Chapter 6

Quotient cordial graphs

In this chapter we introduce the quotient cordial labeling of graphs and investigate the quotient cordial labeling behaviour of path, cycle, star, complete graph etc.

6.1 Quotient cordial labeling

Definition 6.1.1. Let $G$ be a $(p, q)$ graph. Let $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ be an injective map. For each edge $uv$ assign the label $\left[\frac{f(u)}{f(v)}\right]$ (or) $\left[\frac{f(v)}{f(u)}\right]$ according as $f(u) \geq f(v)$ or $f(v) > f(u)$. Then $f$ is called a quotient cordial labeling of $G$ if $|p_e - p_o| \leq 1$ where $p_e$ and $p_o$ respectively denote the number of edges labelled with even integers and number of edges labelled with odd integers. A graph with a quotient cordial labeling is called a quotient cordial graph.

A simple example of a quotient cordial graph is given in Figure 6.1
Theorem 6.1.2. Any path is quotient cordial.

Proof. Let $P_n$ be the path $u_1u_2\ldots u_n$. Assign the label 1 to $u_1$. Then assign $2, 4, 8, \ldots (\leq n)$ to the consecutive vertices until we get $\lfloor \frac{n-1}{2} \rfloor$ edges with label 0. Then choose the least number $\leq n$ that is not used as a label. That is consider the label 3. Assign the label to the next non labelled vertices consecutively by $3, 6, 12, \ldots (\leq n)$ until we get $\lfloor \frac{n-1}{2} \rfloor$ edges with label 0. If not, consider the next least number $\leq n$ that is not used as a label. That is choose 5. Then label the vertices $5, 10, 20, \ldots (\leq n)$ consecutively. If the total number of edges with label 0 is $\lfloor \frac{n-1}{2} \rfloor$, then stop this process, otherwise repeat the same until we get the $\lfloor \frac{n-1}{2} \rfloor$ edges with label 0. Let $S$ be the set of integers less than or equal to $n$ that are not used as a label. Let $t$ be the least integer such that $u_t$ is not labelled. Then assign the label to the vertices $u_t, u_{t+1}, \ldots, u_n$ from the set $S$ in decreasing order. Clearly the above vertex labeling is a quotient cordial labeling. \hfill \Box

Theorem 6.1.3. The complete graph $K_n$ is quotient cordial iff $n \leq 4$.

Proof. Obviously $K_n, n \leq 4$ is quotient cordial. Assume $n > 4$. Suppose $f$ is a quotient cordial labeling of $K_n$.

Case 1. $n$ is odd.

Consider the sets,
\[ S_1 = \left\{ \left[ \frac{n}{n-1} \right], \left[ \frac{n}{n-2} \right], \ldots, \left[ \frac{n}{n+1} \right] \right\} \cup \left\{ \left[ \frac{n}{1} \right] \right\} \]

\[ S_2 = \left\{ \left[ \frac{n-2}{n-3} \right], \left[ \frac{n-2}{n-4} \right], \ldots, \left[ \frac{n-2}{n-1} \right] \right\} \cup \left\{ \left[ \frac{n-2}{1} \right] \right\} \]

\[ \vdots \]

\[ S_{n-1} = \left\{ \left[ \frac{3}{2} \right] \right\} \cup \left\{ \left[ \frac{3}{1} \right] \right\} \]

Clearly, \( S_1 \) contains \( \frac{n+1}{2} \) integers, \( S_2 \) contains \( \frac{n-1}{2} \) integers, \( S_3 \) contains \( \frac{n-3}{2} \) integers, \ldots, \( S_{n-2} \) contains 2 integers. Each \( S_i \) obviously contributes edges with label 1.

Therefore
\[
\begin{align*}
p_o & \geq |S_1| + |S_2| + \ldots + |S_{n-1}| \\
& = \frac{n+1}{2} + \frac{n-1}{2} + \frac{n-3}{2} + \ldots + 2 \\
& = 2 + 3 + \ldots + \frac{n+1}{2} \\
& = \left[ 1 + 2 + 3 + \ldots + \frac{n+1}{2} \right] - 1 \\
& = \frac{\left( \frac{n+1}{2} \right) \left( \frac{n+1}{2} + 1 \right)}{2} - 1 \\
& = \frac{(n+1)(n+3)}{8} - 1 \quad (6.1)
\end{align*}
\]

Next consider the sets,
\[ S'_1 = \left\{ \left\lfloor \frac{n-1}{n-2} \right\rfloor, \left\lfloor \frac{n-1}{n-3} \right\rfloor, \ldots, \left\lfloor \frac{n-1}{n+1} \right\rfloor \right\} \]
\[ S'_2 = \left\{ \left\lfloor \frac{n-3}{n-4} \right\rfloor, \left\lfloor \frac{n-3}{n-5} \right\rfloor, \ldots, \left\lfloor \frac{n-3}{n-1} \right\rfloor \right\} \]
\[ \vdots \]
\[ S'_{\frac{n-3}{2}} = \left\{ \left\lfloor \frac{4}{3} \right\rfloor \right\} \]

Clearly each of the sets \( S'_i \) also contribute edges with label 1.

Therefore
\[
p_o \geq |S'_1| + |S'_2| + \ldots + |S'_{\frac{n-3}{2}}|
\]
\[
= \frac{n-3}{2} + \frac{n-5}{2} + \frac{n-7}{2} + \ldots + 1
\]
\[
= 1 + 2 + 4 + \ldots + \frac{n-3}{2}
\]
\[
= \frac{(n-3)}{2} \left( \frac{n-3}{2} + 1 \right)
\]
\[
= \frac{(n-3)(n-1)}{8}
\]

(6.2)
From (6.1) & (6.2), we get

\[ p_o \geq \frac{(n+1)(n+3)}{8} - 1 + \frac{(n-3)(n-1)}{8} \]
\[ \geq \frac{n^2 + 4n + 3 + n^2 - 4n + 3 - 8}{8} \]
\[ \geq \frac{2n^2 - 2}{8} \]
\[ \geq \frac{n^2 - 1}{4} \]
\[ > \left\lceil \frac{n(n-1)}{4} \right\rceil + 1, \text{ a contradiction to } f \text{ is a quotient cordial labeling.} \]

Case 2. \( n \) is even.

Similar to case 1, we get a contradiction. \( \square \)

Theorem 6.1.4. Every graph is a subgraph of a connected quotient cordial graph.

Proof. Let \( G \) be a \((p,q)\) graph with \( V(G) = \{u_i : 1 \leq i \leq p\} \). Consider the complete graph \( K_p \) with vertex set \( V(G) \). Let \( f(u_i) = i, 1 \leq i \leq p \). By theorem 6.1.2, we get \( e_f(1) > e_f(0) \). Let \( e_f(1) = m + e_f(0), m \in \mathbb{N} \). Consider the two copies of the star \( K_{1,m} \). The supergraph \( G^* \) of \( G \) is obtained from \( K_p \) as follows: Take one star \( K_{1,m} \) and identify the central vertex of the star with \( u_1 \). Take another star \( K_{1,m} \) and identify the central vertex of the same with \( u_2 \). Let \( S_1 = \{x : x \text{ is an even number and } p < x < p + 2m\} \) and \( S_2 = \{x : x \text{ is an odd number and } p < x < p + 2m\} \). Assign the label to the pendent vertices adjacent to \( u_1 \) from the set \( S_1 \) in any order and then assign the label to the pendent vertices adjacent to \( u_2 \) from the set \( S_2 \). Clearly this vertex labeling is a quotient cordial labeling of \( G^* \). \( \square \)

Theorem 6.1.5. Any star \( K_{1,n} \) is quotient cordial.

Proof. Let \( V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\} \) and \( E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\} \).
Assign the label 1 to the central vertex \( u \) and then assign the labels \( 2, 3, \ldots, n + 1 \) to the pendent vertices \( u_1, u_2, \ldots, u_n \). That \( f \) is a quotient cordial labeling follows from the Table 6.1.

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( p_e )</th>
<th>( p_o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>even ( n \equiv 0 ) ( ) (mod ( n ))</td>
<td>( n )</td>
<td>( n )</td>
</tr>
<tr>
<td>odd ( n \equiv 1, 2 ) (mod ( n ))</td>
<td>( n + 1 )</td>
<td>( n - 1 )</td>
</tr>
</tbody>
</table>

Table 6.1:

Theorem 6.1.6. The bistar \( B_{n,n} \) is quotient cordial.

Proof. Let \( V(B_{n,n}) = \{u, u_i, v, v_i : 1 \leq i \leq n\} \) and \( E(B_{n,n}) = \{uv, uu_i, vv_i : 1 \leq i \leq n\} \). Assign the label 1 to \( u \) and assign the label 2 to \( v \). Then assign the labels \( 3, 4, 5, \ldots, n + 2 \) to the vertices \( u_1, u_2, \ldots, u_n \). Next assign the label \( n + 3, n + 4, \ldots, 2n + 2 \) to the pendent vertices \( v_1, v_2, \ldots, v_n \). The edge condition is given in Table 6.2.

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( p_e )</th>
<th>( p_o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0, 1, 2 ) (mod ( n ))</td>
<td>( n + 1 )</td>
<td>( n )</td>
</tr>
<tr>
<td>( n \equiv 3 ) (mod ( n ))</td>
<td>( n )</td>
<td>( n + 1 )</td>
</tr>
</tbody>
</table>

Table 6.2:

Hence \( f \) is a quotient cordial labeling.

\( \square \)

Theorem 6.1.7. Let \( C_3 \) be the cycle \( u_1u_2u_3u_1 \). Let \( G \) be a graph obtained from \( C_3 \) with \( V(G) = V(C_3) \cup \{v_i, w_i, z_i : 1 \leq i \leq n\} \) and \( E(G) = E(C_3) \cup \{u_1v_i, u_2w_i, u_3z_i : 1 \leq i \leq n\} \). Then \( G \) is quotient cordial.

Proof. Define \( f : V(G) \to \{1, 2, 3, \ldots, 3n+3\} \) by \( f(u_1) = 1, f(u_2) = 2, f(u_3) = 3 \).
Case 1. \( n \equiv 0, 2, 3 \) (mod 4).

Define
\[
\begin{align*}
    f(v_i) &= 3i + 1, \quad 1 \leq i \leq n \\
    f(w_i) &= 3i + 3, \quad 1 \leq i \leq n \\
    f(z_i) &= 3i + 2, \quad 1 \leq i \leq n.
\end{align*}
\]

Case 2. \( n \equiv 1 \) (mod 4).

Define
\[
\begin{align*}
    f(v_i) &= 3i + 2, \quad 1 \leq i \leq n \\
    f(w_i) &= 3i + 1, \quad 1 \leq i \leq n \\
    f(z_i) &= 3i + 3, \quad 1 \leq i \leq n.
\end{align*}
\]

The Table 6.3 shows that \( f \) is a quotient cordial labeling.

<table>
<thead>
<tr>
<th>values of n</th>
<th>( p_e )</th>
<th>( p_o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 1, 3 ) (mod 4)</td>
<td>( \frac{3n+3}{2} )</td>
<td>( \frac{3n+3}{2} )</td>
</tr>
<tr>
<td>( n \equiv 0, 2 ) (mod 4)</td>
<td>( \frac{3n+2}{2} )</td>
<td>( \frac{3n+4}{2} )</td>
</tr>
</tbody>
</table>

Table 6.3:

\[\square\]

**Theorem 6.1.8.** Let \( C_3 \) be the cycle \( u_1u_2u_3u_1 \). Let \( G \) be a graph obtained from \( C_3 \) with \( V(G) = V(C_3) \cup \{v_i, w_i, z_i : 1 \leq i \leq n\} \) and \( E(G) = E(C_3) \cup \{u_1v_i, u_2w_i, u_3z_i : 1 \leq i \leq n\} \). Then \( G \) is quotient cordial.

**Proof.** Define \( f : V(G) \to \{1, 2, 3, \ldots, 3n+3\} \) by \( f(u_1) = 1 \), \( f(u_2) = 2 \), \( f(u_3) = 3 \).

Case 1. \( n \equiv 0, 2, 3 \) (mod 4).
Define
\[
\begin{align*}
f(v_i) &= 3i + 1, \ 1 \leq i \leq n \\
f(w_i) &= 3i + 3, \ 1 \leq i \leq n \\
f(z_i) &= 3i + 2, \ 1 \leq i \leq n.
\end{align*}
\]

Case 2. \(n \equiv 1 \text{ (mod 4)}\).

Define
\[
\begin{align*}
f(v_i) &= 3i + 2, \ 1 \leq i \leq n \\
f(w_i) &= 3i + 1, \ 1 \leq i \leq n \\
f(z_i) &= 3i + 3, \ 1 \leq i \leq n.
\end{align*}
\]

The Table 6.4 shows that \(f\) is a quotient cordial labeling.

<table>
<thead>
<tr>
<th>values of (n)</th>
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</thead>
<tbody>
<tr>
<td>(n \equiv 1, 3 \text{ (mod 4)})</td>
<td>(\frac{3n + 3}{2})</td>
<td>(\frac{3n + 3}{2})</td>
</tr>
<tr>
<td>(n \equiv 0, 2 \text{ (mod 4)})</td>
<td>(\frac{3n + 2}{2})</td>
<td>(\frac{3n + 4}{2})</td>
</tr>
</tbody>
</table>

Table 6.4:
Example 6.1.9. A quotient cordial labeling of $G$ obtained from $C_3$ and $K_{1,7}$ is given in Figure 6.2.

![Figure 6.2](image-url)
References


Index

acyclic, 3
addition of edge, 2
bistar, 3
book, 5
butterfly graph, 6
cartesian product, 4
central vertex, 4
closed helm, 4
comb, 5
complement, 3
component, 3
connected, 3
cordial labeling, 8
corona, 5
crown, 5
cycle, 3
degree, 2
degree splitting graph, 6
double comb, 5
double cone, 4
double fan, 4
fan graph, 4
flower graph, 4
friendship graph, 6
gear graph, 4
graceful, 8
graph, 1
  bipartite, 3
  complete, 2
  complete bipartite, 3
  regular, 2
  star, 3
helm, 4
isolated vertex, 2
isomorphic, 3
jelly fish, 6
join of graphs, 4
ladder, 5
lotus inside a circle, 6
path, 3
pendant vertex, 2
planar grid, 5
prime cordial labeling, 8
prism, 5
product cordial graph, 8
product cordial labeling, 8
removal of a vertex, 2
rim vertex, 4
shadow graph, 6
spanning subgraph, 2
splitting graph, 5
spokes, 4
square of a path, 6
subdivided, 5
subgraph, 1
sunflower graph, 4
support, 2
tree, 3
union, 3
walk, 2
wheel, 4
List of Published Papers


- R.Ponraj, M.Maria Adaickalam and R.Kala, Further results on 3-difference cordial graphs, *Accepted for publication in Journal of Combinatorial Mathematics and Combinatorial Computing*.

- R.Ponraj, M.Maria Adaickalam and R.Kala, 3-difference cordiality of some special graphs, *Accepted for publication in Jordan Journal of Mathematics and Statistics*. 
In this paper we introduce new graph labeling called $k$-difference cordial labeling. Let $G$ be a $(p,q)$ graph and $k$ be an integer, $2 \leq k \leq |V(G)|$. Let $f : V(G) \rightarrow \{1, 2, \cdots, k\}$ be a map. For each edge $uv$, assign the label $|f(u) - f(v)|$. $f$ is called a $k$-difference cordial labeling of $G$ if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denote the number of vertices labelled with $x$, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labelled with 1 and not labelled with 1. A graph with a $k$-difference cordial labeling is called a $k$-difference cordial graph. In this paper we investigate $k$-difference cordial labeling behavior of star, $m$ copies of star and we prove that every graph is a subgraph of a connected $k$-difference cordial graph. Also we investigate 3-difference cordial labeling behavior of some graphs.

Key Words: Path, complete graph, complete bipartite graph, star, $k$-difference cordial labeling, Smarandachely $k$-difference cordial labeling.

AMS(2010): 05C78.

§1. Introduction

All graphs in this paper are finite and simple. The graph labeling is applied in several areas of sciences and few of them are coding theory, astronomy, circuit design etc. For more details refer Gallian [2]. Let $G_1, G_2$ respectively be $(p_1,q_1), (p_2,q_2)$ graphs. The corona of $G_1$ with $G_2$, $G_1 \odot G_2$ is the graph obtained by taking one copy of $G_1$ and $p_1$ copies of $G_2$ and joining the $i^{th}$ vertex of $G_1$ with an edge to every vertex in the $i^{th}$ copy of $G_2$. The subdivision graph $S(G)$ of a graph $G$ is obtained by replacing each edge $uv$ by a path $uwv$. The union of two graphs $G_1$ and $G_2$ is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. In [1], Cahit introduced the concept of cordial labeling of graphs. Recently Ponraj et al. [4], introduced difference cordial labeling of graphs. In this way we introduce $k$-difference cordial labeling of graphs. Also in this paper we investigate the $k$-difference cordial labeling behavior of star, $m$ copies of star etc. $\lfloor x \rfloor$ denote the smallest integer less than or equal to $x$. Terms and results not here follows from Harary [3].
§2. \( k \)-Difference Cordial Labeling

**Definition 2.1** Let \( G \) be a \((p, q)\) graph and \( k \) be an integer \( 2 \leq k \leq |V(G)| \). Let \( f : V(G) \rightarrow \{1, 2, \ldots, k\} \) be a function. For each edge \( uv \), assign the label \(|f(u) - f(v)|\). \( f \) is called a \( k \)-difference cordial labeling of \( G \) if \( |v_f(i) - v_f(j)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \), and Smarandachely \( k \)-difference cordial labeling if \( |v_f(i) - v_f(j)| > 1 \) or \( |e_f(0) - e_f(1)| > 1 \), where \( v_f(x) \) denote the number of vertices labelled with \( x \), \( e_f(1) \) and \( e_f(0) \) respectively denote the number of edges labelled with 1 and not labelled with 1. A graph with a \( k \)-difference cordial labeling or Smarandachely \( k \)-difference cordial labeling is called a \( k \)-difference cordial graph or Smarandachely \( k \)-difference cordial graph, respectively.

**Remark 2.2** (1) \( p \)-difference cordial labeling is simply a difference cordial labeling:

(2) 2-difference cordial labeling is a cordial labeling.

**Theorem 2.3** Every graph is a subgraph of a connected \( k \)-difference cordial graph.

**Proof** Let \( G \) be \((p, q)\) graph. Take \( k \) copies of graph \( K_p \). Let \( G_i \) be the \( i \)-th copy of \( K_p \). Take \( k \) copies of the \( K_k \) and the \( i \)-th copies of the \( K_k \) is denoted by \( G'_i \). Let \( V(G_i) = \{u_i^j : 1 \leq j \leq k, 1 \leq i \leq p\} \). Let \( V(G'_i) = \{v_i^j : 1 \leq j \leq k, 1 \leq i \leq p\} \). The vertex and edge set of super graph \( G^* \) of \( G \) is as follows:

\[
V(G^*) = \bigcup_{i=1}^{k} V(G_i) \cup \bigcup_{i=1}^{k} V(G'_i) \cup \{w_i : 1 \leq i \leq k\} \cup \{w\}.
\]

\[
E(G^*) = \bigcup_{i=1}^{k} E(G_i) \cup \{u_i^j v_i^j : 1 \leq j \leq (p/2), 1 \leq i \leq k-1\} \cup \{u_i^{j+1}w, wv_i^j : 1 \leq j \leq p/2\} \cup \{u_1^j u_2^j \leq 1 \leq j \leq k-1\} \cup \{w_1 w_2\}.
\]

Assign the label \( i \) to the vertices of \( G_i \), \( 1 \leq i \leq k \). Then assign the label \( i+1 \) to the vertices of \( G'_i \), \( 1 \leq i \leq k-1 \). Assign the label 1 to the vertices of \( G'_k \). Then assign 2 to the vertex \( w \). Finally assign the label \( i \) to the vertex \( w_i \), \( 1 \leq i \leq k \). Clearly \( v_f(i) = p + (p/2) + 1, i = 1, 3, \ldots, k \), \( v_f(2) = p + (p/2) + 2 \) and \( e_f(1) = k(p/2) + k, e_f(0) = k(p/2) + k + 1 \). Therefore \( G^* \) is a \( k \)-difference cordial graph.

**Theorem 2.4** If \( k \) is even, then \( k \)-copies of star \( K_{1,p} \) is \( k \)-difference cordial.

**Proof** Let \( G_i \) be the \( i \)-th copy of the star \( K_{1,p} \). Let \( V(G_i) = \{u_j, v_i^j : 1 \leq j \leq k, 1 \leq i \leq p\} \) and \( E(G_i) = \{u_j v_i^j : 1 \leq j \leq k, 1 \leq i \leq p\} \). Assign the label \( i \) to the vertex \( u_j \), \( 1 \leq j \leq k \). Assign the label \( i+1 \) to the pendent vertices of \( G_i \), \( 1 \leq i \leq p/2 \). Assign the label \( k-i+1 \) to the pendent vertices of \( G_{1+i} \), \( 1 \leq i \leq p/2 - 1 \). Finally assign the label 1 to all the pendent vertices of the star \( G_k \). Clearly, \( v_f(i) = p + 1, 1 \leq i \leq k \), \( e_f(0) = e_f(1) = kp/2 \). Therefore \( f \) is a \( k \)-difference cordial labeling of \( k \)-copies of the star \( K_{1,p} \).

**Theorem 2.5** If \( n \equiv 0 \pmod{k} \) and \( k \geq 6 \), then the star \( K_{1,n} \) is not \( k \)-difference cordial.

**Proof** Let \( n = kt \). Suppose \( f \) is a \( k \)-difference cordial labeling of \( K_{1,n} \). Without loss of generality, we assume that the label of central vertex is \( r, 1 \leq r \leq k \). Clearly \( v_f(i) = t \),
1 \leq i \leq n \text{ and } i \neq r, v_f(r) = t + 1. \text{ Then } e_f(1) \leq 2t \text{ and } e_f(0) \geq (k - 2)t. \text{ Now } e_f(0) \geq (k - 2)t - 2t \geq (k - 4)t \geq 2, \text{ which is a contradiction. Thus } f \text{ is not a } k\text{-difference cordial.} \Box

Next we investigate 3-difference cordial behavior of some graph.

§3. 3-Difference Cordial Graphs

First we investigate the path.

**Theorem 3.1** Any path is 3-difference cordial.

*Proof* Let \( u_1 u_2 \ldots u_n \) be the path \( P_n \). The proof is divided into cases following.

**Case 1.** \( n \equiv 0 \pmod{6} \).

Let \( n = 6t \). Assign the label 1, 3, 2, 1, 3, 2 to the first consecutive 6 vertices of the path \( P_n \). Then assign the label 2, 3, 1, 2, 3, 1 to the next 6 consecutive vertices. Then assign the label 1, 3, 2, 1, 3, 2 to the next six vertices and assign the label 2, 3, 1, 2, 3, 1 to the next six vertices. Then continue this process until we reach the vertex \( u_n \).

**Case 2.** \( n \equiv 1 \pmod{6} \).

This implies \( n - 1 \equiv 0 \pmod{6} \). Assign the label to the vertices of \( u_i, 1 \leq i \leq n - 1 \) as in case 1. If \( u_{n-1} \) receive the label 2, then assign the label 2 to the vertex \( u_n \); if \( u_{n-1} \) receive the label 1, then assign the label 1 to the vertex \( u_n \).

**Case 3.** \( n \equiv 2 \pmod{6} \).

Therefore \( n - 1 \equiv 1 \pmod{6} \). As in case 2, assign the label to the vertices \( u_i, 1 \leq i \leq n - 1 \). Next assign the label 3 to \( u_n \).

**Case 4.** \( n \equiv 3 \pmod{6} \).

This forces \( n - 1 \equiv 2 \pmod{6} \). Assign the label to the vertices \( u_1, u_2, \ldots u_{n-1} \) as in case 3. Assign the label 1 or 2 to \( u_n \) according as the vertex \( u_{n-2} \) receive the label 2 or 1.

**Case 5.** \( n \equiv 4 \pmod{6} \).

This implies \( n - 1 \equiv 3 \pmod{6} \). As in case 4, assign the label to the vertices \( u_1, u_2, \ldots, u_{n-1} \). Assign the label 2 or 1 to the vertex \( u_n \) according as the vertex \( u_{n-1} \) receive the label 1 to 2.

**Case 6.** \( n \equiv 5 \pmod{6} \).

This implies \( n - 1 \equiv 4 \pmod{6} \). Assign the label to the vertices \( u_1, u_2, \ldots, u_{n-1} \) as in Case 5. Next assign the label 3 to \( u_n \). \Box

**Example 3.2** A 3-difference cordial labeling of the path \( P_9 \) is given in Figure 1.

![Figure 1](image-url)
Corollary 3.3 If \( n \equiv 0, 3 \pmod{4} \), then the cycle \( C_n \) is 3-difference cordial.

Proof The vertex labeling of the path given in Theorem 3.1 is also a 3-difference cordial labeling of the cycle \( C_n \). \( \square \)

Theorem 3.4 The star \( K_{1,n} \) is 3-difference cordial iff \( n \in \{1, 2, 3, 4, 5, 6, 7, 9\} \).

Proof Let \( V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\} \) and \( E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\} \). Our proof is divided into cases following.

Case 1. \( n \in \{1, 2, 3, 4, 5, 6, 7, 9\} \).

Assign the label 1 to \( u \). The label of \( u_i \) is given in Table 1.

<table>
<thead>
<tr>
<th>( n \backslash u_i )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
<th>( u_5 )</th>
<th>( u_6 )</th>
<th>( u_7 )</th>
<th>( u_8 )</th>
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</tr>
</tbody>
</table>

Table 1

Case 2. \( n \notin \{1, 2, 3, 4, 5, 6, 7, 9\} \).

Let \( f(u) = x \) where \( x \in \{1, 2, 3\} \). To get the edge label 1, the pendent vertices receive the label either \( x - 1 \) or \( x + 1 \).

Subcase 1. \( n = 3t \).

Subcase 1a. \( x = 1 \) or \( x = 3 \).

When \( x = 1 \), \( e_f(1) = t \) or \( t + 1 \) according as the pendent vertices receives t’s 2 or (t+1)’s 2. Therefore \( e_f(0) = 2t \) or \( 2t - 1 \). Thus \( e_f(0) - e_f(1) = t - 2 > 1, t > 4 \) a contradiction.

When \( x = 3 \), \( e_f(1) = t \) or \( t + 1 \) according as the pendent vertices receives t’s 2 or (t+1)’s 2. Therefore \( e_f(0) = 2t \) or \( 2t - 1 \). Thus \( e_f(0) - e_f(1) = t or t - 2 \). Therefore, \( e_f(0) - e_f(1) > 1 \), a contradiction.

Subcase 1b. \( x = 2 \).

In this case, \( e_f(1) = 2t \) or \( 2t + 1 \) according as pendent vertices receives t’s 2 or (t-2)’s 2. Therefore \( e_f(0) = t \) or \( t - 1 \). \( e_f(1) - e_f(0) = t or t + 2 \) as \( t > 3 \). Therefore, \( e_f(0) - e_f(1) > 1 \), a contradiction.

Subcase 2. \( n = 3t + 1 \).
Subcase 2a. \( x = 1 \) or \( 3 \).

Then \( e_f(1) = t \) or \( t + 1 \) according as pendent vertices receives \( t \)'s 2 or \( (t+1) \)'s 2. Therefore \( e_f(0) = 2t + 1 \) or \( 2t \). \( e_f(0) - e_f(1) = t + 1 \) or \( t - 1 \) as \( t > 3 \). Therefore, \( e_f(0) - e_f(1) > 3 \), a contradiction.

Subcase 2b. \( x = 2 \).

In this case \( e_f(1) = 2t \) or \( 2t + 1 \) according as pendent vertices receives \( t \)'s 1 and \( t \)'s 3 and \( t \)'s 1 and \( (t+3) \)'s 3. Therefore \( e_f(0) = t + 1 \) or \( t \). \( e_f(1) - e_f(0) = t - 1 \) or \( t \) as \( t > 3 \). Therefore, \( e_f(0) - e_f(1) > 1 \), a contradiction.

Subcase 3. \( n = 3t + 2 \).

Subcase 3a. \( x = 1 \) or \( 3 \).

This implies \( e_f(1) = t + 1 \) and \( e_f(0) = 2t + 1 \). \( e_f(0) - e_f(1) = t \) as \( t > 3 \). Therefore, \( e_f(0) - e_f(1) > 1 \), a contradiction.

Subcase 3b. \( x = 2 \).

This implies \( e_f(1) = 2t + 2 \) and \( e_f(0) = t \). \( e_f(1) - e_f(0) = t + 2 \) as \( t > 1 \). Therefore, \( e_f(1) - e_f(0) > 1 \), a contradiction. Thus \( K_{1,n} \) is 3-difference cordial iff \( n \in \{1, 2, 3, 4, 6, 7, 9\} \).

Next, we research the complete graph.

**Theorem 3.5** The complete graph \( K_n \) is 3-difference cordial if and only if \( n \in \{1, 2, 3, 4, 6, 7, 9, 10\} \).

**Proof** Let \( u_i, 1 \leq i \leq n \) be the vertices of \( K_n \). The 3-difference cordial labeling of \( K_n \), \( n \in \{1, 2, 3, 4, 6, 7, 9, 10\} \) is given in Table 2.

<table>
<thead>
<tr>
<th>( n ) ( \backslash ) ( u_i )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
<th>( u_5 )</th>
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</tbody>
</table>

**Table 2**

Assume \( n \notin \{1, 2, 3, 4, 6, 7, 9, 10\} \). Suppose \( f \) is a 3-difference cordial labeling of \( K_n \).

**Case 1.** \( n \equiv 0 \pmod{3} \).

Let \( n = 3t, t > 3 \). Then \( v_f(0) = v_f(1) = v_f(2) = t \). This implies \( e_f(0) = \binom{t}{1} + \binom{t}{2} + \binom{t}{2} + \)}
\[ t^2 = \frac{5t^2 - 3t}{2}. \] Therefore \( e_f(1) = t^2 + t^2 = 2t^2 \), \( e_f(0) - e_f(1) = \frac{5t^2 - 3t}{2} - 2t^2 > 1 \) as \( t > 3 \), a contradiction.

**Case 2.** \( n \equiv 1 \pmod{3} \).

Let \( n = 3t + 1 \), \( t > 3 \).

**Subcase 1.** \( v_f(1) = t + 1 \).

Therefore \( v_f(2) = v_f(3) = t \). This forces \( e_f(0) = t(t+1) + (t+1) + t(t+1) = \frac{3}{2}(5t^2 + t) \). \( e_f(1) = t(t + 1) + t^2 = 2t^2 + t \). Then \( e_f(0) - e_f(1) = \frac{3}{2}(5t^2 + t) - (2t^2 + t) > 1 \) as \( t > 3 \), a contradiction.

**Subcase 2.** \( v_f(3) = t + 1 \).

Similar to Subcase 1.

**Subcase 3.** \( v_f(2) = t + 1 \).

Therefore \( v_f(1) = v_f(3) = t \). In this case \( e_f(0) = \frac{5t^2 + 3t}{2} \) and \( e_f(1) = t(t + 1) + t(t + 1) = 2t^2 + 2t \). This implies \( e_f(0) - e_f(1) = \frac{5t^2 + 3t}{2} - (2t^2 + 2t) > 1 \) as \( t > 3 \), a contradiction.

**Case 3.** \( n \equiv 2 \pmod{3} \).

Let \( n = 3t + 2 \), \( t \geq 1 \).

**Subcase 1.** \( v_f(1) = t \).

Therefore \( v_f(2) = v_f(3) = t + 1 \). This gives \( e_f(0) = \binom{t+1}{2} + \binom{t+1}{2} + (t+1)(t+1) = \frac{5t^2 + 3t}{2} \) and \( e_f(1) = t(t + 1) + (t+1)^2 = 2t^2 + 3t + 1 \). This implies \( e_f(0) - e_f(1) = \frac{5t^2 + 3t}{2} - (2t^2 + 3t + 1) > 1 \) as \( t \geq 1 \), a contradiction.

**Subcase 2.** \( v_f(3) = t \).

Similar to Subcase 1.

**Subcase 3.** \( v_f(2) = t \).

Therefore \( v_f(1) = v_f(3) = t + 1 \). In this case \( e_f(0) = \binom{t+1}{2} + \binom{t+1}{2} + (t+1)(t+1) = \frac{5t^2 + 5t + 2}{2} \) and \( e_f(1) = t(t + 1) + t(t + 1) = 2t^2 + 2t \). This implies \( e_f(0) - e_f(1) = \frac{5t^2 + 5t + 2}{2} - (2t^2 + 2t) > 1 \) as \( t \geq 1 \), a contradiction.

**Theorem 3.6** If \( m \) is even, the complete bipartite graph \( K_{m,n} \) (\( m \leq n \)) is 3-difference cordial.

**Proof** Let \( V(K_{m,n}) = \{u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\} \) and \( E(K_{m,n}) = \{u_iv_j : 1 \leq i \leq m, 1 \leq j \leq n\} \).
$m, 1 \leq j \leq n$. Define a map $f : V(K_{m,n}) \rightarrow \{1, 2, 3\}$ by

$$f(u_i) = 1, \ 1 \leq i \leq \frac{m}{2},$$
$$f(u_{\frac{m}{2}+i}) = 2, \ 1 \leq i \leq \frac{m}{2},$$
$$f(v_i) = 3, \ 1 \leq i \leq \left\lceil \frac{m+n}{3} \right\rceil,$$
$$f(v_{\left\lceil \frac{m+n}{3} \right\rceil + i}) = 1, \ 1 \leq i \leq \left\lceil \frac{m+n}{3} \right\rceil - \frac{m}{2} - 1 \quad \text{if } m+n \equiv 1, 2 \pmod{3}$$
$$f(v_{\left\lceil \frac{m+n}{3} \right\rceil - \frac{m}{2} + i}) = 2, \ 1 \leq i \leq n - 2 \left\lceil \frac{m+n}{3} \right\rceil + \frac{m}{2} + 1 \quad \text{if } m+n \equiv 1, 2 \pmod{3}$$
$$f(v_{\left\lceil \frac{m+n}{3} \right\rceil - \frac{m}{2} - 1 + i}) = 2, \ 1 \leq i \leq n - 2 \left\lceil \frac{m+n}{3} \right\rceil + \frac{m}{2} \quad \text{if } m+n \equiv 0 \pmod{3}$$

Since $e_f(0) = e_f(1) = \frac{mn}{2}$, $f$ is a 3-difference cordial labeling of $K_{m,n}$. \hfill \square

**Example 3.7** A 3-difference cordial labeling of $K_{5,8}$ is given in Figure 2.

![Figure 2](image)

Next, we research some corona of graphs.

**Theorem 3.8** The comb $P_n \odot K_1$ is 3-difference cordial.

**Proof** Let $P_n$ be the path $u_1u_2\ldots u_n$. Let $V(P_n \odot K_1) = V(P_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1) = E(P_n) \cup \{u_iv_i : 1 \leq i \leq n\}$.

**Case 1.** $n \equiv 0 \pmod{6}$.

Define a map $f : V(G) \rightarrow \{1, 2, 3\}$ by

$$f(u_{6i-5}) = f(u_{6i}) = 1, \ 1 \leq i \leq \frac{n}{6},$$
$$f(u_{6i-4}) = f(u_{6i-1}) = 3, \ 1 \leq i \leq \frac{n}{6},$$
$$f(u_{6i-3}) = f(u_{6i-2}) = 2, \ 1 \leq i \leq \frac{n}{6}.$$

In this case, $e_f(0) = n - 1$ and $e_f(1) = n$.

**Case 2.** $n \equiv 1 \pmod{6}$.

Assign the label to the vertices $u_i, v_i \ (1 \leq i \leq n - 1)$ as in case 1. Then assign the labels 1, 2 to the vertices $u_n, v_n$ respectively. In this case, $e_f(0) = n - 1, e_f(1) = n$. 

Case 3.  \( n \equiv 2 \) (mod 6).

As in Case 2, assign the label to the vertices \(u_i, v_i\) (1 \(\leq i \leq n - 1\)). Then assign the labels 3, 3 to the vertices \(u_n, v_n\) respectively. In this case, \(e_f(0) = n, e_f(1) = n - 1\).

Case 4.  \( n \equiv 3 \) (mod 6).

Assign the label to the vertices \(u_i, v_i\) (1 \(\leq i \leq n - 1\)) as in case 3. Then assign the labels 2, 1 to the vertices \(u_n, v_n\) respectively. In this case, \(e_f(0) = n - 1, e_f(1) = n\).

Case 5.  \( n \equiv 4 \) (mod 6).

As in Case 4, assign the label to the vertices \(u_i, v_i\) (1 \(\leq i \leq n - 1\)). Then assign the labels 2, 3 to the vertices \(u_n, v_n\) respectively. In this case, \(e_f(0) = n - 1, e_f(1) = n\).

Case 6.  \( n \equiv 5 \) (mod 6).

Assign the label to the vertices \(u_i, v_i\) (1 \(\leq i \leq n - 1\)) as in case 5. Then assign the labels 3, 1 to the vertices \(u_n, v_n\) respectively. In this case, \(e_f(0) = n - 1, e_f(1) = n\). Therefore \(P_n \odot K_1\) is 3-difference cordial.  \(\square\)

**Theorem 3.9** \(P_n \odot 2K_1\) is 3-difference cordial.

**Proof** Let \(P_n\) be the path \(u_1u_2\cdots u_n\). Let \(V(P_n \odot 2K_1) = V(P_n) \cup \{v_i, w_i : 1 \leq i \leq n\}\) and \(E(P_n \odot 2K_1) = E(P_n) \cup \{u_iv_i, u_iw_i : 1 \leq i \leq n\}\).

Case 1.  \( n \) is even.

Define a map \(f : V(P_n \odot 2K_1) \to \{1, 2, 3\}\) as follows:

\[
\begin{align*}
    f(u_{2i-1}) &= 1, \quad 1 \leq i \leq \frac{n}{2} \\
    f(u_{2i}) &= 2, \quad 1 \leq i \leq \frac{n}{2} \\
    f(v_{2i-1}) &= 1, \quad 1 \leq i \leq \frac{n}{2} \\
    f(v_{2i}) &= 2, \quad 1 \leq i \leq \frac{n}{2} \\
    f(w_i) &= 3, \quad 1 \leq i \leq \frac{n}{2}.
\end{align*}
\]

In this case, \(v_f(1) = v_f(2) = v_f(3) = n, e_f(0) = \frac{3n}{2}\) and \(e_f(1) = \frac{3n}{2} - 1\).

Case 2.  \( n \) is odd.

Define a map \(f : V(P_n \odot 2K_1) \to \{1, 2, 3\}\) by \(f(u_1) = 1, f(u_2) = 2, f(u_3) = 3, f(v_1) = f(v_3) = 1, f(w_1) = f(w_2) = 3, f(v_2) = f(w_3) = 2,\)

\[
\begin{align*}
    f(u_{2i+2}) &= 2, \quad 1 \leq i \leq \frac{n-3}{2} \\
    f(u_{2i+3}) &= 1, \quad 1 \leq i \leq \frac{n-3}{2} \\
    f(v_{2i+2}) &= 2, \quad 1 \leq i \leq \frac{n-3}{2} \\
    f(v_{2i+3}) &= 1, \quad 1 \leq i \leq \frac{n-3}{2} \\
    f(w_{i+3}) &= 3, \quad 1 \leq i \leq n - 3.
\end{align*}
\]
Clearly, \( v_f(1) = v_f(2) = v_f(3) = n, \) \( e_f(0) = e_f(1) = \frac{3n-1}{2} \).

Next we research on quadrilateral snakes.

**Theorem 3.10** The quadrilateral snakes \( Q_n \) is 3-difference cordial.

**Proof** Let \( P_n \) be the path \( u_1u_2 \cdots u_n \). Let \( V(Q_n) = V(P_n) \cup \{v_i, w_i : 1 \leq i \leq n-1\} \) and \( E(Q_n) = E(P_n) \cup \{u_iv_i, v_iw_i, w_iu_{i+1} : 1 \leq i \leq n-1\} \). Note that \( |V(Q_n)| = 3n-2 \) and \( |E(Q_n)| = 4n-4 \). Assign the label 1 to the path vertices \( u_i, 1 \leq i \leq n \). Then assign the labels 2, 3 to the vertices \( v_i, 1 \leq i \leq n-1 \) respectively. Since \( v_f(1) = n, v_f(2) = v_f(3) = n-1, e_f(0) = e_f(1) = 2n-2 \), \( f \) is a 3-difference cordial labeling.

The next investigation is about graphs \( B_{n,n}, S(K_{1,n}), S(B_{n,n}) \).

**Theorem 3.11** The bistar \( B_{n,n} \) is 3-difference cordial.

**Proof** Let \( V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\} \) and \( E(B_{n,n}) = \{uv, uu_i, vv_i : 1 \leq i \leq n\} \). Clearly \( B_{n,n} \) has 2n+2 vertices and 2n+1 edges.

**Case 1.** \( n \equiv 0 \pmod{3} \).

Assign the label 1, 2 to the vertices \( u \) and \( v \) respectively. Then assign the label 1 to the vertices \( u_i, v_i \) \((1 \leq i \leq \frac{n}{3})\). Assign the label 2 to the vertices \( u_{\frac{i}{3}+i}, v_{\frac{i}{3}+i} \) \((1 \leq i \leq \frac{n}{3})\). Finally assign the label 3 to the vertices \( u_{\frac{i}{3}+i}, v_{\frac{i}{3}+i} \) \((1 \leq i \leq \frac{n}{3})\). In this case \( e_f(1) = n+1 \) and \( e_f(0) = n \).

**Case 2.** \( n \equiv 1 \pmod{3} \).

Assign the labels to the vertices \( u, v, u_i, v_i \) \((1 \leq i \leq n-1)\) as in Case 1. Then assign the label 3, 2 to the vertices \( u_n, v_n \) respectively. In this case \( e_f(1) = n \) and \( e_f(0) = n+1 \).

**Case 3.** \( n \equiv 2 \pmod{3} \).

As in Case 2, assign the label to the vertices \( u, v, u_i, v_i \) \((1 \leq i \leq n-1)\). Finally assign 1, 3 to the vertices \( u_n, v_n \) respectively. In this case \( e_f(1) = n \) and \( e_f(0) = n+1 \). Hence the star \( B_{n,n} \) is 3-difference cordial.

**Theorem 3.12** The graph \( S(K_{1,n}) \) is 3-difference cordial.

**Proof** Let \( V(S(K_{1,n})) = \{u, u_i, v_i : 1 \leq i \leq n\} \) and \( E(S(K_{1,n})) = \{uu_i, u_i v_i : 1 \leq i \leq n\} \). Clearly \( S(K_{1,n}) \) has 2n+1 vertices and 2n edges.

**Case 1.** \( n \equiv 0 \pmod{3} \).

Define a map \( f : V(S(K_{1,n})) \rightarrow \{1, 2, 3\} \) as follows: \( f(u) = 2, \)

\[
\begin{align*}
f(u_i) &= 1, & 1 \leq i \leq t \\
f(u_{t+i}) &= 2, & 1 \leq i \leq 2t \\
f(v_i) &= 3, & 1 \leq i \leq 2t \\
f(v_{2t+i}) &= 1, & 1 \leq i \leq t.
\end{align*}
\]
Case 2. \( n \equiv 1 \pmod{3} \).

As in Case 1, assign the label to the vertices \( u, u_i, v_i \) (\( 1 \leq i \leq n - 1 \)). Then assign the label 1, 3 to the vertices \( u_n, v_n \) respectively.

Case 3. \( n \equiv 2 \pmod{3} \).

As in Case 2, assign the label to the vertices \( u, u_i, v_i \) (\( 1 \leq i \leq n - 1 \)). Then assign the label 2, 1 to the vertices \( u_n, v_n \) respectively. \( f \) is a 3-difference cordial labeling follows from the following Table 3.

<table>
<thead>
<tr>
<th>Values of ( n )</th>
<th>( v_f(1) )</th>
<th>( v_f(2) )</th>
<th>( v_f(3) )</th>
<th>( e_f(0) )</th>
<th>( e_f(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 3t )</td>
<td>( 2t )</td>
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<td>( n = 3t + 2 )</td>
<td>( 2t + 2 )</td>
<td>( 2t + 2 )</td>
<td>( 2t + 1 )</td>
<td>( 3t + 2 )</td>
<td>( 3t + 2 )</td>
</tr>
</tbody>
</table>

**Table 3**

**Theorem 3.13** \( S(B_{n,n}) \) is 3-difference cordial.

Proof Let \( V(S(B_{n,n})) = \{ u, w, v, u_i, v_i, z_i : 1 \leq i \leq n \} \) and \( E(S(B_{n,n})) = \{ uw, uv, u_iw_i, v_iw_i, v_i, z_i : 1 \leq i \leq n \} \). Clearly \( S(B_{n,n}) \) has \( 4n + 3 \) vertices and \( 4n + 2 \) edges.

Case 1. \( n \equiv 0 \pmod{3} \).

Define a map \( f : V(S(B_{n,n})) \to \{ 1, 2, 3 \} \) by \( f(u) = 1 \), \( f(w) = 3 \), \( f(v) = 2 \),

\[
\begin{align*}
f(u_i) &= 2, & 1 \leq i \leq n \\
f(v_i) &= 1, & 1 \leq i \leq n \\
f(z_i) &= 3, & 1 \leq i \leq n \\
f(u_i) &= 1, & 1 \leq i \leq \frac{n}{3} \\
f(u_{\frac{n}{3} + i}) &= 2, & 1 \leq i \leq \frac{n}{3} \\
f(u_{\frac{2n}{3} + i}) &= 3, & 1 \leq i \leq \frac{n}{3}.
\end{align*}
\]

Case 2. \( n \equiv 1 \pmod{3} \).

As in Case 1, assign the label to the vertices \( u, w, v, u_i, v_i, w_i, z_i \) (\( 1 \leq i \leq n - 1 \)). Then assign the label 1, 2, 1, 3 to the vertices \( u_n, w_n, v_n, z_n \) respectively.

Case 3. \( n \equiv 2 \pmod{3} \).

As in Case 2, assign the label to the vertices \( u, w, v, u_i, v_i, w_i, z_i \) (\( 1 \leq i \leq n - 1 \)). Then assign the label 2, 2, 1, 3 to the vertices \( u_n, w_n, v_n, z_n \) respectively. \( f \) is a 3-difference cordial labeling follows from the following Table 4.
Finally we investigate cycles $C_4^{(t)}$.

**Theorem 3.14** $C_4^{(t)}$ is $3$-difference cordial.

**Proof** Let $u$ be the vertices of $C_4^{(t)}$ and $i$th cycle of $C_4^{(t)}$ be $uu_1^i u_2^i u_3^i u$. Define a map $f$ from the vertex set of $C_4^{(t)}$ to the set $\{1, 2, 3\}$ by $f(u) = 1$, $f(u_1^i) = 3$, $1 \leq i \leq t$, $f(u_2^i) = 1$, $1 \leq i \leq t$, $f(u_3^i) = 2$, $1 \leq i \leq t$. Clearly $v_f(1) = t + 1$, $v_f(2) = v_f(3) = t$ and $e_f(0) = e_f(1) = 2t$. Hence $f$ is $3$-difference cordial. \qed

**References**


