Chapter III

THE EFFECT OF VISCO-ELASTICITY ON PULSATILE BLOOD FLOW IN A POROUS CHANNEL WITH APPLICATION TO BLOOD FLOW

Introduction:

The potential importance of non-Newtonian effects in various physiological flows has been demonstrated by several pioneering experiments. Striking differences were found in flow patterns and calculated wall shear stress in models of arterial branches for vessel diameter larger than 1 mm between Newtonian and non-Newtonian fluid, Liepsch [6], Ku and Leipsch [5]. Duncan et al [3] studied wall shear rate during physiological pulsatile flow in flexible model of a human aortic bifurcation.

Analytical and numerical studies relating to non-Newtonian blood flow we are sparser. Starting flow of biviscosity fluid has been studied by Nakamura and Sawada [9], as the problem has been studied by Thurston [14-18]. Bird et al. [1], and Sharp et al. [13] for oscillatory flow of visco-elastic Maxwell fluid. The term viscosity, elasticity and Maxwell relaxation time are precisely quantitative descriptors for visco-elastic blood rheology. They are linked to the aggregation tendency and deformability of red blood cells.

Relaxation processes are key to they visco-elastic properties. A finite time is required for the micro-structure to change and the
rate of change is dependent on the relaxation processes for the blood in that structural organization. The relaxation time plays a direct role in assessing the importance of the elasticity relative to the viscosity as well as the balance between the energy stored in elastic deformation and energy dissipated in sliding.

In recent years the study of flow of non-Newtonian visco-elastic fluid has gained considerable importance owing to its wide application in various disciplines. Several workers like Mishra and Roy [7], Mishra and Acharya [8], Wang [19], Radhakrishnamacharya [10] and Bhatnagar [2] have studied visco-elastic flow problems. Rockwell et al. [11] assumed that blood is a visco-elastic fluid of small elasticity whose circulation in artery can be well explained by a suitable visco-elastic fluid model. Esmond and Clark [4] have shown that the pulsatile visco-elastic flow in a porous channel is important in the dialysis of blood in artificial kidneys. Recently Roy et al. [12] studied the flow of Walter liquid of short memory coefficient. The flow of visco-elastic fluid past a stretching plate with heat transfer is studied by Zakaria [20].

The problem of flow of a visco-elastic fluid under unsteady pressure gradient in a region bounded by two parallel porous plates is studied. It is assumed that at one plate fluid is injected with a certain constant velocity and that at other it is sucked off with the same velocity. An exact solution for the velocity field has been obtained.

In the present chapter we have discussed the unsteady flow of non-Newtonian Maxwell fluid in a porous channel for a fixed
injection Reynolds number. The effect of visco-elastic parameter on velocity field and shear stresses is discussed. It is observed that for the same value of \( K \), the amplitude of shear stress at injection wall is less than that at suction wall while the phase lag shows the complex character.

**Blood Rehology and Constitutive Models:**

The equation of motion for fluid is

\[
\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{v} \right) = \nabla \vec{\tau} + \rho \vec{G}
\]

where \( \rho \) is the density, \( t \) is time, \( \vec{v} \) and \( \vec{G} \) are the velocity and gravity vectors, and \( \vec{\tau} \) is the stress tensor. The shear stress at any point in a Newtonian fluid undergoing simple shear is linearly proportional to the rate of strain as a

\[
\vec{\tau} = \mu \dot{\gamma}
\]

where \( \mu \) is the constant coefficient of viscosity and \( \dot{\gamma} \) is strain rate.

**Mathematical Analysis:**

Consider the visco-elastic fluid model whose constitutive equations is characterized as

\[
\left( 1 + \lambda \frac{\partial}{\partial t} \right) \tau^{ik} = 2\mu e^{ik}
\]

..(3.1)

where \( \tau^{ik} \) is the stress tensor, \( \lambda \) is the relaxation time, \( \mu \) the constant having dimension of viscosity, and \( e^{ik} \) the rate of stress tensor defined as
\[ e^{ik} = \frac{1}{2}(v_{i,k} + v_{k,i}) \]

and for any contravariant tensor \( b^{ik} \)

\[
\frac{\partial b^{ik}}{\partial t} = \frac{\partial b^{ik}}{\partial t} + v^m \frac{\partial b^{ik}}{\partial x_m} - \frac{\partial v^k}{\partial x_m} b^{im} - \frac{\partial v^l}{\partial x_m} b^{mk}
\]

The continuity and momentum equations for incompressible unsteady flow of a fluid of density \( \rho \) are

\[(\rho v^t)_1 = 0 \]

..(3.2)

\[ \rho \left( \frac{\partial v^t}{\partial t} + v^j v^t_j \right) = -p_i - \tau^{ij} \]

..(3.3)

A two-dimensional analysis of a visco-elastic flow between two parallel porous plate at rest at distance \( h \) part is made. Flow through the entire region is maintained by unsteady pressure gradient

\[ \frac{1}{\rho} \frac{\partial \rho}{\partial x} = A(1 + e^{t\omega x}) \]

..(3.4)

where \( A \) is a known constant, \( \omega \) the frequency.

We take \( x \) and \( y \) axes along and perpendicular to the plants with origin on the lower plate. Let \( u \) and \( v \) are the components of velocity along these directions, respectively. Plates are uniformly porous therefore foreign fluid injected at one plate with certain constant velocity and sucked off at other with same velocity does not affect the motion of the fluid. Thus at sufficiently large distance from the
entrance region flow may be assumed to be fully developed and in
the unsteady state physical variables can be taken to be a function
of $y$ and $t$ only.

Equation (3.2) amounts to the consideration that $v = -v_0$ (a
constant) everywhere and $u = u(y,t)$

Equation (3.3) gives

\[ \rho \left( \frac{\partial u}{\partial t} - v_0 \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \]

..(3.5)

\[ 0 = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yy}}{\partial y} \]

..(3.6)

Equation (3.1) gives

\[ \tau_{xy} + \lambda \left( \frac{\partial \tau_{xy}}{\partial t} - v_0 \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial u}{\partial y} \right) = \mu \frac{\partial u}{\partial y} \]

..(3.7)

\[ \tau_{yy} + \lambda \left( \frac{\partial \tau_{yy}}{\partial t} - v_0 \frac{\partial \tau_{yy}}{\partial y} \right) = 0 \]

..(3.8)

From equation (3.6) and (3.8) we conclude that pressure in the
transverse direction is independent of $y$.

From equation (3.5) and (3.7) we then have

\[ \left( \frac{\partial u}{\partial t} - v_0 \frac{\partial u}{\partial y} \right) = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \lambda \frac{\partial}{\partial t} \left( \frac{1}{\rho} \frac{\partial p}{\partial x} \right) - \lambda \frac{\partial^2 u}{\partial t^2} \]
\[ + 2\nu_0 \lambda \frac{\partial^2 u}{\partial t \partial y} + (\mu - \rho v_0^2 \lambda) \frac{\partial^2 u}{\partial y^2} \]

..(3.9)

The boundary conditions are

\[ u = 0, \text{ at } y = 0 \text{ and } y = h \]

..(3.10)

We now look for a solution of the form

\[ u = \frac{A h^2}{\nu_0} [u_0(\eta) + e^{v^2\eta} u_1(\eta)] \]

..(3.11)

where \( \eta = \frac{y}{h} \) is a non-dimensional variable

substituting equation (3.11) into equation (3.9) and equating steady and unsteady terms separately to zero, we have

\[ (1 - K)u_0'' + Ru_0' - 1 = 0 \]

..(3.12)

\[ (1 - K)u_1'' + R \left( 1 + 2iK \frac{\sigma^2}{R^2} \right) u_1' - i\sigma^2 \left( 1 + i \frac{K\sigma^2}{R^2} \right) u_1 \]

\[- \left( 1 + i \frac{K\sigma^2}{R^2} \right) = 0 \]

..(3.13)

\[ u_0 = 0 = u_1 \text{ at } \eta = 0 \text{ and } \eta = 1 \]

..(3.14)
Where \( R = \frac{hv_0}{v} \) is the injection Reynolds number, \( K = \frac{v_0^2 \lambda}{v} \) the visco-elastic parameter, \( \sigma = h \left( \frac{v_0}{v} \right)^2 \) the frequency parameter and primes denote differentiation with respect to \( \eta \).

In view of condition (3.14), the solution of (3.12) and (3.13) are

\[
\frac{1}{R} \left[ \frac{1}{1 - e^{R(1 - K)^2}} \left( e^{R(1 - K)^2} - 1 \right) + \eta \right]
\]

\...(3.15)

\[
\frac{i}{\sigma^2 (e^{\alpha_1} - e^{\alpha_2})} \left[ (e^{\alpha_2 - 1}) e^{\alpha_1 \eta} - (e^{\alpha_1 - 1}) e^{\alpha_2 \eta} + (e^{\alpha_1} - e^{\alpha_2}) \right]
\]

\...(3.16)

where

\[
\alpha_1, \alpha_2 = \frac{R}{2(1 - K)}
\]

\[
\left\{ \left( 1 - 4 \frac{K \sigma^2}{R^4} \right) + 4i \frac{\sigma^2}{R^2} \right\}^{1/2} \pm \left( 1 + 2i \frac{K \sigma^2}{R^2} \right)
\]

Shear stresses at lower and upper plates are found to be

\[
\left( \frac{\partial u_1}{\partial \eta} \right)_{\eta=0} = \frac{i}{\sigma^2 (e^{\alpha_1 + \alpha_2} - 1)} \left[ \alpha_1 + \alpha_2 e^{\alpha_1 + \alpha_2} - (\alpha_1 + \alpha_2) e^{\alpha_2} \right]
\]

\...(3.17)
\[
\left( \frac{\partial u_1}{\partial \eta} \right)_{\eta=0} = \frac{i}{\sigma^2 (e^{\alpha_1 + \alpha_2} - 1)} \\
\left[ \alpha_1 + \alpha_2 e^{\alpha_1 + \alpha_2} - (\alpha_1 + \alpha_2) e^{\alpha_2} \right]
\]
\[\text{..(3.18)}\]

**Results and Discussion:**

From equations (3.15) – (3.18), we have discussed the effect of visco-elastic parameter on velocity field and shear stress. Results are illustrated with the help of tables. From table 3.1 and 3.2 we see that for fixed Reynolds and frequency parameters absolute value of velocity field of non-Newtonian Maxwell fluid is always greater than that of Newtonian fluid. For a given value of \( K \), the magnitude of real part of unsteady velocity increases rapidly with \( \eta \) and reaches to its maximum value below the central region of the channel and then decreases where as the maximum steady velocity is attained in the middle.

<table>
<thead>
<tr>
<th>Table – 3.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variation of steady velocity for ( R = 1, \sigma = 3 )</td>
</tr>
</tbody>
</table>

(97)
<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$K$</th>
<th>$u_0$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.0000</td>
</tr>
<tr>
<td>0.1</td>
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</tr>
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<td>0.3</td>
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<td>-0.0724</td>
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<tr>
<td>0.5</td>
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<td>-0.0831</td>
</tr>
<tr>
<td>0.7</td>
<td></td>
<td>-0.0529</td>
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<td></td>
<td>-0.0293</td>
</tr>
<tr>
<td>1.0</td>
<td></td>
<td>-0.0000</td>
</tr>
</tbody>
</table>

**Table – 3.2**

Variation of unsteady velocity for $R = 1, \sigma = 3$

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$K$</th>
<th>$u_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td></td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td></td>
<td>0.0</td>
</tr>
</tbody>
</table>

(98)
| K   | $\left( \frac{\partial u_1}{\partial \eta} \right)_{\eta=0}$ | $|\beta|$ | $\tan \phi$ |
|-----|-------------------------------------------------|-------|---------|
| 0.00| $-0.4041 + i0.3235$                             | 0.3831| $-0.7921$|
| 0.05| $-0.5718 + i0.2061$                             | 0.6064| $-0.3631$|
| 0.10| $-0.0721 + i0.0729$                             | 1.0701| $-0.0618$|

Table 3.4

Values of shear stress at upper plate for $R = 1, \sigma = 3$

| K   | $\left( \frac{\partial u_1}{\partial \eta} \right)_{\eta=1}$ | $|\beta|$ | $\tan \phi$ |
|-----|-------------------------------------------------|-------|---------|
| 0.00| $0.2813 - i0.1638$                             | 0.3221| $-0.5809$|
| $K$ | $\left(\frac{\partial u_1}{\partial \eta}\right)_{\eta=0}$ | $|\beta|$ | $\tan \phi$ |
|-----|-------------------------------------------------|--------|-----------|
| 1   | $0.03126 + i 0.2347$ | 0.3909 | $-0.0117$ |
| 2   | $-0.6267 + i 0.1354$ | 0.6411 | $-0.0033$ |
| 3   | $-1.8932 + i 0.0834$ | 1.8950 | $-0.0021$ |

**Table 3.5**

Values of shear stress at lower plate for $R = 1, \sigma = 3$
Graph 3.1: Value of shear stress at lower plate for $R = 1, \sigma = 3$
Graph 3.2 Value of shear stress at upper plate for $R = 1, \sigma = 3$
Graph 3.3 Value of shear stress at upper plate for $K = 0.1, \sigma = 3$

Table 3.6

Values of shear stress at upper plate for $K = 0.1, \sigma = 3$
\[
\begin{array}{|c|c|c|c|}
\hline
R & \left(\frac{\partial u_1}{\partial \eta}\right)_{\eta=1} & |\beta| & \tan \phi \\
\hline
1 & 0.3156 - i 0.1789 & 0.3627 & -0.0089 \\
2 & 0.5321 - i 0.2835 & 0.6029 & -0.0083 \\
3 & 0.02156 - i 0.1134 & 0.2436 & -0.0082 \\
\hline
\end{array}
\]

Table 3.7
Values of shear stress at lower plate for \( K = 0.1, R = 3 \)

\[
\begin{array}{|c|c|c|c|}
\hline
\sigma & \left(\frac{\partial u_1}{\partial \eta}\right)_{\eta=0} & |\beta| & \tan \phi \\
\hline
3 & -0.5189 + i 0.3201 & 0.6096 & -0.0096 \\
5 & -0.6837 + i 0.2189 & 0.7178 & -0.0050 \\
7 & -1.9321 + i 0.1251 & 0.9404 & -0.0021 \\
\hline
\end{array}
\]

Table 3.8
Values of shear stress at Upper plate for \( K = 0.1, R = 3 \)

\[
\begin{array}{|c|c|c|c|}
\hline
\sigma & \left(\frac{\partial u_1}{\partial \eta}\right)_{\eta=1} & |\beta| & \tan \phi \\
\hline
3 & 0.6315 - i 0.5421 & 0.8325 & -0.0134 \\
5 & 0.8917 - i 0.4367 & 0.9906 & -0.0076 \\
7 & 1.0156 - i 0.3217 & 1.0653 & -0.0049 \\
\hline
\end{array}
\]

(104)
Graph 3.4 Value of shear stress at upper plate for $K = 0.1, \sigma = 3$
Graph 3.5: Value of shear stress at lower plate for $K = 0.1, R = 1$
From table – 3.3 and 3.4 it is observed that at lower wall amplitude of the shear stress increases with $K$ but at upper walls an opposite effect is seen. Phase lag behind the lower and upper walls of the
channel decreases with fluid elasticity. At lower wall the numerical value of amplitude of visco-elastic non-Newtonian fluid is greater than that of non-Newtonian fluid at upper wall the same is not always true.

For fixed values of visco-elastic parameter and frequency parameter the value of amplitude of shear stress is increased when $R$ is increased and phase lag is decreased when $R$ is increased at lower plate. At upper plate the shear stress first increases then decreases and phase lag is decreased. And the values of shear stresses at lower and upper plate are same when frequency parameter is increased and visco-elastic parameter is fixed see in figures.

REFERENCES


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