Chapter 2

Synchronisation

2.1 Introduction

Synchronisation is a word, coined from two Greek words chronous (meaning time) and sign (meaning common) to imply occurring at the same time. This effect in physical systems was first observed by Christian Huygens who found that two clocks supported from a common wooden support had the same rhythmic motion. Even when the clocks were disturbed they reestablished their rhythms. Later this phenomenon was found and investigated in different man made devices like electronic generators, musical instruments etc. Nature employs synchronisation at different levels in biological systems. Synchronous variation of nuclei, synchronous firing of neurons, adjustment of heart rate with respiration, synchronous flashing of fire flies etc. are some such examples of natural phenomenon. Recently synchronisation has found new meaning and relevance in the context of nonlinear systems that are capable of exhibiting chaotic or complex behaviour. Such systems when coupled are found to reach synchronisation [49–56]. This finds wide application in coupled systems used in communication and image processing. If two non identical oscillators having their own frequencies
$f_1$ and $f_2$ are coupled together, they can oscillate with a common frequency which is termed as frequency entrainment or locking. Whether the two oscillators synchronise or not depends on two factors viz coupling strength and frequency detuning. If there is no coupling then there is no interaction between the oscillators and they cannot be synchronised. So also frequency detuning $\Delta f = f_1 - f_2$ quantifies how different the coupled oscillators are. It is found that if the detuning of the systems ($\Delta f$) is not very large, the frequencies of the two systems become equal or entrained resulting in synchronised behaviour.

Our interest in this work will be mostly centred around synchronisation occurring in nonlinear and chaotic systems. Since systems exhibiting chaos has sensitive dependence to initial conditions, the possibility of synchronising two such systems starting from different initial conditions is certainly an interesting feature with many useful applications. Interestingly synchronisation then is most often accompanied by control of chaos also leading to synchronised states of lower periodicities. However there are cases when synchronised chaotic states are also realised in practise. The systems that make use of this will therefore be extremely useful in communication networks with different stages of subsystems.

Certainly mutual influence through coupling is the physical agent that induces two or more systems to dynamically equivalent behaviour. Hence synchronisation, robustness of synchronisation and nature of the synchronised state depend to a large extent on the nature, mechanism and strength of coupling. These and related concepts will be made evident in the following sections.
2.2 Coupling schemes for synchronisation

It was Pecora and Carrol [11, 12] who first suggested that it is possible to synchronise even chaotic systems by introducing appropriate coupling between them. This has changed our outlook about chaotic systems. After that the synchronisation and control of chaos lead to many new technological applications. Synchronisation in general can be achieved by different techniques depending on the nature and mechanism of the coupling. We begin with a discussion of different types of coupling schemes usually applied to synchronise two or more systems. Although most of these schemes are employed in a large variety of continuous and discrete systems, there are a few specific methods suitable for isolated cases.

2.2.1 Unidirectional coupling

Schematic diagram for unidirectional coupling.

\[
\begin{bmatrix}
1 & 0 \\
\epsilon & 1 - \epsilon
\end{bmatrix}.
\] (2.1)

In unidirectional coupling the signal from one oscillator forces another one. Usually unidirectional coupling is considered for a regular lattice. For two interacting systems the situation of unidirectional coupling is described by the interaction matrix [50].
Then the two systems evolve as follows

\[ \dot{X}(t) = f(X(t)) \]
\[ \dot{Y}(t) = f(Y(t)) - \epsilon X(t) \]

(2.2)

It is found that one can obtain total synchronisation for two identical systems by this one way coupling [57]. The first chaotic system is treated as the drive system which drive the second system to induce synchronisation. For two Duffing Vanderpol oscillators, one acting as the drive system and other as the response system the equations of the drive system are

\[ \dot{X} = \nu(X^3 - \alpha X - Y) \]
\[ \dot{Y} = X - Y - Z \]
\[ \dot{Z} = \beta Y. \]

(2.3)

while the equations of the response system are

\[ \dot{X'} = \nu[(X')^3 - \alpha X' - Y'] + \nu \epsilon (X - X') \]
\[ \dot{Y'} = X' - Y' - Z' \]
\[ \dot{Z'} = \beta Y'. \]

(2.4)

Here \( \epsilon \) is the coupling parameter. From the numerical simulation of (2.4), the synchronised chaotic behaviour between \( X \) and \( X' \) variables for \( \epsilon = 1.0, \alpha = 0.35, \nu = 0 \) and \( \beta = 300 \) is shown in Fig 2.1(a).

The variation of Euclidean error

\[ S(t) = \sqrt{(X - X')^2 + (Y - Y')^2 + (Z - Z')^2} \]

in plotted against time in Fig 2.1(b). It is clear that when synchronisation is achieved \( S(t) = 0 \). Unidirectional coupling through a common driver as shown in schematic diagram is also feasible [58].
Figure 2.1: (a) Plot in \((X - X')\) plane showing synchronised chaotic behaviour; (b) The variation of the Euclidean error \(S(t) = \sqrt{(X - X')^2 + (Y - Y')^2 + (Z - Z')^2}\) plotted against time. When synchronisation is achieved \(S(t) = 0\).

2.2.2 Mutual coupling

Schematic view of mutual coupling

\[\begin{array}{c}
\text{Square} \\
\text{Arrow} \\
\text{Square}
\end{array}\]

The mutual coupling of two systems is introduced via the following dynamics

\[\begin{align*}
\dot{X}(t) &= f(X(t)) + \epsilon(Y(t) - X(t)) \\
\dot{Y}(t) &= f(Y(t)) + \epsilon(X(t) - Y(t)).
\end{align*}\]  

Here the coupling is applied both ways and the systems in general need not be identical. This method is also called closed loop or feedback method and can be classified [59] as an algorithm for suppressing chaos, as it converts chaotic behaviour into a desired periodic behaviour.

In the case of a chaotic oscillator and a weak periodic oscillator, mutual coupling can be introduced between the state variables of both as ex-
pressed below [59].

\[ \ddot{X} + d_1 \dot{X} - \{cY + (1 - c)X\}(1 - X^2) = f_1 \cos wt \]  \hspace{1cm} (2.7)

\[ \ddot{Y} + d_2 \dot{Y} - \{cX + (1 - c)Y\}(1 - Y^2) = f_2 \cos wt. \]  \hspace{1cm} (2.8)

The $X$ system is chaotic, while $Y$ is weakly periodic. The algorithm is used to suppress chaos in a chaotic Duffing Oscillator. Fig 2.2 gives the $X$ values of the Poincare’ points as a function of mutual coupling strength $c$ i.e., bifurcation plot. It is clear that in this case the suppression of chaos is also achieved by mutual coupling. Mutual coupling is utilised in the synchronisation of coupled map lattices where it is referred to as nearest neighbour diffusive coupling [60].

### 2.2.3 Global coupling

This is the most general case where each element interacts with all others and the interaction between two elements does not depend on the distance between them. Such an arrangement is called global coupling scheme. The schematic diagram of global coupling
We consider \( N \) identical chaotic maps interacting with each other through a dissipative coupling.

The system of equations are

\[
X_k(t + 1) = (1 - \epsilon)f(X_k(t)) + \frac{\epsilon}{N} \sum_{j=1}^{N} f(X_j(t))
\]  

(2.9)

The system (2.9) is often called the system of globally coupled maps or system with mean field coupling, because the last term of (2.9) is the mean over all elements of the ensemble. The interaction matrix \( \hat{L} \) can be represented in the form [50].

\[
\hat{L} = (1 - \epsilon)\hat{I} + \frac{\epsilon}{N} \hat{J}
\]

(2.10)

where \( \hat{I} \) is the unit matrix and all elements of the matrix \( \hat{J} \) are equal to one. Complete synchronisation is a regime when all states are equal \( i.e. \), \( X_1 = X_2 = \ldots = X_N \). Numerical simulations of the model in equation (2.9) show that after some transients, clustering is observed \( i.e. \), large groups of systems having exactly the same value.

In addition recent works concentrate on a variety of variations of the above schemes viz small world, random, hierarchical, randomly updated in time, delay coupling, random recurring etc. [61–67]. These are more relevant in higher dimensional and large size neural networks, biological systems like neurons etc.
2.3 Types of Synchronisation

Depending on the nature of the synchronised state the phenomenon of synchronisation is classified into total synchronisation, phase synchronisation, generalised synchronisation and lag synchronisation. The specific characteristics of each of these is discussed below.

2.3.1 Total synchronisation

Total synchronisation occurs due to strong mutual coupling of two chaotic oscillators. This has been observed [49] in driven oscillators and in discrete time systems. Thus by coupling two identical systems like two Lorenz systems, the corresponding variables can be made equal \( i.e., \) when the systems are synchronised the difference \( (X_1 - X_2), (Y_1 - Y_2) \) and \( (Z_1 - Z_2) \) vanishes [50]. The states of the two systems coincide and vary chaotically in time. This situation is called total synchronisation. Total synchronisation is a threshold phenomenon. It occurs only when the coupling exceeds some critical value that is proportional to the Lyapunov exponent of the individual system. Below the threshold, the states of the two chaotic systems are different but close to one another [50]. Fig 2.3 illustrates this in the context of two coupled Lorenz systems. In Fig 2.3a \( Z_1 vs Z_2 \) is plotted which is a straight line of slope one. The time series of the two systems are chaotic in time but completely coinciding as in Fig 2.3b.

Experimentally, total synchronisation was observed in chaotic intensity fluctuations of two \( Nd : Y ag \) lasers with modulated pump beams [51].

The coupling was implemented by overlapping the intercavity laser fields that can be varied. For strong coupling, the intensities become identical although they continue to vary in time chaotically.
Figure 2.3: Total synchronisation in coupled Lorenz models. (a) The states of the systems are identical, as can be easily seen on the plane $z_1$ vs. $z_2$ (b) the trajectory lies on the diagonal $z_1 = z_2$. The time series of two systems are chaotic in time, but completely coinciding.

### 2.3.2 Phase synchronisation

When two nonlinear systems exhibiting complex oscillating behaviour are coupled, they synchronise with a constant phase. To discuss this we consider chaotic waveforms as in Fig 2.4 [50]. Fixing a large time interval $\tau$, we can count the number of cycles within this interval as $N_\tau$ and compute the mean frequency

$$\langle f \rangle = \frac{N_\tau}{\tau}. \tag{2.11}$$

If the coupling of two oscillators is large, then their mean frequencies become equal. Weak coupling does not affect the chaotic nature of both oscillators; the amplitudes remain irregular and uncorrelated whereas the frequencies are adjusted in a fashion such that there is a phase shift between the signals.

This is a case of phase synchronisation of chaotic systems. Very strong coupling tends to make the states of both oscillators identical. As
Figure 2.4: An example of the chaotic oscillation obtained by simulation of the Rössler system (it can be considered as a model of a generalised chemical reaction) [53]. (The Rössler system, as well as other dynamical models discussed in this book (e.g., the Lorenz and the can der Pol models), are usually written in a dimensionless form.). The interval between successive maxima irregularly varies from cycle to cycle, $T_i \neq T_{i+1} \neq T_{i+2}$, as well as the height of the maxima (the amplitude). Although the variability of $T_i$ is in this particular case barely seen, in general it can be rather large; therefore we characterize the rhythm via an average quantity, the mean frequency.

Figure 2.5: Two chaotic signals $x_1, x_2$ originating from nonidentical uncoupled systems. Strong coupling makes the two signals nearly identical.

A result signals coincide and the regime of total synchronisation sets in as shown in Fig 2.5. As an example of phase synchronisation consider two coupled Rössler systems [51, 52]

\[
\begin{align*}
\dot{X}_{1,2} &= -w_{1,2}Y_{1,2} - Z_{1,2} + C(X_{2,1} - X_{1,2}) \\
\dot{Y}_{1,2} &= w_{1,2}X_{1,2} + 0.15Y_{1,2} \\
\dot{Z}_{1,2} &= 0.2 + Z_{1,2}(X_{1,2} - 10)
\end{align*}
\]

(2.12)

Here $w_{1,2} = 1 \pm \Delta w$ govern the frequency mismatch [68] and $c$ the strength of coupling. As the coupling is increased for a fixed mismatch $\Delta w$, we observe a transition from a regime, where the phase rotate with different
Figure 2.6: Phase difference of two coupled Rössler systems verses time for non synchronous \((C = 0.01)\), nearly synchronous \((C = 0.027)\), and synchronous \((C = 0.035)\) states. In the last case the amplitudes \(A_{1,2}\) remain chaotic.

velocities \(\phi_1 - \phi_2 \sim \Delta \Omega t\), to a synchronous state, where the phase difference does not grow with time \(|\phi_1 - \phi_2| < \text{constant}; \Delta \Omega = 0\). This transition is illustrated in Fig 2.6. In contrast to other types of synchronisation in chaotic systems [49, 69], here the instant fields \(X_{1,2}, Y_{1,2}, Z_{1,2}\) do not coincide. Moreover the correlations between the amplitudes of \(X_1\) and \(X_2\) are pretty small, although the phases are completely locked and in this respect the motions are highly incoherent. The amplitudes \(A_1\) and \(A_2\) may be chaotic.

The phenomenon of phase synchronisation find practical applications in cases where a coherent summation of outputs of slightly different generators operating in a chaotic regime is necessary. For this purpose, it is sufficient to synchronise phases, while amplitudes can remain uncorrelated which is an important problem of output summation in arrays of semiconductor lasers [70] and other physical and biological phenomena [71–73]. It is to be noted that phase synchronisation is a characteristic feature of autonomous continuous time systems.
2.3.3 Generalised synchronisation

The phenomenon of total synchronisation mentioned in 2.3.1 is not possible for coupled nonidentical systems. In this case the states cannot coincide exactly but they can be close to each other. For large coupling there can be a functional relation $X_2 = F(X_1)$ between the state of the two systems [74]. The form of the function $F$ may be rather complicated and the procedure of its finding may be nontrivial. Generalised synchronisation is observed for unidirectional coupling – when the first system (driving system) forces the second (driven system), but there is no backward action. Such a situation is called master slave coupling. The onset of generalised synchronisation can be interpreted as the suppression of dynamics of the driven system by the driving system.

Several methods exist for discussing generalised synchronisation between stochastic oscillators such as nearest neighbour method [74–77], but the frequently used method is that of auxiliary system [78–80]. The essence of auxiliary method is the following: Along with driven system $X_{dr}(t)$, auxiliary system $X_{a}(t)$ identical to it is introduced. Initial conditions for auxiliary system is different from the driven system. If generalised synchronisation takes place, the states of the driven and auxiliary systems must be identical. Thus the equivalence of the states of the driven and auxiliary systems [81–83] is a sign of generalised synchronisation between driving and driven oscillators.

To discuss the mechanism of generalised synchronisation is the case of dissimilar unidirectionally coupled dynamical systems, with non dissipative coupling, consider a unidirectionally coupled Lorenz and Rössler oscil-
The driving system is a Rössler oscillator.

\[
\begin{align*}
\dot{X}_d &= -\alpha(Y_d + Z_d) \\
\dot{Y}_d &= -\alpha(X_d + aY_d) \\
\dot{Z}_d &= -\alpha(P + Z_d(X_d - c)) 
\end{align*}
\]  
(2.13)

with parameters \(\alpha = 6, a = 0.2, P = 0.2, c = 5.7\).

The driven system is a Lorenz oscillator

\[
\begin{align*}
\dot{X}_{dr} &= \sigma(Y_{dr} + X_{dr}) \\
\dot{Y}_{dr} &= rX_{dr} - Y_{dr} - X_{dr}Z_{dr} + \epsilon Y_d \\
\dot{Z}_{dr} &= -bZ_{dr} + X_{dr}Y_{dr} 
\end{align*}
\]  
(2.14)

with parameters \(\sigma = 10, r = 28, b = 8/3\).

The parameter \(\alpha\) is the Rössler system serves to change the characteristic scale of oscillations in the Rössler system. The coupling parameter \(\epsilon\) at which generalised synchronisation is established [85] is \(\epsilon_{gs} = 6.66\). Then the oscillation amplitude \(Y_d\), the variable of the driving Rössler oscillator roughly equals 10 dimensionless units and oscillation amplitude \(Y_{dr}\), the variable of the driven Lorenz oscillator is about 20 units. It is seen then that the external force shifts the representative point in the phase space of the driven system towards domain with strong dissipation, as a result of which the intrinsic chaotic dynamics of the system becomes suppressed and the generalised synchronisation condition sets in.
2.3.4 Lag synchronisation

This is another type of synchronised behaviour of coupled chaotic oscillators with slightly mismatched parameter. In lag synchronisation the dynamical variables of two systems become synchronised but with a time lag with respect to each other [86]. Specifically given two slightly different chaotic oscillators \( \frac{dX_1}{dt} = F_1(X_1) \) and \( \frac{dX_2}{dt} = F_2(X_2) \), if there is a coupling between them with a coupling strength \( \epsilon \) then \( X_1(t) \) synchronises with \( X_2(t + \tau) \) in a range of \( \epsilon \) values, where \( \tau \neq 0 \) is the time lag [87] which depends on both \( \epsilon \) and the parameter characterising the difference between the two oscillators. As such, lag synchronisation cannot be observed if two oscillators are completely identical. Since in reality, it is not possible to have identical nonlinear oscillators, it was speculated that the phenomenon of lag synchronisation would be typical in systems of coupled chaotic oscillators [86]. Lag synchronisation relies on a precise timing between the dynamics of the coupled chaotic oscillators. With the increase in the coupling strength the system can undergo several transitions. First the transition from non synchronous state to phase synchronisation occurs. For large coupling lag synchronisation is observed. Finally with a further increase of coupling, the time shift decreases and this regime tends to total synchronisation.

To characterise lag synchronisation, we introduce similarity function \( S_d \) as a time averaged difference between the variables \( X_1 \) and \( X_2 \) taken with the time shift \( \tau \).

\[
S_d^2(\tau) = \frac{\langle [X_2(t + \tau) - X_1(t)]^2 \rangle}{\langle X_1^2(t) \rangle \langle X_2^2(t) \rangle}^{1/2}
\]  \hspace{1cm} (2.15)

and search for its minimum \( \sigma = \min S(\tau) \). If the signals \( X_1 \) and \( X_2 \) are independent, the difference between them is of the same order as the signals themselves, \( S_d(\tau) \sim 1 \) for all \( \tau \).

If \( X_1(t) = X_2(t) \) as in the case of total synchronisation \( S_d(\tau) \) reaches
its minimum $\sigma = 0$ for $\tau = 0$. When the similarity function $S_d(\tau)$ has a minimum for non zero time shift $\tau$, time lag is said to exist between the two systems.

As an example consider two coupled Rössler systems [86].

\begin{align*}
\dot{X}_{1,2} &= -w_{1,2}Y_{1,2} - Z_{1,2} + \epsilon(X_{2,1} - X_{1,2}) \\
\dot{Y}_{1,2} &= w_{1,2}X_{1,2} + aY_{1,2} \\
\dot{Z}_{1,2} &= f + Z_{1,2}(X_{1,2} - c)
\end{align*}

(2.16)

where $a = 0.165$, $f = 0.2$ and $c = 10$. The parameter $w_{1,2} = w_0 \pm \Delta$ determine the mismatch of natural frequency and $\epsilon$ in the coupling parameter. These equations serve as a good model for real systems having a strange attractor that appears in a period doubling cascade for example in electronic circuits [69, 88].

First the transition to phase synchronisation takes place for parameters $w_0 = 0.97$ and $\Delta = 0.02$ [86] and $\epsilon = \epsilon_p = 0.036$. For stronger coupling $\epsilon = \epsilon_l = 0.14$, a transition to lag synchronisation is observed. Similarity function $S_d(\tau)$ is plotted for different values of coupling strength in Fig 2.7. With the increase of coupling a minimum appears indicating the existence of lag synchronisation. It means that the states of the system become identical, but shifted in time with respect to each other. The regime of lag synchronisation can be clearly demonstrated by plotting $X_1(t + \tau)$ and $X_2(t)$.

To check the universal character of the lag synchronisation we discuss the effect in two dynamical models of real physical systems. One is the electronic circuit experimentally studied [88] in the context of total synchronisation, the other is the hybrid laser system experimentally studied [89]. It is observed that there are regimes of chaotic lag synchronisation in both cases with the similarity function having a sharp minimum. Similarity function in coupled circuits [88] attains its minimum for $\tau = 0.21$. For
Figure 2.7: Similarity function $S(\tau)$ for different values of coupling strength $\epsilon$ (1: $\epsilon = 0.01$, 2: $\epsilon = 0.015$, 3: $\epsilon = 0.05$, 4: $\epsilon = 0.075$, 5: $\epsilon = 0.015$, 6: $\epsilon = 0.02$). With the increase of coupling, a minimum appears, indicating the existence of a certain phase shift between interacting systems (curves 3 and 4). In the regime of lag synchronisation (curves 5 and 6), the minimum is extremely small.

the coupled laser system, $\sigma = 0.005$ for $\tau = 0.3$.

We find that in the lag synchronised state, full coherence of non identical systems is achieved due to interaction. This is important for coherent summation of radiation in laser arrays. As real systems can be hardly found fully identical, the lag synchronisation can be more frequently encountered in experiments with coupled systems than total synchronisation.

Recently another interesting case of anticipatory synchronisation is reported in some time delay systems. In this case $X_1(t) = X_2(t - \tau)$ which means the response $X_2(t)$ anticipates the drive $X_1(t)$ with a time shift $\tau$. The similarity function in this case is defined as [90]

$$S_a^2(\tau) = \frac{\langle[X_2(t) - X_1(t + \tau)]^2\rangle}{\langle[X_1^2(t)]\langle[X_2^2(t)]\rangle^{1/2}}.$$  \hspace{1cm} (2.17)

In this case also the system usually goes to complete synchronisation as the coupling strength is increased.
2.4 Characterisation of synchronised state

The synchronised state of two or more coupled systems can be characterised using different methods. Thus analysis of synchronisation error, similarity function, time taken for synchronisation, time to restore synchronisation once perturbated etc. quantify the basic features of the synchronised state.

Total synchronisation can be characterised by plotting the corresponding variables \( X(t) \) and \( Y(t) \) of the two subsystems. Consider two simple systems exhibiting chaotic behaviour [50] viz. logistic map \( f(X) = 4X(1-X) \) and tent map \( f(Y) = 1-2|Y| \). The two maps are described by variable \( X \) and \( Y \). As the dynamics of each variable is chaotic, if they are independent systems, one observe two independent random like behaviour. If we introduce a coupling, it tends to make the states \( X \) and \( Y \) come closer to each other. The two systems are then described by the following equations.

\[
\begin{align*}
X(t+1) &= (1-\epsilon)f(X(t)) + \epsilon f(Y(t)) \\
Y(t+1) &= \epsilon f(X(t)) + (1-\epsilon)f(Y(t)).
\end{align*}
\]

Here \( \epsilon \) is the coupling parameter. For \( \epsilon > 1/2 \), the two variables \( X_n \) and \( Y_n \) are identical and one immediately observes the state where \( X(t) = Y(t) \) for all \( t \). This is shown in Fig 2.8 where \( X(t) \) and \( Y(t) \) is plotted for \( \epsilon \approx 0.51 \).

To characterise the state of synchronisation between two systems the synchronisation error variables \( E \) is plotted against time. The error \( E \) is defined as the difference between the corresponding variables of the two systems \( i.e., \ E = |X - Y| \). Note that in the completely synchronous state the variable \( E \) vanishes.

As an example consider 3 Bragg acousto-optic bistable systems \( S_1 \), \( S_2 \) and \( S_3 \) [58,91–93]. \( S_1 \) is the driving system and \( S_2 \) and \( S_3 \) are two driven
systems. The dynamic equations of all the three systems can be written as

\[ X_{1,n+1}(t) = \pi \{ A - V \sin^2 [X_{1,n}(t) - X_b] \} \]
\[ X_{2,n+1}(t) = \pi \{ A - V_1 \sin^2 \left( \frac{X_{2,n}(t) - KX_{1,n}(t)}{1 + K} - X_b \right) \} \]
\[ X_{3,n+1}(t) = \pi \{ A - V_1 \sin^2 \left( \frac{X_{3,n}(t) - KX_{1,n}(t)}{1 + K} - X_b \right) \} \]

where \( K \) is the driving stiffness and \( V_1 \) is the bifurcation parameter for the driven systems \( S_2 \) and \( S_3 \). For \( K > 0 \), the driving system begins to drive the two driven systems. It is found that for \( K > 0.45 \), synchronised chaotic state can be obtained both between the driven systems and between the driving system and driven systems. Error variable \( E = X_{2,n}(t) - X_{3,n}(t) \) of the two driven systems is plotted against time and when synchronisation is achieved \( E \to 0 \). Similarity functions \( S(\tau) \) defined in equations (2.15) and (2.17) are used to characterise lag synchronisation and anticipatory synchronisation as explained in 2.3.4. Although the synchronised state is always studied asymptotically, there are situations where the time taken for the systems to synchronise is also relevant. Most often, this \( \tau_S \) can be controlled and made minimum by adjusting the coupling strength or other parameter.
of the problem. To test the robustness of the state after it is synchronised, a noise is applied and the time taken for regaining synchronisation is noted. This time $\tau_p$ also thus characterises the synchronised state. The maximum strength of the noise upto which synchronisation can be regained within a finite time, also is an indicator of the stability of the synchronised state. More quantitative characteristics of stability will be discussed in the next section.

2.5 Stability of synchronised state

2.5.1 Transverse Lyapunov exponent and stability

Consider a simple model of coupling of two, one dimensional maps described by the variables $X$ and $Y$ of the form

$$
X(t + 1) = f(X(t)) \\
Y(t + 1) = f(Y(t)).
$$ \hspace{1cm} (2.20)

As the dynamics of each variable are chaotic in the case of non interacting systems one observes two independent random like process without any mutual correlation. Any additional term containing both $X$ and $Y$ on the right hand side of the equation will provide coupling. Coupling tends to make $X$ and $Y$ closer to each other. The coupling does not affect the symmetric synchronous state $X = Y$. A general form of linear coupling operator is

$$
I = \begin{bmatrix}
1 - \epsilon & \epsilon \\
\epsilon & 1 - \epsilon
\end{bmatrix}.
$$ \hspace{1cm} (2.21)

This linear coupling should be combined with the nonlinear mapping as shown in equation (2.18). If the coupling parameter $\epsilon$ is taken as the bifurcation parameter and increased gradually from zero, one can find the critical
coupling $\epsilon_c$ such that for $\epsilon > \epsilon_c$ the synchronous state $X = Y$ is established. The dynamics can be represented in the plane $(X, Y)$, where the points outside the diagonal $X = Y$ represent the non synchronous state. For further characterisation it is convenient to introduce two new variables $U = \frac{X+Y}{2}$, $V = \frac{X-Y}{2}$ Then the variable $U$ is directed along the diagonal $X = Y$, while the variable $V$ corresponds to the direction, transverse to this diagonal. $V$ is this transverse variable and the stability of the synchronous state can be described as the transverse stability of the symmetric attractor. Rewriting equation (2.18) in the variable $U$ and $V$

\[ U(t+1) = \frac{1}{2} [f(U(t) + V(t)) + f(U(t) - V(t))] \]  
\[ V(t+1) = \frac{1 - 2\epsilon}{2} [f(U(t) + V(t)) - f(U(t) - V(t))] \]  
(2.22)

Linearising this system near the completely synchronous chaotic state $U(t)$, $V = 0$ and the linear mapping for small perturbations $u$ and $v$ is obtained as

\[ u(t+1) = f'(U(t))u(t) \]  
\[ v(t+1) = (1 - 2\epsilon)f'(U(t))v(t). \]  
(2.24)

These linearised equations govern the growth and decay of the perturbations of a dynamical system and is quantitatively measured by the Lyapunov exponents of the system [50]. A two dimensional mapping has two Lyapunov exponents. These exponents can be defined as the average logarithmic growth rates of $u$ and $v$.

\[ \lambda_u = \lim_{t \to \infty} \frac{\ln |u(t)| - \ln |u(0)|}{t} = \langle \ln |f'(u)| \rangle \]  
\[ \lambda_v = \lim_{t \to \infty} \frac{\ln |v(t)| - \ln |v(0)|}{t} = \ln |1 - 2\epsilon| + \langle \ln |f'(u)| \rangle \]  
(2.26)

(2.27)

It is found that the longitudinal Lyapunov exponent $\lambda_u$ is nothing else than the Lyapunov exponent of the uncoupled system. The transverse Lyapunov exponent $\lambda_\perp = \lambda_v$ is related to $\lambda$ as

\[ \lambda_\perp = \ln |1 - 2\epsilon| + \lambda. \]  
(2.28)
Thus the average growth or decay of the transverse perturbation \( v \) is governed by the transverse Lyapunov exponent \( \lambda_\perp \). The criteria for the stability of the synchronous state can be formulated as follows.

When \( \lambda_\perp > 0 \): synchronous state is unstable and \( \lambda_\perp < 0 \), corresponds to stable synchronous state. The stability threshold is then defined from the condition \( \lambda_\perp = 0 \); from (2.28)

\[
\ln |1 - 2\epsilon_c| = -\lambda
\]

which gives

\[
\epsilon_c = \frac{1 - e^{-\lambda}}{2}.
\]

This relation can be generalised to other forms of coupling.

In an array of systems the stability of the synchronisation can be determined by calculating the conditional Lyapunov exponent which is the Lyapunov exponent of the sub system. It is found that the system can be synchronised only if the Maximal conditional Lyapunov exponent (\( \lambda_{\text{MCLE}} \)) is less than zero [58].

We can deduce the Maximal Conditional Lyapunov exponent (\( \lambda_{\text{MCLE}} \)) of the driven systems represented by the equations (2.19) as follows.

\[
\lambda_{\text{MCLE}} = \left\{ \lim_{M \to \infty} \frac{1}{M \tau_d} \sum_{n=1}^{M} \log \left| \frac{\pi V_1}{1 + K} \right| \right\}
\]

where \( \tau_d \) is the delay time of the system and \( M \) is the computing time. Fig 2.9 shows \( \lambda_{\text{MCLE}} \) as a function of the driving stiffness \( K \) for different chaotic states of the driven systems. Here \( A = 0.5 \), \( X_b = \pi/6 \), \( V = 0.69 \) and \( V_1 = 0.67, 0.69 \) and 0.71 respectively. Under these parameters, both the driving system and the driven systems are in different chaotic states.
Figure 2.9: The MCLE of the driven systems changes with the driving stiffness $k$ for different driven states: $V_1 = 0.67, 0.69, 0.71$, respectively.

From Fig 2.9 we find that $\lambda_{MCLE}$ varies with $K$ and there exists a minimum value $K_{\text{min}}$ for each value $V_1$ where $\lambda_{MCLE}$ equals zero. When $K < K_{\text{min}}$, $\lambda_{MCLE}$ is positive and the two driven systems cannot be synchronised with each other, and with the driving system either. When $K > K_{\text{min}}$, the two driven systems can reach chaos synchronisation while the driving system can not synchronise to them. Then the driving stiffness $K$ is much larger than $K_{\text{min}}$, the chaos synchronisation can be achieved not only between the driven systems but also between the driving system and driven systems as well.

Thus the negative nature of largest transverse Lyapunov exponent guarantees the stability of the synchronised state and systems in chaos with negative maximal conditional Lyapunov exponent and driven by a chaotic system can go into synchronisation.
2.6 Recent trends in synchronisation

The control of chaos and synchronisation of chaotic dynamics are now established as central topics of study in nonlinear science. Most often controlling is achieved along with synchronisation. However synchronised chaotic states also have practical application since the naturally occurring chaotic orbit on the attractor of a dynamical system can be utilised to carry information. This process may offer practical advantages over the usual periodic carriages, such as the possibility of real time reconstructions of signal drop outs in the communication or even the possibility of multiplexing [49].

Identical chaotically evolving systems can be synchronised under suitable coupling. The notion of chaos synchronisation can be profitably used to develop a safe and reliable cryptographic system for private use [47]. In particular such secure methods are very much in need because cryptography is extremely useful in this age of information revolution.

Another form of synchronisation is observed if a comparatively strong periodic force acts on a chaotic system. This force can suppress chaos and make the system oscillate periodically with a period of the force. This is called chaos destroying synchronisation [94]. This is observed in electron beam scattered electromagnetic wave system.

Synchronisation of coupled lasers have wide applications [43] because laser is used increasingly in the domain of optical communication.

The dynamics of metapopulation in two patches undergoing migration among each other has been analysed using coupled logistic map models [95]. After synchronisation the metapopulation of the response system patches exhibit chaotic oscillations which are exactly similar to that of the drive system patches, irrespective of their initial population sizes. The syn-
chronisation of chaotic metapopulations has an important implication in that it allows one to study the dynamics of chaotic metapopulations of a collection of many sets of response system patches [96] by simply analysing the behaviour of the drive system in single patches. Further it is shown that the slight inhomogeneities in the environment can affect the suggested synchronisation procedure which can be regained by the addition of small external noise [97] to the system and is therefore more biologically realistic.

In a recent work the theory of chaotic synchronisation combined with the theory of information through a chaotic channel [98] is introduced. Chaotic channel is an active medium formed by connected chaotic systems. This subset of a large chaotic net represents the path along which information flows. The possible amount of information exchange between the transmitter where information enters the net and the receiver, the destination of information is proportional to the level of synchronisation between these two special sub systems. If phase synchronisation exists between the elements of the channel, a transmitted message can be fully recovered at a rate smaller than the case if these elements are completely synchronised. This approach could be useful to understand information transmission in more complex system like human brain or other chaotic networks.

Complex networks play an increasing role in the understanding of complex systems. A problem of fundamental importance is the impact of network topology on the dynamics of complex systems, which has been recently studied intensively in the context of synchronisation of coupled oscillators with heterogeneous connection degrees [63]. The networks are not in completely synchronised state for example when the coupling is not strong enough or when the oscillators are in the presence of noise or when the oscillators are non identical. A hierarchical organisation of the synchronisation behaviour with respect to the collective dynamics of the networks is found. Oscillators with more connections are synchronised more closely by the collective dynamics and constitute the dynamical core of the network.
Effects of such synchronisation of complex networks are relevant in many real world systems [99–101].

Synchronisation properties of coupled dynamics on time varying networks and the corresponding time average network has been studied recently [64]. Natural networks are not static in time and are ubiquitous in nature. The synchronisation properties of networks with time varying [101, 102] structure is compared with the synchronisation in static time average networks. It is found that if the Laplacians corresponding to the time varying networks commute with each other then the stability of the synchronised state for the time varying and time average topologies are approximately the same.

Differently coupled dynamical systems oscillating chaotically in time can lead to interesting emerging properties besides synchronisation due to changes in the environment [103]. This feature has been used to simulate the motion of a two legged robot where each leg is a chaotic oscillator and these oscillators are diffusively coupled. The appeal of such a system is the fast adaptation it shows when sudden unexpected dynamic changes in the environment occur. This adaptation process strongly depends on the coupling characteristics of the network.

A key scheme has been proposed by using delay times in communication using chaos synchronisation [104]. As a result the communication system affords to generate a session key easily. Since key generation is regarded as one of the most important algorithms in a crypto system, this method will enhance the importance of chaos synchronisation in communication [65].

Networks with connection delays arise naturally in many areas of science including biology population dynamics, neuroscience etc. Network with delays can synchronise more easily [105]. This synchronisability prop-
ergy is especially relevant for neural networks, circumventing the difficulties in establishing a concept of collective or simultaneous information processing in the presence of delayed information transmission [106]. Furthermore delays shape the dynamics of the synchronised system, leading to the emergence of a variety of new dynamics which individual units are not capable of producing.