Appendix I

Capital Asset Pricing Model (CAPM): A Mathematical Derivation

The Capital Asset Pricing Model (CAPM) of Sharpe (1964) is an equilibrium model built on the theoretical foundations of Markowitz. The basic assumptions on which CAPM is built are the following. There are a finite number of investors in the economy each with an endowment (wealth) that is small compared to the total endowment of all investors. An individual cannot affect the price of an asset by his buying or selling action; hence investors in total determine the price by their actions. Information is freely and instantly available to all investors accordingly all investors in the economy are in agreement concerning the expected returns and covariance of the joint probability distribution of the security returns. Investors make their investment decision utilising the framework developed by Markowitz. There are no transaction costs or other constraints and asset positions are fully divisible. Unlimited short sales are allowed so that investors can take any position positive, zero or negative in an investment.

Based on these assumptions the objective of an investor can be set as maximise the utility function

\[ U_k = E_k - \frac{V_k}{\tau_k} \]  

subject to the full investment criterion

\[ \sum_j X_{ik} = 1 \]
where, \( E_k = \sum_i X_{ik} E(R_i) \) - the expected return of investor \( k \)'s portfolio;

\[ V_k = \sum_i \sum_j X_{ik} X_{jk} \sigma_{ij} \] - volatility of investor \( k \)'s portfolio; \( E(R_i) \) - Expected return of asset \( i \); \( \sigma_{ij} \) - Covariance between return of asset \( i \) and asset \( j \);

\( X_{ik} \) - proportion of investor \( k \)'s portfolio invested in asset \( i \); \( \tau_k \) - Investor \( k \)'s risk tolerance.

Levy and Markowitz (1979) has shown \( U_k \) given by equation (A1.1) provide an excellent approximation to the investors expected utility irrespective of the nature of utility function over wealth and the return distribution. Alternatively, \( U_k \) may also be interpreted as risk adjusted expected return or certainty equivalent expected return and \( \frac{V_k}{\tau_k} \) as risk penalty.

By applying Lagrangian multiplier method one can easily arrive at the optimality condition as

\[ E(R_i) - \frac{2}{\tau_k} \sigma_{ik} = \lambda_k \]  \hspace{1cm} (A1.3)  

where, \( \sigma_{ik} = \sum_j X_{jk} \sigma_{ji} \) is the covariance of asset \( i \) with investor \( k \)'s optimal portfolio and \( \lambda_k \) is the Lagrangian multiplier.

Equation (A1.3) implies that an optimal portfolio can be obtained by selecting assets having marginal utility equal to \( \lambda_k \), which is the marginal utility
of wealth for investor $k$\textsuperscript{1}. If this were not the case it would be possible to reallocate funds from a security with a lower marginal utility to one with a higher marginal utility, thereby increasing the utility without violating the full investment constraint.

Since by assumption all investors have homogeneous expectations regarding the expected return and covariance of securities, the equilibrium condition of the market can be obtained by aggregating the conditions that must hold when each investor obtain an optimal portfolio taking into account the relative amount of wealth each has invested. Aggregating equation (A1.3) by assigning weights $w_k$, the investor $k$’s invested wealth expressed as the proportion of total wealth invested by all investors, it reduces to

$$E(R_i) - \frac{2}{\tau_m} \sigma_{im} = \lambda_m$$  \hspace{1cm} (A1.4)$$

where, $\tau_m = \sum_k w_k \tau_k$ - the societal risk tolerance; $\sigma_{im} = \sum_k w_k \sigma_{ik}$ - the covariance of asset $i$ with market portfolio, which include all assets, with each represented in proportion to its outstanding value; and $\frac{\sum_k w_k \tau_k \lambda_k}{\tau_m}$ - the weighted average of the value of $\lambda_k$, can be interpreted as societal marginal utility of wealth.

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\textsuperscript{1} The left hand side of equation (A1.3) is the partial derivative of $U_k$ with respect to $X_{ik}$ and $\dot{\lambda}_k$ is the partial derivative of the Lagrangian function with respect to the total invested wealth.
Thus in equilibrium the expected return of an asset is given by

\[ E(R_i) = \lambda_m + \frac{2}{\tau_m} \sigma_{im} \]  

(A1.5)

Usually this relationship is expressed in terms of an asset's beta, a scaled measure obtained by dividing the covariance between security's return and return on the market portfolio by the variance of return on the market portfolio

\[ \beta_{im} = \frac{\sigma_{im}}{\sigma_m^2} \]. That is

\[ E(R_i) = \lambda_m + \frac{2\sigma^2}{\tau_m} \beta_{im} \]  

(A1.6)

which can further be reduced to

\[ E(R_i) = \lambda_m + [E(R_m) - \lambda_m] \beta_{im} \]  

(A1.7)

This is the basic equation of CAPM, which says the expected return of all securities, and portfolios in equilibrium is positively\(^2\) linearly related to the market risk measured in terms of the market beta (but not to the total risk as often believed). This is the most important single conclusion derived from CAPM. From equation (A1.7) it can be easily seen that \( \lambda_m \) is the expected return on an asset for which the market beta equal to zero. If a risk less asset is available and can be held in positive or negative amounts as assumed by the model at rate \( R_f \), then \( \lambda_m \) can be replaced by \( R_f \) and CAPM equation reduces to

\[ E(R_i) = R_f + [E(R_m) - R_f] \beta_{im} \]  

(A1.8)

\(^2\) It is assumed that societal marginal utility of wealth is always less than average market return.
If the assumption of unlimited risk less lending and borrowing is dropped, $\lambda_m$ can be replaced by $E(R_0)$ the expected return on a zero beta portfolio in that case CAPM equation becomes,

$$E(R_i) = E(R_0) + [E(R_m) - E(R_0)]\beta_{im} \quad (A1.9)$$

This form of CAPM introduced in Black (1972) is called Black’s “Two-Factor form of CAPM or Zero-Beta version of CAPM.

CAPM is a fundamental contribution to understanding the manner in which capital markets function. The model says systematic risk is the only important ingredient in determining the expected return of an asset and these two are linearly related. The unsystematic risk that can be eliminated through diversification plays no role in determining the expected return. Thus investors get rewarded for bearing systematic risk only, not the total risk. As per the model every investor’s optimal portfolio is equivalent to an investment in market portfolio plus a lending or borrowing at risk free rate. For investors having higher risk appetite an optimal investment is a negative position in risk free asset and a positive position in market portfolio, while as for investors with less risk appetite an optimal investment is a positive position both in risk free asset and in market portfolio.
Appendix II

Dickey-Fuller Unit Root Test

The ‘unit root test’ of Dickey and Fuller (1979) is one of the popular tests for testing non-stationarity of a time series. A time series \( \{Y_t\} \) is non-stationary if

\[
Y_t = Y_{t-1} + U_t \quad \text{(A2.1)}
\]

where \( U_t \) is the stochastic error term with mean zero, constant variance and non-auto correlated. Such an error term is usually referred to as ‘white noise error term’ in the literature. Hence to test the stationarity of a time series one consider the regression model

\[
Y_t = \rho Y_{t-1} + U_t \quad \text{(A2.2)}
\]

and test the hypothesis \( H_0 : \rho = 1 \). The rejection of this null hypothesis implies the underlying time series is stationary. Equation (A2.2) can be equivalently expressed as

\[
\Delta Y_t = \rho \Delta Y_{t-1} + U_t \quad \text{(A2.3)}
\]

where \( \Delta Y_t = Y_t - Y_{t-1} \) and \( \delta = \rho - 1 \). In this case the null hypothesis becomes \( H_0 : \delta = 0 \). Under the null hypothesis the conventionally computed t-statistic does not follow student’s t distribution and hence it is known as \( \tau \)-statistic whose critical values are tabulated in Dickey and Fuller (1979) on the basis of Monte Carlo simulation.
It is possible that a time series could behave as non-stationary with a drift term. This means the value of $Y_t$ may not centre to zero and hence a constant term is usually added to equation (A2.3) to examine the stationarity of the time series. Further a linear trend variable could also be added to the equation to test stationarity after removing the trend effect. Thus for theoretical and practical reasons the Dickey-Fuller unit root test is usually applied to two regressions of the following form.

\[ \Delta Y_t = \beta_1 + \rho Y_{t-1} + U_t \]  \hspace{1cm} (A2.4)

\[ \Delta Y_t = \beta_1 + \beta_2 t + \rho Y_{t-1} + U_t \]  \hspace{1cm} (A2.5)

In each case the null hypothesis is $H_0 : \delta = 0$

It has been proved that one can use the regression (A2.5) to examine whether the trend in a time series is deterministic or stochastic. In estimating this regression if we find the given time series has a unit root, that is non-stationary, the time series exhibits stochastic trend.
Appendix III

The GRS Test

Gibbons, Ross and Shanken (1989) argued that the specific fit of CAPM is to be evaluated in a multivariate set up. They proved that the multivariate approach could lead to more appropriate conclusions than those based on the traditional univariate approach. According to them when (4.1) is estimated on \( p \) well-diversified portfolios, for the empirical validity of CAPM the hypothesis to be tested is

\[
H_0 : \alpha = 0
\]  

(A3.1)

where, \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_p) \) is the \( px1 \) vector of intercepts and 0 is the \( px1 \) null vector.

This hypothesis can be tested using Hotelling’s \( T^2 \) statistic\(^1\) given by

\[
T^2 = \frac{N}{1 + \hat{\theta}_m} \hat{\alpha}' \hat{\Sigma} \hat{\alpha}
\]  

(A3.2)

where, \( N \) - the number of time series observations;

\[
\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2, ..., \hat{\alpha}_p)' - \text{the estimated intercept vector};
\]

\[
\hat{\theta}_m = \frac{\bar{r}_m}{s_m};
\]

\( \bar{r}_m \) - Sample mean of the excess return series given by \( R_{mt} - R_{mt} \);

\( s_m \) - Sample standard deviation of the excess return series and

\(^1\) For details of the Hotelling’s \( T^2 \) test refer Anderson (1984)
\( \hat{\Sigma} \) - Unbiased estimate of the covariance matrix

This \( T^2 \) statistic follows Hotelling’s \( T^2 \) distribution with \( p \) and \( N-2 \) degrees of freedom and the GRS statistic for testing the hypothesis is

\[
F = \frac{T^2}{N-2} \frac{N - p - 1}{p} \tag{A3.3}
\]

Under the null hypothesis this GRS statistic follows \( F \) distribution with \( p \) and \( N-p-1 \) degrees of freedom, hence its significance can be tested using the concerned \( F \) distribution.
Appendix IV

Variance Inflation Factor

Variance inflation factor (VIF) is an index, which measures the impact of collinearity among the explanatory variables in a multiple regression model.

The VIF value of the explanatory variable $X_i$ is defined as

$$VIF_i = \frac{1}{1 - R_i^2}$$

(A4.1)

where $R_i^2$ is the coefficient of determination of the auxiliary regression obtained by regressing $X_i$ on the remaining explanatory variables. The larger value of $VIF_i$, the more collinear the variable $X_i$. As a rule of thumb if $VIF_i$ exceeds 10 ($R_i^2 > 0.90$) the variable is highly collinear.

Some authors use ‘tolerance’ as a measure of multicollinearity. It is defined as

$$TOL_i = \frac{1}{VIF_i}$$

(A4.2)

If $TOL_i = 1$, $X_i$ is not correlated with other variable and if it is zero there is perfect correlation.