of absolute biases, the ML estimator for sample sizes (20,20) and (50,50) dominates over the gamma prior. For \( n=5 \), the ML estimator dominates over Bayes estimator under gamma prior for sample sizes (20,20) and (50,50) in terms of both MSEs and absolute biases.

**Variations of parameters: (For uniform and Jeffreys priors)**

(iv) **Variations in model parameters**: In Table 3.4, the variations in model parameters \((\alpha_1, \alpha_2, \lambda_1, \lambda_2)\) are considered for non-informative type uniform and Jeffreys priors by keeping rest of the augmentation parameters fixed constant as \( n=2, m=2, p=2.5 \) . Three sets of variations are considered for model parameters \((\alpha_1, \alpha_2, \lambda_1, \lambda_2)\) as \((\alpha_1 = 0.5, \alpha_2 = 1.5, \lambda_1 = 0.75, \lambda_2 = 1.5)\), \((\alpha_1 = 1.5, \alpha_2 = 2.75, \lambda_1 = 1.75, \lambda_2 = 2.5)\) and \((\alpha_1 = \alpha_2 = 1.5, \lambda_1 = \lambda_2 = 2.5)\). The following observations are made from Table 3.4.

**Findings**: For first set of variation in model parameter \((\alpha_1 = 0.5, \alpha_2 = 1.5, \lambda_1 = 0.75, \text{and } \lambda_2 = 1.5)\), it is observed from the table that the Bayes estimators for uniform and Jeffreys priors under both the loss functions perform better with smaller MSEs than the ML estimators. As compared between both the priors the uniform priors perform better than Jeffreys prior. For both second and third sets of model parameter variation \((\alpha_1 = 1.5, \alpha_2 = 2.75, \lambda_1 = 1.75, \lambda_2 = 2.5)\) and \((\alpha_1 = \alpha_2 = 1.5, \lambda_1 = \lambda_2 = 2.5)\), it is noticed that the Bayes estimators for both uniform as well as Jeffreys priors perform better with smaller MSEs and absolute biases as compared to ML estimators. Between both the priors, the uniform prior gives overall better results with minimum MSEs and absolute biases than the Jeffreys prior.

(v) **Variations in augmentation parameter ‘m’**: Table 3.5 presents the average estimates, MSEs and absolute biases of ML and Bayes estimators under non-informative priors (uniform and Jeffreys priors) for the variations in augmentation parameter ‘m’ as 1.5 and 3.5 by keeping rest of the parameters fixed as
\((\alpha_1 = \alpha_2 = 1.5, \lambda_1 = \lambda_2 = 2.5, p = 2.5; n = 2)\). The following observations are made from the table.

**Findings:** It is observed from the table that the augmented strength reliability increases for increasing values of ‘\(m\)’. For both the variations \(m=1.5\) and \(m=3.5\), it is noticed that the Bayes estimators for both the non-informative priors gives more accurate estimates of augmented strength reliability with lesser MSEs and absolute biases than the ML estimator. Among Bayes estimator, the uniform prior gives the minimum MSE than Jeffreys prior. The effect of loss functions are almost similar.

(vi) **Variations in augmentation parameter ‘\(n\)’**: In Table 3.6, the average estimates, MSEs and absolute biases of ML estimator as well as Bayes estimator for uniform and Jeffreys priors under SELF and LLF are presented for the variations in augmentation parameter ‘\(n\)’ as 3 and 5 by fixing rest of the model parameters constants as \((\alpha_1 = \alpha_2 = 1.5, \lambda_1 = \lambda_2 = 2.5, p = 2.5; m = 2)\).

**Findings:** It is found from the Table that the augmented strength reliability increases as the value of ‘\(n\)’ increases. It is observed from the table that for both the variations of \(n\) (3,5), the Bayes estimators under uniform as well as Jeffreys priors gives far better results as compared to ML estimator with minimum MSEs and absolute biases.

### 3.6 Conclusion

In this chapter, the augmented strength reliability is derived under the generalized case of ASP by assuming that the strength of the equipment and the common random stress imposed on it, are independently but non identically distributed as two-parameter gamma distribution with real valued unknown shape parameters. The ML and Bayes estimation of generalized augmented strength reliability are also carried out. In Bayesian context, the informative priors (gamma and inverted gamma) as well as non-informative priors (uniform and Jeffreys) are considered and the Bayes estimators are computed under each of the loss function SELF and LLF.
Even, the assumptions of identical stress-strength distributions may not be realistic and feasible, but it is attempted due to mathematical simplicity in Chapter 2. In this chapter, the choice of non-identical gamma stress-strength distributions are considered with four unknown parameters \((\alpha_1, \alpha_2, \lambda_1, \lambda_2)\), because in real life situations many stresses e.g. temperature, pressure, voltage etc. are applied to the system and are beyond the control of human hand. The effects of different sets of values of scale and shape parameters on augmented strength reliability and its estimation are recorded in the tables.

A numerical illustration is carried out to validate the proposed estimators of augmented strength reliability through simulated as well as real data sets. A simulation based comparison of different estimators of augmented strength reliability is carried out on the basis of their mean square errors and absolute biases through MCMC technique. It is observed from the given Tables that the Bayes estimators perform significantly better with minimum MSEs and absolute biases as compared to ML estimators. Moreover, the choice of gamma prior under informative prior set up is considered to be better for this case as compared to inverted gamma prior, whereas, the uniform prior dominates over the Jeffreys prior under non-informative prior setup.

It is also noticed that the strength reliability gets enhanced for increasing values of augmentation parameters ‘m’ and ‘n’. Thus, the proposed ASP may be suggestive for enhancing the strength reliability of a weaker equipment.
### Tables for informative Priors (gamma and Inverted gamma priors)

**Table 3.1:** Average estimates, MSE and Absolute bias of the estimator of augmented strength reliability ($R_i$) under SELF and LLF for gamma and inverted gamma priors for different $(n_1,n_2)$ with variation in Model Parameters when $n = 2, m = 2; a_1 = 0.50, b_1 = 0.75, a_2 = 0.60, b_2 = 0.35, c_1 = 0.75, c_2 = 0.63, d_1 = 0.80, d_2 = 0.72, p = 2.5$

<table>
<thead>
<tr>
<th>$(n_1,n_2)$</th>
<th>MLE</th>
<th>Bayes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MLE</td>
<td>Gamma Prior</td>
</tr>
<tr>
<td></td>
<td>SELF</td>
<td>LLF</td>
</tr>
<tr>
<td></td>
<td>Estimate</td>
<td>MSE</td>
</tr>
<tr>
<td>$(20,20)$</td>
<td>0.867305</td>
<td>0.952621</td>
</tr>
<tr>
<td></td>
<td>0.023933</td>
<td>0.006453</td>
</tr>
<tr>
<td></td>
<td>0.045065</td>
<td>0.042411</td>
</tr>
<tr>
<td>$(20,40)$</td>
<td>0.890417</td>
<td>0.948568</td>
</tr>
<tr>
<td></td>
<td>0.013536</td>
<td>0.007431</td>
</tr>
<tr>
<td></td>
<td>0.021953</td>
<td>0.021698</td>
</tr>
<tr>
<td>$(40,20)$</td>
<td>0.869175</td>
<td>0.958775</td>
</tr>
<tr>
<td></td>
<td>0.019372</td>
<td>0.002295</td>
</tr>
<tr>
<td></td>
<td>0.043195</td>
<td>0.042963</td>
</tr>
<tr>
<td>$(50,50)$</td>
<td>0.89584</td>
<td>0.997556</td>
</tr>
<tr>
<td></td>
<td>0.007106</td>
<td>0.006624</td>
</tr>
<tr>
<td></td>
<td>0.016530</td>
<td>0.013346</td>
</tr>
<tr>
<td>$(50,40)$</td>
<td>0.879371</td>
<td>0.964760</td>
</tr>
<tr>
<td></td>
<td>0.044513</td>
<td>0.000360</td>
</tr>
<tr>
<td></td>
<td>0.090954</td>
<td>0.005565</td>
</tr>
</tbody>
</table>

$(\alpha_1 = 0.5, \alpha_2 = 1.5, \lambda_1 = 0.75, \lambda_2 = 1.5)$

$R_i = 0.9123701$

$(\alpha_1 = 1.5, \alpha_2 = 2.75, \lambda_1 = 1.75, \lambda_2 = 2.5)$

$R_i = 0.9730883$

$(\alpha_1 = \alpha_2 = 1.5, \lambda_1 = \lambda_2 = 2.5)$

$R_i = 0.9743247$
<table>
<thead>
<tr>
<th>(20,40)</th>
<th>Estimate</th>
<th>MSE</th>
<th>Abs.bias</th>
<th>Estimate</th>
<th>MSE</th>
<th>Abs.bias</th>
<th>Estimate</th>
<th>MSE</th>
<th>Abs.bias</th>
<th>Estimate</th>
<th>MSE</th>
<th>Abs.bias</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.917889</td>
<td>0.999695</td>
<td>0.999694</td>
<td>0.929552</td>
<td>0.929511</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.021901</td>
<td>0.000863</td>
<td>0.000863</td>
<td>0.001698</td>
<td>0.001702</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.052436</td>
<td>0.02937</td>
<td>0.029369</td>
<td>0.040772</td>
<td>0.040814</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(40,20)</td>
<td>Estimate</td>
<td>MSE</td>
<td>Abs.bias</td>
<td>Estimate</td>
<td>MSE</td>
<td>Abs.bias</td>
<td>Estimate</td>
<td>MSE</td>
<td>Abs.bias</td>
<td>Estimate</td>
<td>MSE</td>
<td>Abs.bias</td>
</tr>
<tr>
<td></td>
<td>0.889315</td>
<td>0.980651</td>
<td>0.980619</td>
<td>0.933495</td>
<td>0.933473</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.035261</td>
<td>0.000168</td>
<td>0.000182</td>
<td>0.000178</td>
<td>0.001380</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.081009</td>
<td>0.040230</td>
<td>0.020319</td>
<td>0.036819</td>
<td>0.036852</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(50, 50)</td>
<td>Estimate</td>
<td>MSE</td>
<td>Abs.bias</td>
<td>Estimate</td>
<td>MSE</td>
<td>Abs.bias</td>
<td>Estimate</td>
<td>MSE</td>
<td>Abs.bias</td>
<td>Estimate</td>
<td>MSE</td>
<td>Abs.bias</td>
</tr>
<tr>
<td></td>
<td>0.929230</td>
<td>0.969057</td>
<td>0.968621</td>
<td>0.933342</td>
<td>0.933326</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.011574</td>
<td>0.000998</td>
<td>0.000104</td>
<td>0.001386</td>
<td>0.001387</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.041095</td>
<td>0.001268</td>
<td>0.001704</td>
<td>0.036983</td>
<td>0.036999</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Variation of m (1.5, 3.5)**

Table 3.2: Average estimates, MSE and Absolute bias of the estimator of augmented strength reliability ($R_3$) under SELF and LLF for gamma and inverted gamma priors for different $(n_1, n_2)$ with variation in $m$ when $\alpha_1 = \alpha_2 = 1.5$, $\lambda_1 = \lambda_2 = 2.5$, $n = 2; a_1 = 0.50, b_1 = 0.75, a_2 = 0.60$, $b_2 = 0.35, c_1 = 0.75, c_2 = 0.63, d_1 = 0.80, d_2 = 0.72, p = 2.5$
### Variation of $n$ (3, 5)

Table 3.3: Average estimates, MSE and Absolute bias of the estimator of augmented strength reliability ($R_3$) under SELF and LLF for gamma and inverted gamma priors for different $(n_1, n_2)$ with variation in $n$ when $\alpha_1 = \alpha_2 = 1.5, \lambda_1 = \lambda_2 = 2.5, m = 2; a_1 = 0.50, b_1 = 0.75, a_2 = 0.60,$ $b_2 = 0.35, c_1 = 0.75, c_2 = 0.63, d_1 = 0.80, d_2 = 0.72, p = 2.5$.

<table>
<thead>
<tr>
<th>$(n_1, n_2)$</th>
<th>Estimate</th>
<th>MSE</th>
<th>Abs.bias</th>
<th>MLE</th>
<th>MSE</th>
<th>Abs.bias</th>
<th>Bayes</th>
<th>MSE</th>
<th>Abs.bias</th>
</tr>
</thead>
<tbody>
<tr>
<td>(50, 50)</td>
<td>0.979380</td>
<td>0.002601</td>
<td>0.015942</td>
<td>0.979380</td>
<td>0.002601</td>
<td>0.015942</td>
<td>Gamma Prior</td>
<td>0.978812</td>
<td>0.002780</td>
</tr>
<tr>
<td></td>
<td>0.99993</td>
<td>0.000021</td>
<td>0.004609</td>
<td>0.99993</td>
<td>0.000021</td>
<td>0.004609</td>
<td>Inv. Gamma Prior</td>
<td>0.99993</td>
<td>0.000021</td>
</tr>
<tr>
<td></td>
<td>0.015160</td>
<td>0.015150</td>
<td></td>
<td>0.015160</td>
<td>0.015150</td>
<td></td>
<td>SELF</td>
<td>0.99993</td>
<td>0.000021</td>
</tr>
<tr>
<td></td>
<td>0.000278</td>
<td>0.004609</td>
<td></td>
<td>0.000278</td>
<td>0.004609</td>
<td></td>
<td>LLF</td>
<td>0.99993</td>
<td>0.000021</td>
</tr>
</tbody>
</table>

$n = 3; R_3 = 0.996952$

<table>
<thead>
<tr>
<th>$(n_1, n_2)$</th>
<th>Estimate</th>
<th>MSE</th>
<th>Abs.bias</th>
<th>MLE</th>
<th>MSE</th>
<th>Abs.bias</th>
<th>Bayes</th>
<th>MSE</th>
<th>Abs.bias</th>
</tr>
</thead>
<tbody>
<tr>
<td>(20, 20)</td>
<td>0.947869</td>
<td>0.019329</td>
<td>0.049083</td>
<td>0.947869</td>
<td>0.019329</td>
<td>0.049083</td>
<td>Gamma Prior</td>
<td>0.917178</td>
<td>0.049083</td>
</tr>
<tr>
<td></td>
<td>0.910556</td>
<td>0.007681</td>
<td>0.000006</td>
<td>0.910556</td>
<td>0.007681</td>
<td>0.000006</td>
<td>Inv. Gamma Prior</td>
<td>0.999334</td>
<td>0.000006</td>
</tr>
<tr>
<td></td>
<td>0.013787</td>
<td>0.002382</td>
<td></td>
<td>0.013787</td>
<td>0.002382</td>
<td></td>
<td>SELF</td>
<td>0.999334</td>
<td>0.000006</td>
</tr>
<tr>
<td></td>
<td>0.013883</td>
<td>0.002382</td>
<td></td>
<td>0.013883</td>
<td>0.002382</td>
<td></td>
<td>LLF</td>
<td>0.999334</td>
<td>0.000006</td>
</tr>
</tbody>
</table>

$n = 5; R_3 = 0.999976$

<table>
<thead>
<tr>
<th>$(n_1, n_2)$</th>
<th>Estimate</th>
<th>MSE</th>
<th>Abs.bias</th>
<th>MLE</th>
<th>MSE</th>
<th>Abs.bias</th>
<th>Bayes</th>
<th>MSE</th>
<th>Abs.bias</th>
</tr>
</thead>
<tbody>
<tr>
<td>(20, 20)</td>
<td>0.983765</td>
<td>0.004502</td>
<td>0.017174</td>
<td>0.983765</td>
<td>0.004502</td>
<td>0.017174</td>
<td>Gamma Prior</td>
<td>0.778261</td>
<td>0.017174</td>
</tr>
<tr>
<td></td>
<td>0.750622</td>
<td>0.000006</td>
<td>0.068165</td>
<td>0.750622</td>
<td>0.000006</td>
<td>0.068165</td>
<td>Inv. Gamma Prior</td>
<td>0.750622</td>
<td>0.000006</td>
</tr>
<tr>
<td></td>
<td>0.013406</td>
<td>0.002382</td>
<td></td>
<td>0.013406</td>
<td>0.002382</td>
<td></td>
<td>SELF</td>
<td>0.750622</td>
<td>0.000006</td>
</tr>
<tr>
<td></td>
<td>0.013406</td>
<td>0.002382</td>
<td></td>
<td>0.013406</td>
<td>0.002382</td>
<td></td>
<td>LLF</td>
<td>0.750622</td>
<td>0.000006</td>
</tr>
</tbody>
</table>

108
### Tables for non-informative Priors (uniform and Jeffreys priors)

**Table 3.4:** Average estimates, MSE and Absolute bias of the estimator of augmented strength reliability ($R_1$) under SELF and LLF for uniform and Jeffreys priors for different ($n_1, n_2$) with variation in Model Parameters when $n = 2, m = 2, p = 2.5$

<table>
<thead>
<tr>
<th>$(n_1, n_2)$</th>
<th>MLE Estimate</th>
<th>MLE MSE</th>
<th>MLE Abs.bias</th>
<th>Bayes Uniform Prior</th>
<th>Bayes Jeffreys Prior</th>
<th>( R_1 = 0.9123701 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(20, 20)</td>
<td>0.867305</td>
<td>0.023933</td>
<td>0.045065</td>
<td>0.867202</td>
<td>0.009753</td>
<td>0.813716 0.813379</td>
</tr>
<tr>
<td></td>
<td>0.867202</td>
<td>0.002558</td>
<td>0.045681</td>
<td>0.866689</td>
<td>0.098654</td>
<td>0.002558 0.098991</td>
</tr>
<tr>
<td>(20, 40)</td>
<td>0.890417</td>
<td>0.013356</td>
<td>0.021953</td>
<td>0.869242</td>
<td>0.010137</td>
<td>0.811775 0.811455</td>
</tr>
<tr>
<td></td>
<td>0.869242</td>
<td>0.002558</td>
<td>0.043664</td>
<td>0.868706</td>
<td>0.010095</td>
<td>0.002615 0.100912</td>
</tr>
<tr>
<td>(40, 20)</td>
<td>0.869175</td>
<td>0.019372</td>
<td>0.043195</td>
<td>0.861425</td>
<td>0.002824</td>
<td>0.813066 0.812889</td>
</tr>
<tr>
<td></td>
<td>0.861425</td>
<td>0.002629</td>
<td>0.050945</td>
<td>0.863516</td>
<td>0.002851</td>
<td>0.002558 0.099481</td>
</tr>
<tr>
<td>(50, 50)</td>
<td>0.895840</td>
<td>0.007106</td>
<td>0.023357</td>
<td>0.861425</td>
<td>0.000729</td>
<td>0.816775 0.816629</td>
</tr>
<tr>
<td></td>
<td>0.861425</td>
<td>0.000745</td>
<td>0.051183</td>
<td>0.863516</td>
<td>0.000749</td>
<td>0.009140 0.009167</td>
</tr>
</tbody>
</table>

| \( \alpha_1 = 0.5, \alpha_2 = 1.5, \lambda_1 = 0.75, \lambda_2 = 1.5 \) |
| \( R_1 = 0.9730883 \) |

<table>
<thead>
<tr>
<th>$(n_1, n_2)$</th>
<th>MLE Estimate</th>
<th>MLE MSE</th>
<th>MLE Abs.bias</th>
<th>Bayes Uniform Prior</th>
<th>Bayes Jeffreys Prior</th>
<th>( R_1 = 0.9730883 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(20, 20)</td>
<td>0.905227</td>
<td>0.030279</td>
<td>0.067862</td>
<td>0.949731</td>
<td>0.000736</td>
<td>0.930253 0.930188</td>
</tr>
<tr>
<td></td>
<td>0.949731</td>
<td>0.000745</td>
<td>0.042835</td>
<td>0.949507</td>
<td>0.000749</td>
<td>0.001893 0.001899</td>
</tr>
<tr>
<td>(20, 40)</td>
<td>0.936048</td>
<td>0.012474</td>
<td>0.037040</td>
<td>0.955109</td>
<td>0.000345</td>
<td>0.924826 0.924768</td>
</tr>
<tr>
<td></td>
<td>0.955109</td>
<td>0.000331</td>
<td>0.048262</td>
<td>0.954726</td>
<td>0.000345</td>
<td>0.002392 0.002397</td>
</tr>
<tr>
<td>(40, 20)</td>
<td>0.902813</td>
<td>0.031989</td>
<td>0.070275</td>
<td>0.946105</td>
<td>0.000736</td>
<td>0.928023 0.927932</td>
</tr>
<tr>
<td></td>
<td>0.946105</td>
<td>0.000749</td>
<td>0.041191</td>
<td>0.945862</td>
<td>0.000749</td>
<td>0.001731 0.001734</td>
</tr>
<tr>
<td>(50, 50)</td>
<td>0.943144</td>
<td>0.005920</td>
<td>0.029944</td>
<td>0.945346</td>
<td>0.000695</td>
<td>0.930727 0.930678</td>
</tr>
<tr>
<td></td>
<td>0.945346</td>
<td>0.000695</td>
<td>0.042410</td>
<td>0.945159</td>
<td>0.000695</td>
<td>0.001612 0.001612</td>
</tr>
</tbody>
</table>

| \( \alpha_1 = \alpha_2 = 1.5, \lambda_1 = \lambda_2 = 2.5 \) |
| \( R_1 = 0.9703247 \) |

<table>
<thead>
<tr>
<th>$(n_1, n_2)$</th>
<th>MLE Estimate</th>
<th>MLE MSE</th>
<th>MLE Abs.bias</th>
<th>Bayes Uniform Prior</th>
<th>Bayes Jeffreys Prior</th>
<th>( R_1 = 0.9703247 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(20, 20)</td>
<td>0.879371</td>
<td>0.044513</td>
<td>0.090954</td>
<td>0.975951</td>
<td>0.000331</td>
<td>0.928023 0.927932</td>
</tr>
<tr>
<td></td>
<td>0.975951</td>
<td>0.000031</td>
<td>0.042395</td>
<td>0.975807</td>
<td>0.000031</td>
<td>0.001731 0.001801</td>
</tr>
</tbody>
</table>

109
<table>
<thead>
<tr>
<th>(20,40)</th>
<th>Estimate</th>
<th>MSE</th>
<th>Abs.bias</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.917889</td>
<td>0.021901</td>
<td>0.052436</td>
</tr>
<tr>
<td></td>
<td>0.976367</td>
<td>0.000039</td>
<td>0.005903</td>
</tr>
<tr>
<td></td>
<td>0.976227</td>
<td>0.001871</td>
<td>0.043228</td>
</tr>
<tr>
<td></td>
<td>0.976227</td>
<td>0.001879</td>
<td>0.043319</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(40,20)</th>
<th>Estimate</th>
<th>MSE</th>
<th>Abs.bias</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.889315</td>
<td>0.035261</td>
<td>0.081009</td>
</tr>
<tr>
<td></td>
<td>0.973485</td>
<td>0.000011</td>
<td>0.003161</td>
</tr>
<tr>
<td></td>
<td>0.973407</td>
<td>0.0017</td>
<td>0.041136</td>
</tr>
<tr>
<td></td>
<td>0.973407</td>
<td>0.001704</td>
<td>0.041186</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(50, 50)</th>
<th>Estimate</th>
<th>MSE</th>
<th>Abs.bias</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.92923</td>
<td>0.011574</td>
<td>0.041095</td>
</tr>
<tr>
<td></td>
<td>0.97247</td>
<td>0.000005</td>
<td>0.002145</td>
</tr>
<tr>
<td></td>
<td>0.972404</td>
<td>0.000005</td>
<td>0.002080</td>
</tr>
<tr>
<td></td>
<td>0.972404</td>
<td>0.000005</td>
<td>0.002080</td>
</tr>
</tbody>
</table>

Table 3.5: Average estimates, MSE and Absolute bias of the estimator of augmented strength reliability \(R_k\) under SELF and LLF for uniform and Jeffreys priors for different \((n_1, n_2)\) with variation in \(m\) when \(\alpha_1 = \alpha_2 = 1.5, \lambda_1 = \lambda_2 = 2.5, p = 2.5; n = 2\)

\[
m = 1.5; \quad R_3 = 0.9344424
\]

<table>
<thead>
<tr>
<th>((n_1, n_2))</th>
<th>MLE</th>
<th>Bayes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Uniform Prior</td>
<td>Jeffreys Prior</td>
</tr>
<tr>
<td>SELF</td>
<td>LLF</td>
<td>SELF</td>
</tr>
<tr>
<td>(20, 20)</td>
<td>Estimate</td>
<td>0.835698</td>
</tr>
<tr>
<td>MSE</td>
<td>0.05996</td>
<td>0.00144</td>
</tr>
<tr>
<td>Abs.bias</td>
<td>0.098744</td>
<td>0.011706</td>
</tr>
<tr>
<td>(20,40)</td>
<td>Estimate</td>
<td>0.864121</td>
</tr>
<tr>
<td>MSE</td>
<td>0.038400</td>
<td>0.000073</td>
</tr>
<tr>
<td>Abs.bias</td>
<td>0.070321</td>
<td>0.008219</td>
</tr>
<tr>
<td>(40,20)</td>
<td>Estimate</td>
<td>0.823151</td>
</tr>
<tr>
<td>MSE</td>
<td>0.063121</td>
<td>0.000021</td>
</tr>
<tr>
<td>Abs.bias</td>
<td>0.111292</td>
<td>0.004530</td>
</tr>
<tr>
<td>(50, 50)</td>
<td>Estimate</td>
<td>0.882894</td>
</tr>
<tr>
<td>MSE</td>
<td>0.021576</td>
<td>0.000022</td>
</tr>
<tr>
<td>Abs.bias</td>
<td>0.051549</td>
<td>0.004560</td>
</tr>
</tbody>
</table>

\[
m = 3.5; \quad R_3 = 0.9953218
\]

| (20, 20) | Estimate | 0.950953 | 0.996687 | 0.996676 | 0.974978 | 0.974944 |
| MSE | 0.017208 | 0.000002 | 0.000002 | 0.000416 | 0.000418 |
| Abs.bias | 0.044369 | 0.001365 | 0.001354 | 0.020344 | 0.020377 |
| (20,40) | Estimate | 0.972226 | 0.996704 | 0.996692 | 0.974917 | 0.974884 |
| MSE | 0.004655 | 0.000002 | 0.000002 | 0.000425 | 0.000426 |
| Abs.bias | 0.023095 | 0.001382 | 0.00137 | 0.020405 | 0.020438 |
| (40,20) | Estimate | 0.957699 | 0.995891 | 0.995884 | 0.97728 | 0.977264 |
| MSE | 0.013127 | 0.000002 | 0.000001 | 0.000326 | 0.000326 |
| Abs.bias | 0.037623 | 0.000569 | 0.000562 | 0.018042 | 0.018058 |
| (50, 50) | Estimate | 0.97938 | 0.995862 | 0.995857 | 0.978037 | 0.978025 |
| MSE | 0.002601 | 0.000001 | 0.000001 | 0.000299 | 0.000300 |
| Abs.bias | 0.015942 | 0.000540 | 0.000535 | 0.013284 | 0.014297 |
Table 3.6: Average estimates, MSE and Absolute bias of the estimator of augmented strength reliability \( (R_s) \) under SELF and LLF for uniform and Jeffreys priors for different \( (n_1, n_2) \) with variation in \( n \) when \( (\alpha = \alpha_2 = 1.5, \lambda_1 = \lambda_2 = 2.5, p = 2.5; m = 2) \)

<table>
<thead>
<tr>
<th>((n_1, n_2))</th>
<th>MLE</th>
<th>Bayes</th>
<th>Uniform Prior</th>
<th>Jeffreys Prior</th>
</tr>
</thead>
<tbody>
<tr>
<td>|</td>
<td></td>
<td>SELF</td>
<td>LLF</td>
<td>SELF</td>
</tr>
</tbody>
</table>

\( n = 3; \ R_s = 0.996952 \)

| \((20, 20)\) | Estimate | 0.947869 | 0.997435 | 0.997416 | 0.981504 | 0.98149 |
| | MSE | 0.019329 | 0.000021 | 0.000023 | 0.000239 | 0.00024 |
| | Abs.bias | 0.049083 | 0.000483 | 0.000464 | 0.015448 | 0.015462 |

| \((20, 40)\) | Estimate | 0.967762 | 0.997311 | 0.997291 | 0.981837 | 0.981824 |
| | MSE | 0.006988 | 0.000003 | 0.000002 | 0.000229 | 0.000230 |
| | Abs.bias | 0.02919 | 0.000359 | 0.000339 | 0.015115 | 0.015128 |

| \((40, 20)\) | Estimate | 0.946784 | 0.997020 | 0.997012 | 0.983677 | 0.983671 |
| | MSE | 0.020218 | 0.000045 | 0.000045 | 0.000177 | 0.000177 |
| | Abs.bias | 0.050168 | 0.000068 | 0.000060 | 0.013275 | 0.013281 |

| \((50, 50)\) | Estimate | 0.979778 | 0.997066 | 0.997061 | 0.98333 | 0.983325 |
| | MSE | 0.002412 | 0.000042 | 0.000052 | 0.000186 | 0.000186 |
| | Abs.bias | 0.017174 | 0.000114 | 0.000109 | 0.013622 | 0.013627 |

\( n = 5; \ R_s = 0.999976 \)

| \((20, 20)\) | Estimate | 0.983765 | 0.999686 | 0.999671 | 0.999066 | 0.999066 |
| | MSE | 0.005483 | 0.000012 | 0.000002 | 0.000001 | 0.000001 |
| | Abs.bias | 0.016211 | 0.000290 | 0.000305 | 0.000910 | 0.00091 |

| \((20, 40)\) | Estimate | 0.995475 | 0.999718 | 0.999704 | 0.999064 | 0.999064 |
| | MSE | 0.003837 | 0.000003 | 0.000029 | 0.000001 | 0.000001 |
| | Abs.bias | 0.004502 | 0.000258 | 0.000272 | 0.000912 | 0.000912 |

| \((40, 20)\) | Estimate | 0.988018 | 0.999921 | 0.999920 | 0.99914 | 0.99914 |
| | MSE | 0.003472 | 0.000001 | 0.000001 | 0.000001 | 0.000001 |
| | Abs.bias | 0.011958 | 0.000056 | 0.000056 | 0.000836 | 0.000836 |

| \((50, 50)\) | Estimate | 0.998528 | 0.999938 | 0.999937 | 0.999172 | 0.999172 |
| | MSE | 0.000034 | 0.000002 | 0.000002 | 0.000001 | 0.000001 |
| | Abs.bias | 0.001448 | 0.000038 | 0.000039 | 0.000804 | 0.000804 |
Table 3.7: Fitting summary for Carbon fiber breaking strength data for example 2

<table>
<thead>
<tr>
<th>Data</th>
<th>Model</th>
<th>K-S</th>
<th>LogL</th>
<th>AIC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>Gamma</td>
<td>0.0879</td>
<td>-53.01</td>
<td>110.0204</td>
<td>114.4886</td>
</tr>
<tr>
<td>Y</td>
<td>Gamma</td>
<td>0.0872</td>
<td>-56.75</td>
<td>117.4958</td>
<td>121.7820</td>
</tr>
<tr>
<td>X</td>
<td>log-normal</td>
<td>0.1173</td>
<td>-58.21</td>
<td>120.4179</td>
<td>124.8861</td>
</tr>
<tr>
<td>Y</td>
<td>log-normal</td>
<td>0.1059</td>
<td>-60.98</td>
<td>125.9492</td>
<td>130.2355</td>
</tr>
<tr>
<td>X</td>
<td>Exponential</td>
<td>0.3632</td>
<td>-95.08</td>
<td>192.1612</td>
<td>194.3953</td>
</tr>
<tr>
<td>Y</td>
<td>Exponential</td>
<td>0.2746</td>
<td>-77.51</td>
<td>157.0216</td>
<td>159.1648</td>
</tr>
</tbody>
</table>

Figure 3.1: fitted and empirical cumulative distribution function for data sets x and y