Chapter 1

Transport through Random Media

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Transport through Random Media

Propagation of matter in a random environment has its own qualitative and quantitative characteristics depending upon the type and nature of randomness. Waves in random media typically get localized and only back scattering occurs. This is true for light as well as matter waves. In this thesis we theoretically investigate some hitherto unexplored scenarios of transport through random media. In this thesis we focus on deriving the full probability distribution of reflection coefficient with the expectation that some experimental group will experimentally verify it.

1.1 Introduction

Lasing of light with feedback given by Anderson localization of light had been discussed theoretically and is practically seen in random media with lasing molecules present [1]. Theoretical models of reflection statistics
have been obtained by considering an imaginary part in the refractive index. Pradhan and Kumar [2] considered a constant imaginary part and derived the statistics of reflection coefficient of light. The amplification by the lasing molecules being randomly distributed was treated in the situation where the randomness of the background and the randomness of the lasing molecules were correlated, as is relevant for a large class of active random media [3].

1.2 Invariant Imbedding

The classical theory of transport focuses on the fluxes inside the transporting material and ends up with Boltzmann equation or linear response theory. Another method put forward by Landauer [4] is mainly concerned with external quantities such as reflected and transmitted currents. The later has been used extensively for the study of Anderson localisation. For instance transmission of light through a random medium can be expressed simply in terms of reflection and transmission amplitudes across the considered random medium. An interesting account of this method that has been used since the time of Stokes [5] can be found in reference [6].

The general principle of invariant imbedding method is to address directly the emergent quantities - the reflection and transmission coefficients. This can be done through the reduction of the initial (boundary-value) problem to a Cauchy (initial-value) problem, relative to the imbedding parameters. The reflection and transmission coefficients can then be described by a closed set of non-linear differential equations. When
randomness is present, one can use the stochastic differential equations obtained from the invariant imbedding approach, in order to derive directly Fokker-Planck equations for the probability distributions of emergent quantities [7]. This approach can be summarised as:

i) The differential equation of the wave field is first transformed into integral form. This first step can always be performed. We just need the expression of the Green functions in the region outside the layer.

ii) We use the boundary conditions to relate to the field at the boundary to R and T. Successive derivatives of the integral equation with respect to the Imbedding parameters lead to the desired equations.

The main advantage of this method is to provide a direct access to the physical quantities of main interest, through an initial value equation (Cauchy problem) [8].

1.2.1 Model

Consider a one-dimensional active disordered medium of length L with a random complex refractive index \( \eta(x) \), \( 0 \leq x \leq L \). For simplicity polarization effects are neglected. The complex amplitude of the field \( \Psi \) is assumed to be in the stationary regime, corresponding to the solutions of the type, \( \Psi(x, t) = e^{i k^2 t} \Phi(x) \). This will lead to following equation for \( \Phi(x) \),

\[
\frac{d^2 \Phi(x)}{dx^2} + k^2 [1 + \eta(x)] \Phi(x) = 0
\]  

(1.1)
where $\hat{k} = \frac{2\pi}{\lambda} \hat{k}$, $\hat{k}$ is the unit vector in the direction of wave and $\eta(x) = -\frac{V(x)}{k^2}$. Let a plane wave $\Phi_0(x, t) = e^{i k(L-x)}$ with wave number $k$ be incident on the right, $x = L$. The physical quantities of main interest are the complex reflection coefficient ($R$) and transmission coefficient ($T$) across the layer. A direct access to these coefficients is possible as they can be solutions of ordinary differential equations with respect to the imbedding parameter $L$. To the right and left of the medium the field has the form,

$$\Phi(x) = e^{i k (L-x)} + R e^{i k (x-L)}$$

$$\Phi(x) = T e^{-i k x}.$$  

From the boundary conditions (continuity of $\Phi(x)$ and $\frac{d\Phi}{dx}$) on the layer boundaries we get,

$$\Phi(L) = 1 + R,$$

$$\Phi'(L) = -i k (1 - R) = -i k (2 - \Phi(L)).$$

$$\Phi(0) = T,$$

$$\Phi'(0) = -i k T = -i k \Phi(0).$$

Eqns. (1.1) and (1.3) define a well posed problem and we want to find the complex reflection coefficient $R(L)$. The main first step is to transform the boundary value problem to an integral equation

$$\Phi(x; L) = e^{i k (L-x)} + \frac{i k}{2} \int_0^L dx' e^{i k |x-x'|} \eta(x') \Phi(x'; L)$$
where we have used the notation $\Phi(x;L) \equiv \Phi(x)$. This is the first step in invariant imbedding method. Differentiating Eq. (1.4) with respect to $L$ will lead to

$$
\frac{\partial \Phi(x;L)}{\partial L} = A(L)\Phi(x;L) + B(x;L) 
$$

(1.5)

where

$$
A(L) = i k + \frac{i k}{2} \eta(L).\Phi_L, \Phi_L \equiv \Phi(x = L;L) 
$$

(1.6)

and

$$
B(x;L) = \frac{i k}{2} \int_0^L dx' e^{ik|x-x'|} \eta(x') \left[ \frac{\partial \Phi(x';L)}{\partial L} - A(L) \Phi(x';L) \right] 
$$

(1.7)

Then using Eq. (1.5)

$$
B(x;L) = \frac{i k}{2} \int_0^L dx' e^{ik|x-x'|} \eta(x') B(x';L) 
$$

(1.8)

This shows in particular that $B(x;L)$ satisfies the same Eq. (1.4) as $\Phi(x;L)$ but without source term. This implies that $B(x;L) = 0$ and Eq. (1.5) becomes

$$
\frac{\partial \Phi(x;L)}{\partial L} = A(L)\Phi(x;L) 
$$

(1.9)

with the boundary conditions $\Phi(x;L)|_{x=L} = \Phi_x$ and $\Phi(L;L) = \Phi_L$. Taking the derivative of $\Phi_L$ with respect to $L$

$$
\frac{\partial \Phi_L}{\partial L} = \frac{\partial \Phi(x;L)}{\partial x} \bigg|_{x=L} + \frac{\partial \Phi(x;L)}{\partial L} \bigg|_{x=L} = -ik(2 - \Phi_L) + A(L)\Phi_L 
$$

(1.10)

and using the boundary conditions, (1.3) and (1.9) we get

$$
\frac{dR(L)}{dL} = 2ikR(L) + \frac{ik}{2} \eta(L)[1 + R(L)]^2. 
$$

(1.11)

Equation (1.11) is a stochastic differential equation and we are interested in the corresponding Fokker-Planck equation for the probability distribution $P(r,\theta;L)$ which can be readily obtained by following standard procedures.
1.3 Medium with Uncorrelated Disorders

We consider cases of photon transport through random medium with complex refractive index. In chapter 2, we consider the situation where the randomness of the background and the randomness of the imaginary part are not correlated as in laser dye doped gels or intralipid suspensions. The complex refractive index $\eta(x)$ is taken as equal to $\eta_r(x) + i (\eta_i(x) + a)$ where $\eta_r(x)$ is a Gaussian random function with zero mean, $\eta_i(x)$ is dichotomic Markovian process with mean zero. $\eta_i^2(x)$ is non-random with value $\Delta^2$. By taking the complex reflection coefficient the Langevin equation reduces to two stochastic differential equations. These two coupled stochastic differential equations will produce a flow of density $Q(r,\theta;L)$ in the $(r,\theta)$ space according to the stochastic Liouville equation by changing the length of the sample. The probability distribution of $r, \theta$ parametrized by $L$, $P(r,\theta;L)$ is got by averaging over the stochastic aspect, i.e. over all realizations of the random potentials $\eta_r(x)$ and $\eta_i(x)$. We assume that the disorder is weak and hence do a random phase approximation by taking $\theta$ to be uniformly distributed. We define,

$$W(r,\theta) = \left\langle \left\langle Q \right\rangle \right\rangle_{\eta_r} \quad \text{and} \quad W_1(r,\theta) = \left\langle \eta_i \left\langle Q \right\rangle \eta_r \right\rangle_{\eta_i}$$

(1.12)

$$P(r) = \left\langle \left\langle Q \right\rangle \right\rangle_{\eta_r} \quad \text{and} \quad P_1(r) = \left\langle \eta_i \left\langle Q \right\rangle \eta_r \right\rangle_{\theta}$$

(1.13)

as the probability distribution for $r$ after $\theta$ averaging. The stochastic coupled equations satisfied by $P$ and $P_1$ goes over to the one in Ref. [3] in the
Gaussian limit of the imaginary part. We consider the asymptotic limit and find the analytic solution. Only the numerical solution of the full-coupled stochastic equations can be obtained. The asymptotic probability distribution is different from the correlated case.

1.4 Negative Refractive Index

In 1967, Victor Veselago [9] realized that one of Maxwell's famous equations - which describe the interplay of electromagnetic waves and matter - has a special solution when both the electric permittivity and magnetic permeability are negative corresponding to a material with a negative refractive index. These materials are referred to as meta-materials or ‘left handed' because the inverted response reverses the energy flow associated with a ray of light. In normal materials the constituent atoms and molecules determine electrical and magnetic properties; they are much smaller than the wavelength of light so only the average response of the atoms matters. In the meta-materials an intermediate or meta-structure is engineered on a scale somewhere between atomic dimensions and the wavelength of radiation. Researchers have succeeded in fabricating a series of artificial materials with negative index of refraction since 2002. They could be used to make better lens that could have a host of applications in optoelectronics. In chapter 3, a theoretical investigation of negative refractive index materials that have random distribution of refractive index is done. This can be physically
realised in many ways. One simple way is to randomly stack negative refractive index materials of different index values in slab geometry.

In order to derive the probability distribution for the reflection coefficient for transport through negative random refractive index materials we consider the general case that includes a random imaginary part also. We take both real and imaginary parts of the refractive index to be Gaussian random functions. Then following the similar procedure as in chapter 2, we derive the Fokker-Planck equation satisfied by the probability distribution for the reflection coefficient averaged over its phase. The asymptotic solution is obtained numerically and analytically.

In chapter 4, the magnitudes and phases of reflection coefficient for propagation through a negative refractive index medium is numerically studied. We then do an ensemble averaging to get the average values of the transport coefficients for propagation through a random negative refractive index medium.

1.5 Solar Neutrinos

A neutrino appears as a reaction product in proton-proton chain responsible for converting hydrogen to helium in the central core of a star like the sun. The CNO cycle also produces neutrinos in the core of the star. They can supply information directly about nuclear processes within the stellar core. Neutrinos are elementary particles that have no mass and carry
no electric charge. Only the weak force is significant in the interaction of the neutrino with matter. Because stellar matter is virtually transparent to neutrinos, they can come promptly out of the core of the star with a high probability of not interacting at all with the matter in the star.

The weakly interacting nature of the neutrino is a mixed blessing for neutrino astrophysics. As neutrinos interact weakly with matter only few interact with detectors making stellar neutrino experiment hard to do. The probability for a neutrino to interact with an earth-based detector is so small. The sun is the only star close enough to provide adequate neutrino flux at the earth for any measurements to be feasible. So only solar neutrinos can be studied with the present technology.

In spite of experimental difficulties, data obtained indicated that the observed neutrino flux from the sun was about one-third of prediction based on solar models suggesting nonstandard neutrino properties. This discrepancy constitutes the Solar Neutrino Problem. Studies indicate that there are astrophysical situations for which fluctuations might significantly influence neutrino propagation despite the extreme weakness of neutrino interactions [10-15].

The long-standing deficit of solar neutrinos (SNP) has now been observed by operating experiments [16-20]. The main essence of the SNP is the strong deficit of the beryllium neutrinos [21]. On the other hand, the high-energy boron neutrinos are moderately suppressed, while the low
energy ones are almost undepleted. This strongly suggests that any astrophysical solution fails [21] in reconciling the experimental data with the Standard Solar Model (SSM) predictions [22-25].

Coupling of neutrinos to matter is important in the role of neutrino propagation in stellar as well as other astrophysical situations. In chapter 5, we have a general formulation for the probability distribution and the various moments of neutrino correlations as it propagates through random media. The evolution equation for $X$, the multi-flavour wave function or density matrix is given in the interaction representation,

$$i \frac{\partial X}{\partial t} = \rho(t) \hat{L}(t) X$$

with $X(0) = X_0$. $\hat{L}(t)$ is some linear operator that is a commutator for density matrix and $\rho(t)$ is Gaussian process with its first and second moments given by $\langle \rho(t) \rangle = 0$ and $\langle \rho(t) \rho(t') \rangle = g \delta(t, t')$. We consider ensembles of randomness and then average over them to get the probability distribution of $X$, $P(X,t)$, over the ensemble. Using Novikov's theorem [26] for Gaussian white noise we get,

$$\frac{\partial P}{\partial t} = -g \sum_{i,j=1}^{N} \frac{\partial}{\partial X_i} \left( L_{ij}(t) X_j \sum_{n,m=1}^{N} (L_{nm}(t) X_m P) \right)$$

Once we solve for $P(X,t)$ the probability distribution, the various moments of $X(t)$ can be obtained. Or using this equation we can derive the equations satisfied by the moments. We now consider the special case of two-flavour
neutrino system in a random environment. This could be neutrino propagation in sun with random density fluctuations. We derive a set of coupled equations for the various moments. This is another approach for studying neutrino propagation through random media. Similar coupled equations have been derived for higher moments. The second order moments we derive are standard results. We derive third order moments that in principle can be exactly solved. This method can be generalized to higher order moments.

Neutrino oscillations in the presence of matter and magnetic fields have been an area of intense study for approximately the last ten years. In the Mikheyev–Smirnov–Wolfenstein (MSW) effect, electron neutrinos on their journey from the core, are resonantly transformed into muon or tau neutrinos [27,28]. If neutrinos are Majorana fermions with transition magnetic moments, they can undergo a magnetic resonant transformation into muon or tau antineutrinos [29]. Neutrinos from supernovae explosions can be transformed from one flavor to another as they pass through the outer part of the star [30]. In many stellar situations, the matter density and/or magnetic fields may fluctuate about a mean value. Some well-known examples where fluctuations are likely to exist include the magnetic field and the matter density in the solar convective zone and also the turbulence of the post-supernovae matter that has been blown off by the explosion. A general approach to the neutrino oscillations in inhomogeneous matter was
developed in Ref. [31]. A Study of matter fluctuations which are not random, but harmonic [32,33], or occur as a jump-like change in the solar density [34], are available in the literature. Matter currents and density changes affect neutrino flavor oscillations in a similar way and have also been examined in Ref. [33]. If the magnetic field in a polarized medium has a domain structure with different strength and direction in different domains, the modification of the potential felt by neutrino due to polarized electron will have a random character [34]. Most neutrinos emitted from the core are produced by a neutral current process, and so the luminosities are approximately the same for all flavors. The energy spectra are approximately Fermi-Dirac with a zero chemical potential [35]. In chapter 5, we also discuss the neutrino propagation by discussing the time independent Dirac equation with random potential and derive the equation satisfied by the probability distribution of reflection coefficient.
Bibliography


