CHAPTER 7

Analysis of $M^X/G(1,b)/1$ Retrial Queueing Models with Different Vacation Policies

7.1 Introduction

In this chapter, two batch service retrial queueing models with different vacation policies are discussed. In model 1, bulk arrival and batch service retrial queueing system with working and non-working vacation is analysed. In model 2, state dependent arrival in bulk arrival and batch service retrial queueing system with immediate Bernoulli feedback, threshold and multiple vacations is analysed.

7.2 An $M^X/G(1,b)/1$ Retrial Queueing System with Server Failure and Two Types of Vacations

Stochastic modelling of queueing systems with vacations has gained lot of attention among researchers because it deals with problems existing in real time systems like manufacturing industries, communication network and production line systems, particularly in reducing congestion and optimum cost management. Function of media access control in wireless network is one of the application for working vacation queueing model. Queueing model with vacations has been studied by many experts which includes bulk arrival and batch service queue with vacation interruption by Haridass and Arumuganathan (2012) and a survey on queueing systems with vacations by Doshi (1986).

Bulk arrival retrial queueing model was first introduced by Falin(1976). In this chapter he proposed a mechanism to deal with bulk arrival retrial queues. Modelling queueing system with working vacation was first designed (Servi and Finn, 2002). In this article, working vacation is defined as an arrival of customers during vacation will be served in different service rates instead of waiting for service until the vacation completion. Also they obtained queue length distribution of M/M/1/Wv classical queueing system.

Queueing system with working vacation has been studied by many authors in recent times. In particular, Zhang and Hou (2010) derived steady state results of a service status and queue length distribution of a Non-Markovian queuing system with working vacation and vacation interruption. Gao and Liu (2013) adapted Bernoulli schedule control to interrupt vacation in M/G/1 single working vacation queueing system. Yang and Wu (2015) used Matrix-geometric method to derive queue length distribution M/M/1 queue system with working vacation and 'N' policy. Also they extended the model with breakdown and cost optimization.

In the above literature, working vacation queueing models were discussed only in continuous time systems, whereas Shweta Upadhyaya (2015) studied bulk arrival discrete time retrial queueing system with working vacation. This article concentrates on joint optimal values of vacation returning rate and service rate of the server during working vacation by using direct search method based on heuristic approach.

The excited literature review in working vacation queueing models are concerned with systems that can serve one at a time. Despite, in many practical situations, batch service of varying batches will be provided to customers, for example in inventory models, transportation systems and communication networks, etc. Retrial queueing system with batch service has been analysed by using supplementary variable technique (Haridass et al., 2012).

In this section, server failure, working and non-working vacation are introduced for bulk arrival and batch service retrial queueing model with constant retrial policy.

7.2.1 Model Description

In this section, bulk arrival and batch service queueing model with server failure, two type of vacation and constant retrial rate are considered, the two types of vacation are working and non-working vacation. Server leaves for vacation whenever the orbit is empty. In vacation queueing models the server will not provide any service. But in working vacation queueing model the server provides service in lower service rate rather than remaining idle during vacation. Customers are arriving into the system in bulk as primary customers, according to Poisson arrival rate. Primary customers are served in batches under general bulk service rule with minimum of one and maximum of ‘b’ number of customers. If an arriving batch of customers with batch size ‘τ’$(1 \leq \tau \leq b)$ finds that the server is free, then entire batch will be served immediately. On the other hand, if the batch size is more than ‘b’ then service will be provided for only ‘b’ customers, and the remaining $'\tau - b'$ customers will join the orbit. Additionally if an arriving batch of customers finds that the server is busy then entire batch joins the orbit to explore service again. The customers in orbit attempt for service one by one with constant retrial rate ‘γ’. In the service completion epoch, if there are no customers in the orbit then the server leaves for working vacation. During non-working vacation (secondary job) period the server does not serve any customers. Moreover in working vacation period, arriving customers are served in a service rate lower than the regular service rate. If there are no customers in the orbit even at the working vacation completion then the server leaves for non-working vacation. The server may get failure while serving customers, but the service will not be interrupted and it will be continued for current service by doing some technical arrangements. Renewal time is defined as time needed to repair the server or proper maintenance of the server. After completing a service, if the server got failure with probability φ then renewal of service station will be considered. In the time of renewal completion or if there is no breakdown with probability $1 - \varphi$, the orbit is empty then the server leaves for working vacation. If the slow service ends prior to the working vacation period then the server remains idle until the working vacation period ends. On the other hand, if slow service exceeds the working vacation period then the slow service rate will be changed into regular service rate. On completion of working vacation, if the orbit size is still zero then the server leaves for non-working vacation. Schematic representation of the proposed model is given below.

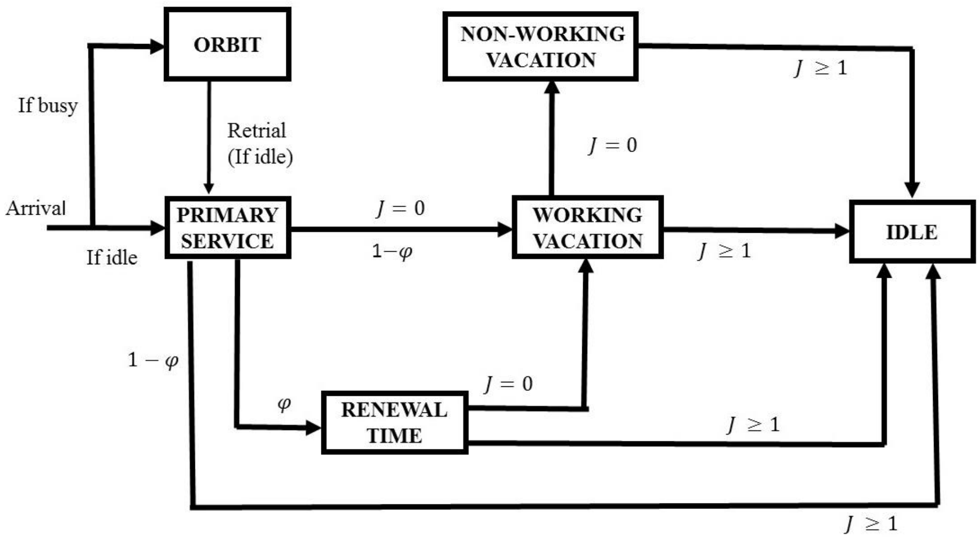

Fig. 7.1 Schematic Representation of the Queueing Mode1: $J -$ Orbit Size

For the proposed model, the probability generating function (PGF) of the steady state orbit size distribution at an arbitrary time epoch is obtained by using supplementary variable technique. Various performance measures are derived. Some special cases are also discussed. Numerical solution for particular values of parameters is presented.

7.2.2 Mathematical Model

Let X be the group size random variable of the arrival, λ be the Poisson arrival rate, g_k be the probability that 'k' customers arrive in a batch, γ be the constant retrial rate of the customer from the orbit. Service rate is μ_a when the server is not in working vacation and the service rate is $\mu_b (\mu_b < \mu_a)$ when the server is in working vacation. Working vacation duration follows exponential distribution with parameter η.

Let N_q (t) - number of customers waiting for service at time t.

N_s(t) - number of customers under the service at time t.

$X(z)$ - probability generating function of X.

$N(t)$ - number of customers in the orbit at time t.

Table 7.1 Notations

	Cumulative Distributive function	Probability Density function	Laplace -Stieltjes transform	Remaining time
Service	$S_b(x)$	$s_b(x)$	$\tilde{S}_b(\theta)$	${s_b}^0$(x)
Working vacation	$S_v(x)$	$s_v(x)$	$\tilde{S}_v(\theta)$	${s_v}^0$(x)
Non-working vacation	$V(x)$	$v(x)$	$\tilde{V}(\theta)$	$V^0(x)$
Renewal time	$R(x)$	$r(x)$	$\tilde{R}(\theta)$	$R^0(x)$

The server states at time t are defined as follows,

$$C(t) = \begin{cases} 0, \text{ if the server is busy with regular service} \\ 1. \text{ if the server is busy during working vacation} \\ 2, \text{ if the server is idle during working vacation} \\ 3, \text{ if the server is in non-working vacation} \\ 4, \text{ if the server is in renewal period} \\ 5, \text{ if the server is idle} \end{cases}$$

The state probabilities are defined as follows:

$$P_{ij}(x,t)dt = Pr\{N_s(t) = i, N_q(t) = j, x \leq {S_b}^0(t) \leq x + dt, C(t) = 0\}$$

$$1 \leq i \leq b, j \geq 0$$

$$Q_n(x,t)dt = Pr\{N(t) = n, x \leq {S_v}^0(t) \leq x + dt, C(t) = 1\}, \qquad n \geq 0$$

$$W_n(t)dt = Pr\{N(t) = n, C(t) = 2\}, \qquad n \geq 0$$

$$B_n(t)dt = Pr\{N(t) = n, x \leq V^0(t) \leq x + dt, C(t) = 3\}, \qquad n \geq 0$$

$$R_n(t)dt = Pr\{N(t) = n, x \leq R^0(t) \leq x + dt, C(t) = 4\}, \qquad n \geq 0$$

$$D_n(t)dt = Pr\{N(t) = n, C(t) = 5\}, \; n \geq 0$$

The following equations are obtained by using supplementary variable technique

$$D_j(t+\Delta t) = D_j(t)\left(1-\lambda\Delta t-\left(1-\delta_{j,0}\right)\gamma\Delta t\right)+(1-\varphi)\sum_{m=1}^{b} A_{mj}(0,t)\Delta t \tag{7.1}$$

$$+R_j(0,t)\Delta t + W_j(t)\eta\Delta t + V_j(0,t)\Delta t$$

$$P_{1j}(x-\Delta t, t+\Delta t) = P_{1j}(x,t)(1-\lambda\Delta t)+\gamma D_{j+1}(t)s_b(x)\Delta t \tag{7.2}$$

$$+D_j(t)\lambda g_1 s_b(x)\Delta t + \left(\int_0^\infty Q_j(y,t)dy\right)\eta s_b(x)\Delta t \qquad j\geq 0$$

$$P_{i0}(x-\Delta t, t+\Delta t) = P_{i0}(x,t)(1-\lambda\Delta t)+D_0(t)\lambda g_i s_b(x)\Delta t \qquad 2\leq i\leq b \tag{7.3}$$

$$P_{ij}(x-\Delta t, t+\Delta t) = P_{ij}(x,t)(1-\lambda\Delta t)+D_j(t)\lambda g_i s_b(x)\Delta t \tag{7.4}$$

$$+\sum_{k=1}^{j} P_{i\ j-k}(x,t)\,\lambda g_k\Delta t \qquad 2\leq i\leq b-1,\ j\geq 1$$

$$P_{bj}(x-\Delta t, t+\Delta t) = P_{bj}(x,t)(1-\lambda\Delta t)+\sum_{k=1}^{j} P_{b\ j-k}(x,t)\,\lambda g_k\Delta t \qquad j\geq 1 \tag{7.5}$$

$$+\sum_{k=0}^{j} D_{j-k}(t)\,\lambda g_{b+k}s_b(x)\Delta t$$

$$Q_n(x-\Delta t, t+\Delta t) = Q_n(x,t)(1-\lambda\Delta t-\eta\Delta t)+\sum_{k=1}^{n} Q_{n-k}(x,t)\lambda g_k\Delta t\left(1-\delta_{n,0}\right)$$

$$+\gamma W_{n+1}(t)s_v(x)\Delta t + W_j(t)\lambda g_1 s_v(x)\Delta t \tag{7.6}$$

$$W_0(t+\Delta t) = W_0(t)(1-\lambda\Delta t-\eta\Delta t)+Q_0(0,t)\Delta t \tag{7.7}$$

$$+(1-\varphi)\sum_{m=1}^{b} P_{m0}(0,t)\Delta t$$

$$W_n(t+\Delta t) = W_n(1-\lambda\Delta t-\eta\Delta t-\gamma\Delta t)+Q_n(0,t)\Delta t \tag{7.8}$$

$$B_0(x-\Delta t, t+\Delta t) = B_0(x,t)(1-\lambda\Delta t)+Q_0(0,t)v(x)\Delta t+W_0(0)v(x)\Delta t \tag{7.9}$$

$$B_n(x-\Delta t, t+\Delta t) = B_n(x,t)(1-\lambda\Delta t)+\sum_{k=1}^{n} B_{n-k}(x,t)\,\lambda g_k\Delta t \qquad n\geq 0 \tag{7.10}$$

$$R_0(x-\Delta t, t+\Delta t) = R_0(x,t)(1-\lambda\Delta t)+\varphi\sum_{m=1}^{b} P_{m0}(0,t)r(x)\Delta t \tag{7.11}$$

$$R_n(x-\Delta t, t+\Delta t) = R_n(x,t)(1-\lambda\Delta t) + \varphi \sum_{m=1}^{b} P_{mn}(0,t) r(x)\Delta t \quad (7.12)$$

$$+ \sum_{k=1}^{n} R_{n-k}(x,t)\, \lambda g_k \Delta t$$

7.2.3 Steady State Orbit Size Distribution

Dividing the above equations by Δt and allowing $\Delta t \to 0$, the steady state orbit size equations are obtained as follows:

$$0 = -D_j\left(\lambda + \left(1-\delta_{j,0}\right)\gamma\right) + (1-\varphi) \sum_{m=1}^{b} A_{mj}(0) \quad (7.13)$$
$$+R_j(0) + W_j(0)\eta + V_j(0)$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} P_{1j}(x) = -\lambda P_{1j}(x) + \gamma D_{j+1} s_b(x) + D_j \lambda g_1 s_b(x) + \left(\int_0^{\infty} Q_j(y)dy\right)\eta s_b(x) \quad (7.14)$$
$$j \geq 0$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} P_{i0}(x) = -\lambda P_{i0}(x) + D_0 \lambda g_i s_b(x) \qquad 2 \leq i \leq b \quad (7.15)$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} P_{ij}(x) = -\lambda P_{ij}(x) + D_j \lambda g_i s_b(x) + \sum_{k=1}^{j} P_{i\ j-k}(x)\, \lambda g_k \quad (7.16)$$
$$2 \leq i \leq b-1, \quad j \geq 1$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} P_{bj}(x) = -\lambda P_{bj}(x) + \sum_{k=1}^{j} P_{b\ j-k}(x)\, \lambda g_k + \sum_{k=0}^{j} D_{j-k}\, \lambda g_{b+k} s_b(x) \quad j \geq 1 \quad (7.17)$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} Q_n(x) = -(\lambda+\eta) Q_n(x) + \sum_{k=1}^{n} Q_{n-k}(x) \lambda g_k \left(1-\delta_{n,0}\right) + \gamma W_{n+1} s_v(x) \quad (7.18)$$
$$+W_j \lambda g_1 s_v(x)$$

$$0 = -(\lambda+\eta) W_0(x) + Q_0(0) + (1-\varphi) \sum_{m=1}^{b} P_{m0}(0) \quad (7.19)$$

$$0 = -(\lambda+\eta+\gamma) W_n(x) + Q_n(0) \quad (7.20)$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} B_0(x) = -\lambda B_0(x) + Q_0(0) v(x) + W_0 v(x) \quad (7.21)$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} B_n(x) = -\lambda B_n(x) + \sum_{k=1}^{n} B_{n-k}(x)\, \lambda g_k \qquad n \geq 0 \quad (7.22)$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} R_0(x) = -\lambda R_0(x) + \varphi \sum_{m=1}^{b} P_{m0}(0) r(x) \quad (7.23)$$

$$-\frac{d}{dx}R_n(x) = -\lambda R_n(x) + \varphi \sum_{m=1}^{b} P_{mn}(0) r(x) + \sum_{k=1}^{n} R_{n-k}(x)\,\lambda g_k \qquad n \geq 1 \quad (7.24)$$

The Laplace – Stieltjes transform of $A_{in}(x)$, $Q_{jn}(x)$ and $B_n(x)$ are defined as

$$\tilde{P}_{in}(\theta) = \int_0^\infty e^{-\theta x}\, P_{in}(x) dx \qquad \tilde{Q}_n(\theta) = \int_0^\infty e^{-\theta x}\, Q_n(x) dx$$

$$\tilde{R}_n(\theta) = \int_0^\infty e^{-\theta x}\, R_n(x) dx$$

Taking Laplace – Stieltjes transform on both sides from Eqn. 7.13 to Eqn. 7.24, we get the following equations

$$(\theta - \lambda)\tilde{P}_{1j}(\theta) = P_{1j}(0) - \gamma D_{j+1}\tilde{S}_b(\theta) - D_j \lambda g_1 \tilde{S}_b(\theta) - \tilde{Q}_j(0)\eta \tilde{S}_b(\theta) \; j \geq 0 \qquad (7.25)$$

$$(\theta - \lambda)\tilde{P}_{io}(\theta) = P_{i0}(0) - D_0 \lambda g_i \tilde{S}_b(\theta) \qquad 2 \leq i \leq b \qquad (7.26)$$

$$(\theta - \lambda)\tilde{P}_{ij}(\theta) = P_{ij}(0) - D_j \lambda g_i \tilde{S}_b(\theta) - \sum_{k=1}^{j} \tilde{P}_{i\, j-k}(\theta)\,\lambda g_k \qquad (7.27)$$

$$2 \leq i \leq b-1, j \geq 1$$

$$(\theta - \lambda)\tilde{P}_{bj}(x) = P_{bj}(0) - \sum_{k=1}^{j} \tilde{P}_{b\, j-k}(\theta)\,\lambda g_k - \sum_{k=0}^{j} D_{j-k}\,\lambda g_{b+k}\tilde{S}_b(\theta) \quad j \geq 1 \quad (7.28)$$

$$(\theta - (\lambda + \eta))\tilde{Q}_n(\theta) = Q_n(0) - \sum_{k=1}^{n} \tilde{Q}_{n-k}(\theta)\lambda g_k\big(1 - \delta_{n,0}\big) - \gamma W_{n+1}(0)\tilde{S}_v(\theta) \quad (7.29)$$
$$-W_j \lambda g_1 \tilde{S}_v(\theta)$$

$$\theta \tilde{B}_0(\theta) - B_0(0) = \lambda \tilde{B}_0(\theta) - (Q_0(0) + W_0(0))\tilde{V}(\theta) \qquad (7.30)$$

$$\theta \tilde{B}_n(\theta) - B_n(0) = \lambda \tilde{B}_n(\theta) - \sum_{k=1}^{n} \tilde{B}_{n-k}(\theta)\,\lambda g_k \qquad (7.31)$$

$$\theta \tilde{R}_0(\theta) - R_0(0) = \lambda \tilde{R}_0(\theta) - \varphi \sum_{m=1}^{b} P_{m0}(0)\tilde{R}(\theta) \qquad (7.32)$$

$$\theta \tilde{R}_n(\theta) - R_n(0) = \lambda \tilde{R}_n(\theta) - \varphi \sum_{m=1}^{b} P_{mn}(0)\tilde{R}(\theta) - \sum_{k=1}^{n} \tilde{R}_{n-k}(\theta)\,\lambda g_k \qquad (7.33)$$

7.2.4 Probability Generating Function

To derive the steady state probability generating function of an orbit size, the following probability generating functions are defined.

$$\tilde{P}_i(z,\theta) = \sum_{j=0}^{\infty} \tilde{P}_{ij}(\theta) z^j \qquad P_i(z,0) = \sum_{j=0}^{\infty} P_{ij}(0) z^j \qquad 2 \leq i \leq b$$

$$\tilde{B}_j(z,\theta) = \sum_{j=0}^{\infty} \tilde{B}_j(\theta) z^j \qquad B_j(z,0) = \sum_{j=0}^{\infty} Q_j(0) z^j \qquad j \geq 1 \qquad (7.34)$$

$$\tilde{R}(z,\theta) = \sum_{n=0}^{\infty} \tilde{R}_n(\theta)\, z^n \qquad R(z,0) = \sum_{n=0}^{\infty} R_n(0) z^n$$

$\tilde{Q}(z,\theta) = \sum_{n=0}^{\infty} Q_n(\theta) z^n \qquad W(z) = \sum_{n=0}^{\infty} W_n z^n \qquad D(z) = \sum_{j=0}^{\infty} D_j z^j$

The following equations are obtained by multiplying the equations from Eqn. 7.25 to Eqn. 7.32 with suitable powers of z^n and summing over n, then by using Eqn. 7.34, we have,

$$(\lambda+\gamma)D(z) - \gamma D_0 = (1-\varphi)\sum_{m=1}^{b} P_m(z,0) + R(z,0) + \eta W(z,0) + B(z,0) \tag{7.35}$$

$$(\lambda+\eta+\gamma)W(z) = \gamma W_0 + Q(z,0) + (1-\varphi)P_0 \tag{7.36}$$

$$(\theta-(\lambda+\eta)+\lambda X(z))\tilde{Q}(z,\theta) = Q(z,0) - (\gamma/z)(W(z)-W_0)\tilde{S}_v(\theta) - \lambda g_1 W(z)\tilde{S}_v(\theta) \tag{7.37}$$

$$(\theta-\lambda)\tilde{P}_1(z,\theta) - P_1(z,0) = -\gamma\tilde{S}_b(\theta)\frac{1}{z}(D(z)-D_0) - \lambda g_1 D(z)\tilde{S}_b(\theta) - \tilde{Q}(z,0)\eta\tilde{S}_b(\theta) \tag{7.38}$$

$$(\theta-\lambda+\lambda X(z))\tilde{P}_i(z,\theta) = P_i(z,0) - \lambda g_i D(z)\tilde{S}_b(\theta) \quad 2 \le i \le b-1 \tag{7.39}$$

$$(\theta-\lambda+\lambda X(z))\tilde{P}_b(z,\theta) = P_b(z,0) - \lambda g_b D_0\tilde{S}_b(\Theta) - \sum_{k=0}^{\infty}\lambda g_{b+k} z^k D(z)\tilde{S}_b(\theta) \tag{7.40}$$

$$\big(\theta-\lambda+\lambda X(z)\big)\tilde{B}(z,\theta) = B(z,0) - (Q_0+W_0)\tilde{V}(\theta) \tag{7.41}$$

$$\big(\theta-\lambda+\lambda X(z)\big)\tilde{R}(z,\theta) = R(z,0) - \varphi\tilde{R}(\theta)\sum_{m=1}^{b} P_m(z,0) \tag{7.42}$$

From Eqn. 7.36

$$W(z) = \frac{W_0\left(\gamma-\frac{\gamma}{z}\left(\tilde{S}_v\left((\lambda+\eta)-\lambda X(z)\right)\right)\right)+(1-\varphi)P_0}{\left((\lambda+\eta+\gamma)-\left(\lambda g_1+\frac{\gamma}{z}\right)\tilde{S}_v\left((\lambda+\eta)-\lambda X(z)\right)\right)} \tag{7.43}$$

Substituting $\sum_{m=1}^{b} P_m(z,0)$, $R(z,0)$, $W(z)$ *and* $B(z,0)$ in Eqn. 7.35, we get

$$D(z) = \frac{D_0 z^b \left\{ \begin{matrix} \gamma z + \left((1-\phi)+\phi\tilde{R}\left(\lambda-\lambda x(z)\right)\right) \\ \times\left(\lambda z g_b \tilde{S}_b\left(\lambda-\lambda x(z)\right)-\gamma\tilde{S}_b(\lambda)\right)\left(z\left(-(\lambda+\eta)+\lambda X(z)\right)\right) \end{matrix} \right\} + z^{b+1}\left(\begin{matrix} \left(\eta W(z)+(Q_0+W_0)\tilde{V}\left(\lambda-\lambda X(z)\right)\right)\left(-(\lambda+\eta)+\lambda X(z)\right) \\ -\left(\left(\frac{\gamma}{z}\right)(W(z)-W_0)+W(z)\lambda g_1\right)\left(\tilde{S}_v\left((\lambda+\eta)-\lambda X(z)\right)-1\right) \end{matrix} \right)}{\left(\begin{matrix} z^{b+1}(\lambda+\gamma)\left(\theta-(\lambda+\eta)+\lambda X(z)\right)-\left((1-\phi)+\phi\tilde{R}\left(\lambda-\lambda x(z)\right)\right) \\ \times\left(\begin{matrix} \tilde{S}_b(\lambda)(\gamma+\lambda z g_1)z^b \\ +\lambda\tilde{S}_b\left(\lambda-\lambda x(z)\right)\left(z^{b+1}\sum_{i=2}^{b-1} g_i - z\left(X(z)-\sum_{j=1}^{b-1} g_j z^j\right)\right) \end{matrix} \right)\left(-(\lambda+\eta)+\lambda X(z)\right) \end{matrix} \right)} \quad (7.44)$$

Let $P(z)$ be the PGF of the orbit size at an arbitrary time, then

$$\mathrm{P}(\mathrm{z}) = \widetilde{\mathrm{P}}_1(\mathrm{z},0) + \sum_{\mathrm{m}=1}^{\mathrm{b}-1} \widetilde{\mathrm{P}}_{\mathrm{m}}(\mathrm{z},0) + \widetilde{\mathrm{P}}_{\mathrm{b}}(\mathrm{z},0) + \widetilde{\mathrm{Q}}(\mathrm{z},0) \quad (7.45)$$

$$+ \widetilde{\mathrm{R}}(\mathrm{z},0) + \mathrm{D}(\mathrm{z}) + \widetilde{\mathrm{B}}(\mathrm{z},0) + \mathrm{W}(\mathrm{z})$$

Substituting $\theta = \lambda$ in Eqn. 7.38, $\theta = (\lambda+\eta) - \lambda X(z)$ in Eqn. 7.39 and $\theta = \lambda - \lambda\,\mathrm{X}(\mathrm{z})$ in the equations from Eqn. 7.39 to Eqn. 7.42 and using Eqn. 7.43 and Eqn. 7.44, after doing some algebra, the PGF of the orbit size is defined in Eqn. 7.45 is simplified as

$$P(z) = \frac{D_0\left\{\left(z^b F(z)+G(z)\right)A(z)+H(z)K(z)\right\}+(\Psi(z)+M(z))K(z)}{K(z)\left(-(\lambda+\eta)+\lambda X(z)\right)(-\lambda)\left(-\lambda+\lambda X(z)\right)} \quad (7.46)$$

where

$$F(z) = \gamma z + \left((1-\varphi)+\varphi\tilde{R}\left(\lambda-\lambda x(z)\right)\right)\left(\lambda z g_b \tilde{S}_b\left(\lambda-\lambda x(z)\right)-\gamma\tilde{S}_b(\lambda)\right)$$
$$\times z\left(-\lambda-\eta+\lambda X(z)\right)$$

$$G(z) = z^{b+1}\left(\begin{matrix} \left(\eta W(z)+(Q_0+W_0)\tilde{V}\left(\lambda-\lambda X(z)\right)\right)\left(-(\lambda+\eta)+\lambda X(z)\right) \\ -\left(\left(\frac{\gamma}{z}\right)(W(z)-W_0)+W(z)\lambda g_1\right)\left(\tilde{S}_v\left((\lambda+\eta)-\lambda X(z)\right)-1\right) \end{matrix} \right)$$

$$K(z) = z^{b+1}(\lambda+\gamma)\left(-(\lambda+\eta)+\lambda X(z)\right)-\left((1-\varphi)+\varphi\tilde{R}\left(\lambda-\lambda x(z)\right)\right)$$
$$\times\left(\begin{matrix} \tilde{S}_b(\lambda)(\gamma+\lambda z g_1)z^b + \lambda\tilde{S}_b\left(\lambda-\lambda x(z)\right) \\ \times\left(z^{b+1}\sum_{i=2}^{b-1} g_i - z\left(X(z)-\sum_{j=1}^{b-1} g_j z^j\right)\right) \end{matrix} \right)\left(-\lambda-\eta+\lambda X(z)\right)$$

$$A(z)=\left[\begin{array}{c}\big(-(\lambda+\eta)+\lambda X(z)\big)(-\lambda)\big(-\lambda+\lambda X(z)\big)\\ +\left(\frac{\gamma}{z}+\lambda g_1\right)\begin{pmatrix}\big(\tilde{S}_b(\lambda)-1\big)\big(-\lambda+\lambda X(z)\big)\big(-(\lambda+\eta)+\lambda X(z)\big)\\ +\varphi\big(\tilde{R}\big(\lambda-\lambda x(z)\big)-1\big)(-\lambda)\tilde{S}_b(\lambda)\end{pmatrix}\\ +\big(\sum_{i=2}^{b-1}\lambda g_i+\sum_{k=0}^{\infty}\lambda g_{b+k}z^k\big)+L_1\end{array}\right]$$

$$L_1=\left[\begin{array}{c}\big(-(\lambda+\eta)+\lambda X(z)\big)(-\lambda)\big(\tilde{S}_b\big(\lambda-\lambda X(z)\big)-1\big)\\ +\varphi\big(\tilde{R}\big(\lambda-\lambda x(z)\big)-1\big)(-\lambda)\tilde{S}_b(\lambda-\lambda X(z))\end{array}\right]$$

$$H(z)=-\frac{\gamma}{z}\Big(\big(\tilde{S}_b(\lambda)-1\big)\big(-\lambda+\lambda X(z)\big)+\varphi\big(\tilde{R}\big(\lambda-\lambda x(z)\big)-1\big)(-\lambda)\tilde{S}_b(\lambda)\Big)$$
$$+\big(-(\lambda+\eta)+\lambda X(z)\big)(-\lambda)\begin{pmatrix}\lambda g_b\big(\tilde{S}_b\big(\lambda-\lambda X(z)\big)-1\big)\\ +\varphi\big(\tilde{R}\big(\lambda-\lambda x(z)\big)-1\big)\tilde{S}_b(\lambda-\lambda X(z))\end{pmatrix}$$

$$\Psi(z)=(\tilde{V}\big(\lambda-\lambda X(z)\big)-1)(Q_0+W_0)\big(-(\lambda+\eta)+\lambda X(z)\big)(-\lambda)$$
$$-\left(\frac{\gamma}{z}(W(z)-W_0)+W(z)\lambda g_1\right)\times L_2$$

$$L_2=\big(\tilde{S}_v\big((\lambda+\eta)-\lambda X(z)\big)-1\big)\begin{pmatrix}(\tilde{S}_b(\lambda)-1)\big(-\lambda+\lambda X(z)\big)\\ +\tilde{S}_b(\lambda)\varphi\big(\tilde{R}\big(\lambda-\lambda x(z)\big)-1\big)(-\lambda)\end{pmatrix}$$

$$M(z)=\left(\left(\frac{\gamma}{z}\right)(W(z)-W_0)+W(z)\lambda g_1\right)\big(\tilde{S}_v\big((\lambda+\eta)-\lambda X(z)\big)-1\big)$$
$$\times(-\lambda)\big(-\lambda+\lambda X(z)\big)\big(-(\lambda+\eta)+\lambda X(z)\big)(-\lambda)\big(-\lambda+\lambda X(z)\big)W(z)$$

The steady condition is derived by using the normalizing condition $\lim_{z\to 1} P(z)=1$. Hence the steady state condition is derived as

$$\rho=-\lambda\eta E(X)(\lambda+\eta+\gamma-(\gamma+\lambda g_1)\tilde{S}_v(\eta)<1$$

The unknown constant D_0 is obtained as

$$D_0=\lim_{z\to 1}\frac{K(z)-(\eta)(-\lambda)(\lambda E(X))-\big(\Psi(z)+M(z)\big)K'(z)-\big(\Psi'(z)+M'(z)\big)}{\big(z^bF'(z)+bz^{b-1}F(z)+G'(z)\big)A(z)+\big(z^bF(z)+G(z)\big)A'(z)+H'(z)K(z)+K'(z)H(z)}$$

7.2.5 Performance Measures

In this section some useful performance measures such as expected orbit length, expected waiting time, probability that the server is busy with and without working vacation, probability that the server is on non-working vacation, probability that the server is idle during working vacation, probability that the server is not busy and the server is not on working vacation and the probability that the server is on renewal period are derived.

7.2.5.1 Probability that the Server is not Busy during Working Vacation

$$P(NBWV) = \frac{W_0\gamma\left(1-\tilde{S}_v(\eta)\right)+(1-\varphi)P_0}{(\lambda+\eta+\gamma)-(\lambda g_1+\gamma)\tilde{S}_v(\eta)}$$

7.2.5.2 Probability that the Server is not Busy and the Server is not on Working Vacation

$$P(I) = \frac{\begin{array}{c} D_0\left(\gamma-\eta\left(\lambda g_b-\gamma\tilde{S}_b(\lambda)\right)\right)-\eta(Q_0+W_0)+\gamma W_0(\tilde{S}_v(\eta)-1) \\ +S1(\eta-(\gamma+\lambda g_1)(\tilde{S}_v(\eta)-1)) \end{array}}{\eta\left(\tilde{S}_b(\lambda)(\gamma+\lambda g_1)+\lambda\left(\sum_{i=2}^{b-1} g_i-\left(1-\sum_{j=1}^{b-1} g_j\right)\right)\right)-\eta(\lambda+\gamma)}$$

where $S1 = \frac{W_0\gamma\left(1-\tilde{S}_v(\eta)\right)+(1-\varphi)P_0}{\lambda\left(1-g_1\tilde{S}_v(\eta)\right)+\gamma\left(1-\tilde{S}_v(\eta)\right)+\eta}$

7.2.5.3 Probability that the Server is Busy during Working Vacation

$$P(BWV) = \frac{(\tilde{S}_v(\eta)-1)\begin{bmatrix} \left(W_0\gamma\left(1-\tilde{S}_v(\eta)\right)+(1-\varphi)P_0\right)(\gamma+\lambda g_1) \\ -W_0\gamma\left(\lambda\left(1-g_1\tilde{S}_v(\eta)\right)+\gamma\left(1-\tilde{S}_v(\eta)\right)+\eta\right) \end{bmatrix}}{\eta\left(\lambda\left(1-g_1\tilde{S}_v(\eta)\right)+\gamma\left(1-\tilde{S}_v(\eta)\right)+\eta\right)}$$

7.2.5.4 Probability that the Server is Busy without Working Vacation

$$P(B) = \frac{\begin{array}{c} (\lambda E(X))(\tilde{S}_b(\lambda)-1)\left[(D(1)(\gamma+\lambda g_1)-\gamma D_0)(-\eta)-\left(S1\left((\gamma+\lambda g_1)-\gamma W_0\right)(\tilde{S}_v(\eta)-1)\right)\right] \\ -\eta\lambda^2 E(S_b)E(X)\sum_{i=2}^{b-1}\lambda g_i D(1)-\eta\lambda^2 E(S_b)E(X)\left(\lambda g_b D_0+\sum_{k=0}^{\infty}\lambda g_{b+k} D(1)\right) \end{array}}{\eta\lambda^2 E(X)}$$

where $D(1) = \frac{\begin{array}{c} D_0\left(\gamma-\eta\left(\lambda g_b-\gamma\tilde{S}_b(\lambda)\right)\right)-\eta(Q_0+W_0)+\gamma W_0(\tilde{S}_v(\eta)-1) \\ +S1(\eta-(\gamma+\lambda g_1)(\tilde{S}_v(\eta)-1)) \end{array}}{\eta\left(\tilde{S}_b(\lambda)(\gamma+\lambda g_1)+\lambda\left(\sum_{i=2}^{b-1} g_i-\left(1-\sum_{j=1}^{b-1} g_j\right)\right)\right)-\eta(\lambda+\gamma)}$

$$S1 = \frac{W_0\gamma\left(1-\tilde{S}_v(\eta)\right)+(1-\varphi)P_0}{\lambda\left(1-g_1\tilde{S}_v(\eta)\right)+\gamma\left(1-\tilde{S}_v(\eta)\right)+\eta}$$

7.2.5.5 Probability that the Server is on Non-Working Vacation

$$P(V) = (Q_0 + W_0)E(B)$$

7.2.5.6 Probability that the Server is on Renewal Process

$$P(R) = E(R)\begin{bmatrix} \tilde{S}_b(\lambda)\left(D(1)(\gamma+\lambda g_1) - \gamma D_0 - (S1\left((\gamma+\lambda g_1) - \gamma W_0)(\tilde{S}_v(\eta)-1)\right)\right) \\ -\eta\left(\sum_{i=2}^{b-1}\lambda g_i D(1) + \eta(\lambda g_b D_0 + \sum_{k=0}^{\infty}\lambda g_{b+k}\ D(1))\right) \end{bmatrix}$$

7.2.5.7 Expected Orbit Length

$$E(Q) = \frac{v_1u_2 - u_1v_2}{{v_1}^2} + \frac{h_1(w_2+w_3+w_4+w_5) - h_2w_1}{2(h_1)^2}$$

where

$v_1 = -\eta(s_1)$ $v_2 = s_1(\lambda E(X)) - \eta(s_2)$ $s_1 = (\lambda+\eta+\gamma) - (\lambda g_1+\gamma)\tilde{S}_v(\eta)$

$s_2 = \gamma\tilde{S}_v(\eta) - (\lambda g_1+\gamma){\tilde{S}_v}'(\eta)(-\lambda E(X))$ $h_1 = \lambda^2\eta E(X)$

$h_2 = \lambda^2\eta X''(1) - 2\lambda^3(E(X))^2$

$$u_1 = \gamma\left(1-\tilde{S}_v(\eta)\right)(-\eta) - \tilde{S}_v(\eta-1)\gamma s_1 + (1-\varphi)P_0\,\lambda E(X)$$
$$+(1-\varphi)P_0\left(\eta + (\tilde{S}_v(\eta)-1)\right)w(1)(\lambda g_1+\gamma)\,s_1$$

$$u_2 = \tilde{S}_v(\eta-1)\begin{pmatrix} W(1)(\lambda g_1+\gamma)s_2 - \gamma W(1)s_1 \\ +W'(1)s_1(\lambda g_1+\gamma) \end{pmatrix}\ {\tilde{S}_v}'(\eta)(-\lambda E(X))W(1)(\lambda g_1+\gamma)s_1$$

$$w_1 = (\tilde{S}_b(\lambda)-1)\lambda E(X)(-k_1\eta + k_2) + \lambda^2\eta\left(E(S_b)E(X)k_3 + E(V)E(X)(Q_0+W_0)\right)$$
$$+E(R)\,\lambda\,E(X)\,\varphi\left(\tilde{S}_b(\lambda)(k_1+k_2)+k_3\right)$$

$$w_2 = 2E(R)\,\lambda\,E(X)\,\varphi\left(\tilde{S}_b(\lambda)({k_1}'+{k_2}') + {\tilde{S}_b}'(\lambda)(k_1+k_2) + {k_3}' + \lambda E(S_b)E(X)k_3\right)$$
$$+\left\{E(R)\lambda X''(1) + E(R^2)\,\lambda^2\left(E(X)\right)^2\right\}\varphi\left(\tilde{S}_b(\lambda)(k_1+k_2)+k_3\right)$$

$$w_3 = (Q_0+W_0)\left(2\lambda^3E(V)(E(X))^2 - \left(E(V)\lambda X''(1) + E(V^2)\,\lambda^2\left(E(X)\right)^2\right)\lambda\eta\right)$$
$$+2\lambda E(S_b)\left(\left(E(X)\right)^2\right)(-k_3\,\lambda^2 - {k_3}'\lambda\eta)$$

$$-\left(\mathrm{E}(S_b)\lambda X''(1) + E\left(S_b{}^2\right)\lambda^2\left(E(X)\right)^2\right) k_3\lambda\eta\mathrm{E}(\mathrm{X})$$

$$w_4 = (k_2\lambda X''(1) + 2\lambda\mathrm{E}(\mathrm{X}))\left(\tilde{S}_b(\lambda) - 1\right) + 2k_2\lambda\mathrm{E}(\mathrm{X})\tilde{\mathrm{S}}_\mathrm{b}{}'(\lambda)$$

$$w_5 = \left(\tilde{S}_b(\lambda) - 1\right)k_1\left(2\,\lambda^2\left(E(X)\right)^2 - \lambda\eta X''(1)\right)$$
$$-\lambda\eta\mathrm{E}(\mathrm{X})\left(\left(\tilde{S}_b(\lambda) - 1\right)(k_1{}' + k_1{}'') + (\tilde{\mathrm{S}}_\mathrm{b}{}'(\lambda) + \tilde{\mathrm{S}}_\mathrm{b}{}''(\lambda))k_1 + 2\tilde{\mathrm{S}}_\mathrm{b}{}'(\lambda)k_1{}'\right)$$

$$k_1 = D(1)(\gamma + \lambda g_1) - \gamma D_0 \quad k_1{}' = -\gamma D(1) + D'(1)(\gamma + \lambda g_1) + \gamma D_0$$

$$k_1{}'' = 2\gamma D(1) - 2\gamma D'(1) + D''(1)(\gamma + \lambda g_1) + 2\gamma D_0$$

$$k_2 = (W(1)(\gamma + \lambda g_1) - \gamma W_0)(\tilde{\mathrm{S}}_\mathrm{v}(\eta) - 1)$$

$$k_3 = D(1)\lambda\left(\sum_{i=2}^{b-1} g_i + \sum_{k=0}^{\infty} g_{b+k}\right) + \lambda g_b D_0$$

$$k_2{}' = -(W(1)(\gamma + \lambda g_1) - \gamma W_0)\tilde{\mathrm{S}}_\mathrm{v}{}'(\eta)\lambda\mathrm{E}(\mathrm{X}) + \left((\gamma + \lambda g_1)W'(1) + \gamma W_0 - \gamma W(1)\right)(\tilde{\mathrm{S}}_\mathrm{v}(\eta) - 1)$$

$$k_3{}' = D(1)(\lambda\sum_{k=0}^{\infty} k g_{b+k}) + D'(1)\lambda\left(\sum_{i=2}^{b-1} g_i + \sum_{k=0}^{\infty} g_{b+k}\right)$$

7.2.5.8 Average Waiting Time in the Retrial Queue

$$E(W) = \frac{E(Q)}{\lambda E(X)}$$

7.2.6 Special Cases

The proposed model is developed with the assumption that the service time is arbitrary. However, to analyse real time systems, suitable distribution is required. This section presents some special cases of the system by indicating bulk service time as exponential distribution, hyper exponential distribution and Erlangian distribution.

Case. 1: Exponential bulk service time

The probability density function of exponential service time is $A(x) = e^{-\mu x}$, where μ a parameter is. Therefore

$$\tilde{S}_v\left(\lambda - \lambda x(z)\right) = \left(\frac{\mu_b}{\mu_b + (\lambda - \lambda x(z))}\right) \quad \tilde{S}_b\left(\lambda + \eta - \lambda x(z)\right) = \left(\frac{\mu_a}{\mu_a + ((\lambda + \eta) - \lambda x(z))}\right)$$

The PGF of the orbit size for exponential service time is derived by substituting the expression for $\tilde{S}_b(\lambda-\lambda x(z))$ and $\tilde{S}_v(\lambda-\lambda x(z))$ in Eqn. 7.46

$$P(z)=\frac{D_0\left\{\left(z^bF(z)+G(z)\right)A(z)+H(z)K(z)\right\}+(\Psi(z)+M(z))K(z)}{K(z)\left(-(\lambda+\eta)+\lambda X(z)\right)(-\lambda)\left(-\lambda+\lambda X(z)\right)}$$

where

$$F(z)=\left(\gamma+\left((1-\varphi)+\varphi\tilde{R}\left(\lambda-\lambda x(z)\right)\right)\begin{pmatrix}\left(\frac{\mu_a}{\mu_a+\left((\lambda+\eta)-\lambda x(z)\right)}\right)\\ \lambda z g_b-\gamma\left(\frac{\mu_a}{\mu_a+\lambda}\right)\end{pmatrix}\left(-\lambda-\eta+\lambda X(z)\right)\right)z$$

$$G(z)=z^{b+1}\begin{pmatrix}\left(\eta W(z)+(Q_0+W_0)\tilde{V}\left(\lambda-\lambda X(z)\right)\right)\left(-(\lambda+\eta)+\lambda X(z)\right)\\ -\left(\left(\frac{\gamma}{z}\right)(W(z)-W_0)+W(z)\lambda g_1\right)\left(\left(\frac{\mu_b}{\mu_b+(\lambda-\lambda x(z))}\right)-1\right)\end{pmatrix}$$

$$K(z)=z^{b+1}(\lambda+\gamma)(-(\lambda+\eta)+\lambda X(z))-\left((1-\phi)+\phi\tilde{R}\left(\lambda-\lambda x(z)\right)\right)$$

$$\times\begin{pmatrix}\left(\frac{\mu_a}{\mu_a+\lambda}\right)(\gamma+\lambda z g_1)z^b+\lambda\left(\frac{\mu_a}{\mu_a+\left((\lambda+\eta)-\lambda x(z)\right)}\right)\\ \times\left(z^{b+1}\sum_{i=2}^{b-1}g_i-z\left(X(z)-\sum_{j=1}^{b-1}g_jz^j\right)\right)\end{pmatrix}\left(-\lambda-\eta+\lambda X(z)\right)$$

$$A(z)=\begin{bmatrix}\left(-(\lambda+\eta)+\lambda X(z)\right)(-\lambda)\left(-\lambda+\lambda X(z)\right)\\ +\left(\frac{\gamma}{z}+\lambda g_1\right)\begin{pmatrix}\left(\left(\frac{\mu_a}{\mu_a+\lambda}\right)-1\right)\left(-\lambda+\lambda X(z)\right)\left(-(\lambda+\eta)+\lambda X(z)\right)\\ +\phi\left(\tilde{R}\left(\lambda-\lambda x(z)\right)-1\right)(-\lambda)\left(\frac{\mu_a}{\mu_a+\lambda}\right)\end{pmatrix}\\ +\left(\sum_{i=2}^{b-1}\lambda g_i+\sum_{k=0}^{\infty}\lambda g_{b+k}z^k\right)L_1\end{bmatrix}$$

$$L_1=\begin{bmatrix}\left(-(\lambda+\eta)+\lambda X(z)\right)(-\lambda)\left(\left(\frac{\mu_a}{\mu_a+\left((\lambda+\eta)-\lambda x(z)\right)}\right)-1\right)\\ +\phi\left(\tilde{R}\left(\lambda-\lambda x(z)\right)-1\right)(-\lambda)\left(\frac{\mu_a}{\mu_a+\left((\lambda+\eta)-\lambda x(z)\right)}\right)\end{bmatrix}$$

$$H(z)=-\frac{\gamma}{z}\left(\left(\left(\frac{\mu_a}{\mu_a+\lambda}\right)-1\right)\left(-\lambda+\lambda X(z)\right)+\phi\left(\tilde{R}\left(\lambda-\lambda x(z)\right)-1\right)(-\lambda)\left(\frac{\mu_a}{\mu_a+\lambda}\right)\right)$$

$$+\left(-(\lambda+\eta)+\lambda X(z)\right)(-\lambda)\begin{pmatrix}\lambda g_b\left(\left(\frac{\mu_a}{\mu_a+\left((\lambda+\eta)-\lambda x(z)\right)}\right)-1\right)\\ +\phi\left(\tilde{R}\left(\lambda-\lambda x(z)\right)-1\right)\left(\frac{\mu_a}{\mu_a+\left((\lambda+\eta)-\lambda x(z)\right)}\right)\end{pmatrix}$$

$$\Psi(z)=(\tilde{V}\left(\lambda-\lambda X(z)\right)-1)(Q_0+W_0)\left(-(\lambda+\eta)+\lambda X(z)\right)(-\lambda)$$

$$-\left(\frac{\gamma}{z}(W(z)-W_0)+W(z)\lambda g_1\right)$$

$$\times\left(\left(\frac{\mu_b}{\mu_b+(\lambda-\lambda x(z))}\right)-1\right)\begin{pmatrix}\left(\left(\frac{\mu_a}{\mu_a+\lambda}\right)-1\right)(-\lambda+\lambda X(z))\\ +\left(\frac{\mu_a}{\mu_a+\lambda}\right)\phi\left(\tilde{R}\left(\lambda-\lambda x(z)\right)-1\right)(-\lambda)\end{pmatrix}$$

$$M(z)=\left(\left(\frac{\gamma}{z}\right)(W(z)-W_0)+W(z)\lambda g_1\right)\left(\left(\frac{\mu_b}{\mu_b+(\lambda-\lambda x(z))}\right)-1\right)(-\lambda)\left(-\lambda+\lambda X(z)\right)$$

$$\left(-(\lambda+\eta)+\lambda X(z)\right)(-\lambda)\left(-\lambda+\lambda X(z)\right)W(z)$$

Case. 2: Hyper exponential bulk service time

When the service time follows hyper exponential distribution with probability density function, then $a(x)=cde^{-dx}+(1-c)\,fe^{-fx}$, where d and f are parameters, then,

$$\tilde{S}_b\left(\lambda-\lambda x(z)\right)=\left(\frac{dc}{d+\left(\lambda-\lambda x(z)\right)}\right)+\left(\frac{f(1-c)}{f+\left(\lambda-\lambda x(z)\right)}\right)$$

$$\tilde{S}_v\left(\lambda+\eta-\lambda x(z)\right)=\left(\frac{dc}{d+\left(\lambda+\eta-\lambda x(z)\right)}\right)+\left(\frac{f(1-c)}{f+\left(\lambda+\eta-\lambda x(z)\right)}\right)$$

The PGF of the orbit size for hyper exponential service time is derived by substituting the expression for $\tilde{S}_b\left(\lambda-\lambda x(z)\right)$ and $\tilde{S}_v\left(\lambda+\eta-\lambda x(z)\right)$in Eqn. 7.46.

$$P(z)=\frac{D_0\left\{\left(z^bF(z)+G(z)\right)A(z)+H(z)K(z)\right\}+(\Psi(z)+M(z))K(z)}{K(z)\left(-(\lambda+\eta)+\lambda X(z)\right)(-\lambda)\left(-\lambda+\lambda X(z)\right)}$$

where

$$F(z)=\gamma z+\begin{pmatrix}(1-\phi)\\ +\phi\tilde{R}\left(\lambda-\lambda x(z)\right)\end{pmatrix}\begin{pmatrix}\lambda z g_b\left(\frac{dc}{d+\left(\lambda-\lambda x(z)\right)}+\frac{f(1-c)}{f+\left(\lambda-\lambda x(z)\right)}\right)\\ -\gamma\left(\frac{dc}{d+\lambda}+\frac{f(1-c)}{f+\lambda}\right)\end{pmatrix}z\begin{pmatrix}-\lambda\\ -\eta+\lambda X(z)\end{pmatrix}$$

$$G(z)=z^{b+1}\begin{pmatrix}\left(\eta W(z)+(Q_0+W_0)\tilde{V}\left(\lambda-\lambda X(z)\right)\right)\left(-(\lambda+\eta)+\lambda X(z)\right)\\ -\left(\left(\frac{\gamma}{z}\right)(W(z)-W_0)+W(z)\lambda g_1\right)\left(\left(\frac{dc}{d+\left(\lambda+\eta-\lambda x(z)\right)}+\frac{f(1-c)}{f+\left(\lambda+\eta-\lambda x(z)\right)}\right)-1\right)\end{pmatrix}$$

$$K(z)=z^{b+1}(\lambda+\gamma)\left(-(\lambda+\eta)+\lambda X(z)\right)-\left((1-\phi)+\phi\tilde{R}\left(\lambda-\lambda x(z)\right)\right)$$

$$\times\begin{pmatrix}\left(\frac{dc}{d+\lambda}+\frac{f(1-c)}{f+\lambda}\right)(\gamma+\lambda z g_1)z^b+\lambda\left(\frac{dc}{d+\left(\lambda-\lambda x(z)\right)}+\frac{f(1-c)}{f+\left(\lambda-\lambda x(z)\right)}\right)\\ \times\left(z^{b+1}\sum_{i=2}^{b-1}g_i-z\left(X(z)-\sum_{j=1}^{b-1}g_jz^j\right)\right)\end{pmatrix}\left(-\lambda-\eta+\lambda X(z)\right)$$

$$A(z)=\begin{bmatrix}\left(-(\lambda+\eta)+\lambda X(z)\right)(-\lambda)\left(-\lambda+\lambda X(z)\right)\\ +\left(\frac{\gamma}{z}+\lambda g_1\right)\begin{pmatrix}\left(\left(\frac{dc}{d+\lambda}+\frac{f(1-c)}{f+\lambda}\right)-1\right)\left(-\lambda+\lambda X(z)\right)\left(-(\lambda+\eta)+\lambda X(z)\right)\\ +\phi\left(\tilde{R}\left(\lambda-\lambda x(z)\right)-1\right)(-\lambda)\left(\frac{dc}{d+\lambda}+\frac{f(1-c)}{f+\lambda}\right)\end{pmatrix}\\ +\left(\sum_{i=2}^{b-1}\lambda g_i+\sum_{k=0}^{\infty}\lambda g_{b+k}z^k\right)L_1\end{bmatrix}$$

$$L_1=\begin{bmatrix}\left(-(\lambda+\eta)+\lambda X(z)\right)(-\lambda)\left(\begin{pmatrix}\dfrac{dc}{d+\left(\lambda-\lambda X(z)\right)}\\ +\dfrac{f(1-c)}{f+\left(\lambda-\lambda x(z)\right)}\end{pmatrix}-1\right)\\ +\phi\left(\tilde{R}\left(\lambda-\lambda x(z)\right)-1\right)(-\lambda)\begin{pmatrix}\dfrac{dc}{d+\left(\lambda-\lambda x(z)\right)}\\ +\dfrac{f(1-c)}{f+\left(\lambda-\lambda x(z)\right)}\end{pmatrix}\end{bmatrix}$$

$$H(z)=-\frac{\gamma}{z}\begin{pmatrix}\left(\left(\frac{dc}{d+\lambda}+\frac{f(1-c)}{f+\lambda}\right)-1\right)\left(-\lambda+\lambda X(z)\right)\\ +\phi\left(\tilde{R}\left(\lambda-\lambda x(z)\right)-1\right)(-\lambda)\left(\frac{dc}{d+\lambda}+\frac{f(1-c)}{f+\lambda}\right)\end{pmatrix}+L_2$$

$$L_2=\begin{pmatrix}\lambda g_b\left(\left(\frac{dc}{d+\left(\lambda-\lambda x(z)\right)}+\frac{f(1-c)}{f+\left(\lambda-\lambda x(z)\right)}\right)-1\right)\\ +\phi\left(\tilde{R}\left(\lambda-\lambda x(z)\right)-1\right)\left(\frac{dc}{d+\left(\lambda-\lambda x(z)\right)}+\frac{f(1-c)}{f+\left(\lambda-\lambda x(z)\right)}\right)\end{pmatrix}\left(-(\lambda+\eta)+\lambda X(z)\right)(-\lambda)$$

$$\Psi(z)=(\tilde{V}\left(\lambda-\lambda X(z)\right)-1)(Q_0+W_0)\left(-(\lambda+\eta)+\lambda X(z)\right)(-\lambda)$$

$$-\left(\frac{\gamma}{z}(W(z)-W_0)+W(z)\lambda g_1\right)\times\left(\left(\frac{dc}{d+\left(\lambda+\eta-\lambda x(z)\right)}+\frac{f(1-c)}{f+\left(\lambda+\eta-\lambda x(z)\right)}\right)-1\right)L_3$$

$$L_3=\begin{pmatrix}\left(\left(\frac{dc}{d+\lambda}+\frac{f(1-c)}{f+\lambda}\right)-1\right)\left(-\lambda+\lambda X(z)\right)\\ +\left(\frac{dc}{d+\lambda}+\frac{f(1-c)}{f+\lambda}\right)\phi\left(\tilde{R}\left(\lambda-\lambda x(z)\right)-1\right)(-\lambda)\end{pmatrix}$$

$$M(z) = \left(\left(\frac{\gamma}{z}\right)(W(z) - W_0) + W(z)\lambda g_1\right)\left(\left(\begin{array}{c}\frac{dc}{d+(\lambda+\eta-\lambda x(z))}\\ +\frac{f(1-c)}{f+(\lambda+\eta-\lambda x(z))}\end{array}\right) - 1\right)(-\lambda)(-\lambda + \lambda X(z))$$

$$\times\left(-(\lambda+\eta) + \lambda X(z)\right)(-\lambda)\left(-\lambda + \lambda X(z)\right)W(z)$$

Case. 3: K-Erlangian bulk service time

Let us consider that service time follows K- Erlang distribution with probability density function

$a(x) = \frac{(k\mu)^k x^{k-1} e^{-(k\mu x)}}{(k-1)!}$, k > 0; where μ is the parameter, then

$$\tilde{S}_b\left(\lambda - \lambda x(z)\right) = \left(\frac{k\mu_a}{k\mu_a + (\lambda - \lambda x(z))}\right)^k \qquad \tilde{S}_v\left(\lambda + \eta - \lambda x(z)\right) = \left(\frac{k\mu_b}{k\mu_b + (\lambda + \eta - \lambda x(z))}\right)^k$$

The PGF of the orbit size with K-Erlangian bulk service time is derived by substituting the expression for $\tilde{S}_b\left(\lambda - \lambda x(z)\right)$ and $\tilde{S}_b\left(\lambda - \lambda x(z)\right)$ in Eqn. 7.46..

7.2.7 Numerical Illustrations

In this section, theoretical results such are justified with suitable numerical results. In order to study an effect of arrival rate and retrial rate with respect to expected orbit length and waiting the following assumptions are made.

Service time (working vacation) $-\mu_b$ Regular service time $-\mu_a$,

Renewal rate with parameter $-$ ξ   Vacation time $-$η.

An influence of retrial rate on expected orbit length and expected waiting time is given in Table 7.2, Fig. 7.2 and Fig. 7.3 by assuming that service time follows exponential, Erlang-2 and hyper exponential.

In Table 7.3 an effect of arrival rate with respect to performance measures is specified with the assumption that service time follows exponential, Erlang-2 and hyper exponential.

In Table 7.4 an effect of service rate with respect to performance measures is specified with the assumption that service time follows exponential, Erlang-2 and hyper exponential.

Results given from Table 7.5 to Table 7.8 are obtained with the assumption that service time follows exponential, Erlang-2 and hyper exponential by specifying the following parameters $\lambda = 0.7, \mu_b = 1.7, \mu_v = 0.9, \eta = 2, \xi = 3, \varphi = 0.5$.

Table 7.2 Retrial rate versus Performance Measures

$$\begin{pmatrix} \lambda = 0.5, \mu_b = 1.5, \mu_v = 0.6 \\ \eta = 2, \xi = 3, \varphi = 0.5 \end{pmatrix}$$

γ	Exponential		Erlang-2		Hyper Exponential	
	E(Q)	E(W)	E(Q)	E(W)	E(Q)	E(W)
0.7	0.5032	2.7361	0.4795	2.5248	0.5013	2.7483
0.9	0.4563	2.5236	0.4123	2.2321	0.44429	2.4629
1.2	0.4032	2.2793	0.3847	1.9487	0.4028	2.1743
1.4	0.3698	1.8362	0.3158	1.6271	0.3695	1.8421
1.6	0.3052	1.4972	0.2569	1.2783	0.3057	1.7605
1.8	0.2638	1.2729	0.2174	1.1837	0.2649	1.5583
2.0	0.2145	1.0934	0.1625	0.8341	0.2168	1.4982
2.2	0.1937	0.8623	0.1294	0.6327	0.2041	1.4134
2.4	0.1734	0.5219	0.0874	0.3389	0.1841	1.2789

Table 7.3 Arrival Rate versus Performance Measures

$$\begin{pmatrix} \gamma = 0.5, \mu_b = 1.5, \mu_v = 0.6 \\ \eta = 2, \xi = 3, \varphi = 0.5 \end{pmatrix}$$

λ	Exponential		Erlang-2		Hyper Exponential	
	E(Q)	E(W)	E(Q)	E(W)	E(Q)	E(W)
0.5	7.4639	15.9238	7.3982	14.3197	7.4021	14.9168
1.0	8.3268	18.3126	8.1527	17.6224	8.2456	16.7341
1.5	10.9761	19.2391	10.8653	18.1317	10.9048	18.6427
2.0	13.9632	21.6392	13.4371	20.66124	13.3662	20.9254
2.5	14.5310	23.5812	14.4123	21.5542	14.5011	21.8272
3.0	16.3417	24.9623	16.2217	22.2396	16.2817	23.1769
3.5	17.9340	26.3421	17.8271	24.3347	17.8783	25.8264
4.0	19.3214	29.2852	19.2440	26.9853	19.3019	27.2543
4.5	21.4523	31.7623	21.3223	28.6738	21.4011	29.3468

Table 7.4 Service Rate versus Performance Measures

$$\begin{pmatrix} \lambda = 0.5, & \gamma = 0.6, & \mu_v = 0.6 \\ \eta = 2.5, & \xi = 3.5, & \varphi = 0.5 \end{pmatrix}$$

μ_b	**Exponential**		**Erlang-2**		**Hyper Exponential**	
	E(Q)	**E(W)**	**E(Q)**	**E(W)**	**E(Q)**	**E(W)**
1.3	0.2742	2.2981	0.2126	2.1562	0.1922	2.1137
1.5	0.2364	2.0762	0.1841	2.0362	0.1628	2.0133
1.7	0.2059	1.8347	0.1572	1.7962	0.1337	1.6642
1.9	0.1723	1.7489	0.1197	1.7133	0.0923	1.5484
2.1	0.1497	1.6371	0.0735	1.6042	0.0621	1.429
2.2	0.0926	1.4189	0.0348	1.3115	0.0128	1.2953

Table 7.5 Retrial Rate versus Probability that the Server is Busy with Working Vacation

$\boldsymbol{\gamma}$	P(BWV)		
	Exponential	**Erlang-2**	**Hyper Exponential**
0.3	0.0463	0.0461	0.0463
0.5	0.0474	0.0474	0.0474
0.7	0.0475	0.0475	0.0475
0.9	0.0475	0.0475	0.0475
1.1	0.0475	0.0475	0.0475

Table 7.6 Retrial Rate versus Probability that the Server is Busy but not in Working Vacation

γ	P(BNWV)		
	Exponential	**Erlang-2**	**Hyper Exponential**
0.3	0.0972	0.0982	0.0978
0.5	0.0991	0.0998	0.0995
0.7	0.1232	0.1255	0.1018
0.9	0.1521	0.1542	0.1027
1.1	0.1635	0.1652	0.1039

Table 7.7 Retrial Rate versus Probability that the Server is not Busy during Working Vacation

γ	P(NBWV)		
	Exponential	**Erlang-2**	**Hyper Exponential**
0.3	0.0047	0.0049	0.0046
0.5	0.0045	0.0047	0.0048
0.7	0.0043	0.0044	0.0046
0.9	0.0040	0.0043	0.0044
1.1	0.0038	0.0039	0.0044

Table 7.8 Retrial Rate versus Probability that the Server is Idle

γ	$P(I)$		
	Exponential	**Erlang-2**	**Hyper Exponential**
0.3	0.0922	0.1242	0.1165
0.5	0.0737	0.1242	0.1154
0.7	0.0469	0.1242	0.1132
0.9	0.0243	0.1232	0.1108
1.1	0.0096	0.1202	0.1089

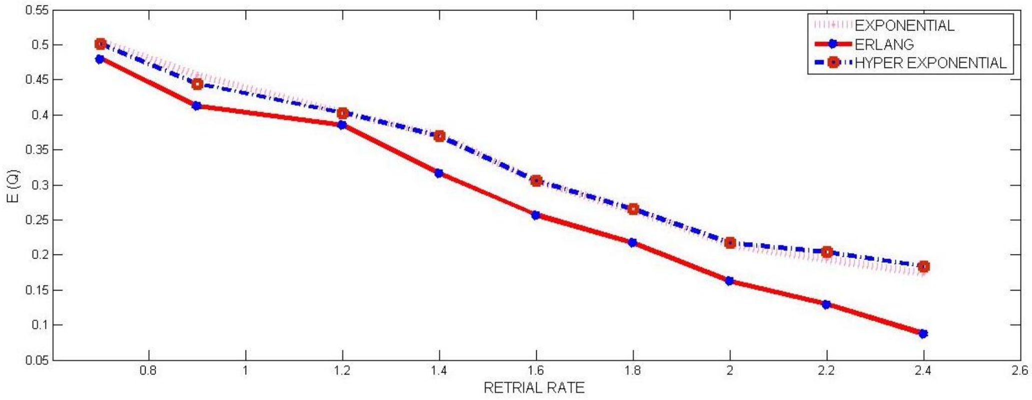

Fig. 7.2 Retrial Rate versus Expected Orbit Length (E (Q))

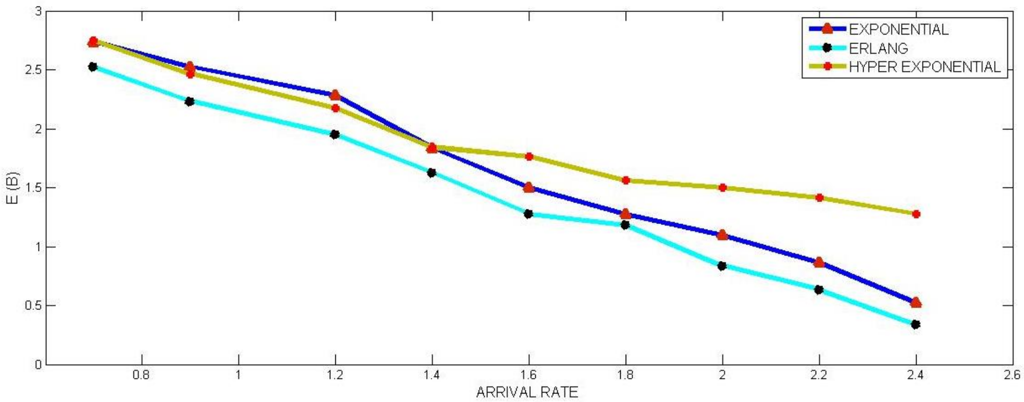

Fig. 7.3 Retrial Rate versus Expected Waiting Time in the Orbit (E (Q))

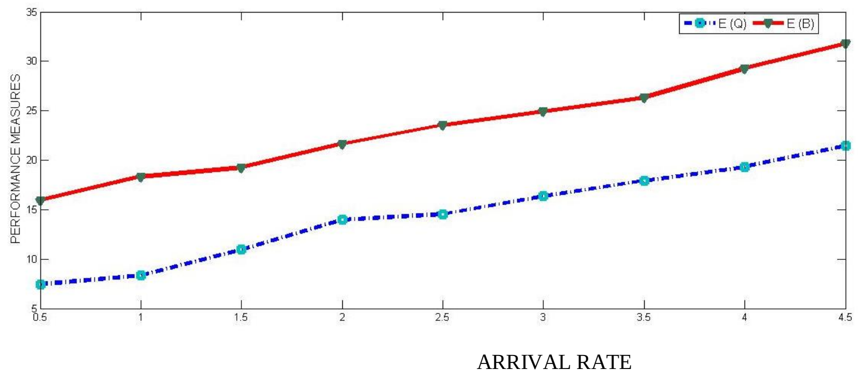

Fig. 7.4 Arrival Rate versus Performance Measures

7.3 State Dependent Arrival in Bulk Retrial Queueing System with Immediate Bernoulli Feedback, Multiple Vacations and Threshold

Mathematical modelling of queueing system with vacations is much more flexible and useful while dealing with real time congestion problems. During vacation, queueing systems the server directs to do few supplementary works at the period of its idle time, which may improve the server performance. The application of queueing system with vacations can be established from manufacturing industries, production line systems, inventory management, communication networks, etc.

Analysis of retrial queues with secondary jobs (vacations) has been taken into account by many of the researchers. Some of their works includes different models of retrial queueing systems given (Falin and Templeton, 1997). A brief survey and an overview of retrial queues have been explained (Artalejo, 1999). Falin (1976) was first introduced batch arrival retrial queueing system with the following rule: "If the server is busy at the arrival epoch, then the whole batch joins the retrial group, whereas the server is free, then one of the arriving units starts its service and the rest join the retrial group".

In order to model and analyse queueing system with retrials, the method of retrials is required. Many real time applications which exist in network systems illustrate that there are chances in queueing system with retrials such that retrial rate is independent of number of items (if any) present in the orbit, which is called constant retrial policy. Such kind of retrial policy was first constructed by Fayolle (1986), who investigated "M/M/1 retrial queue, where the queue will be formed by the retrial group of customers and request for service is possible only for customers at the head of the orbit queue after an exponentially distributed retrial time" with rate 'v'. Atencia et al. (2008) have analysed bulk retrial queue with constant retrial rate and server breakdowns. Recently Jailaxmi et al. (2017) examined M/G/1 retrial queue with modified vacations, collision and general retrial policy.

In all the above queueing models, server is able to serve one unit at a time. But in many practical situations service is rendered in batches with different batch size. This type of models have applications in inventory systems, manufacturing industries, communication networks, etc.

In classical queueing systems, many of the researchers have contributed in the study of $M^x/G(a,b)/1$ queueing models. Bulk queue with setup times, closedown times, multiple vacations and threshold have been studied (Arumuganathan and Jeyakumar, 2005). Haridass and Arumuganathan (2012) analysed bulk arrival and batch service queueing model with interrupted vacation. In all the above models they derived various performance characteristics of queueing system. They extended their analysis with cost optimization.

Only few works have carried out in batch service queueing system with retrials. Batch service retrial queueing model with constant retrial rate has been analysed by Haridass et al. (2012). They derived some important performance measures. Cost analysis is also carried out in their work.

In the service completion, a batch of customers may seek for further service and adds to the head of the queue and this system is called queueing system with feedback. Queueing model with customer feedback will occur in many real time situations. Choi et al. (2003) analysed queueing system with different types of feedback, FCFS policy and gated vacations. Krishna Kumar *et al.* (2002) studied feedback queue with varying arrival rates and threshold policy. Recently Badamchi Zadeh (2015) derived various performance measures of batch arrival and multi-phase queueing system with random feedback in service and single vacation policy. All the above feedback queueing models are considered as classical queueing system. Only few works are carried out in retrial queueing models with feedback. Madan et al. (2006) analysed bulk queue with feedback and optional vacation policy. Krishna Kumar and Raja (2006) have studied multi server queue with retrials, balking and feedback.

In all the feedback retrial queueing models, batch service is not taken into consideration. Addressing this a queueing system called state dependent arrival in batch service retrial queueing model with active Bernoulli feedback, multiple vacations, threshold and constant retrial rate is constructed.

7.3.1 Model Description

In this section state dependent arrival in bulk queueing system with retrial, active Bernoulli feedback, multiple vacations and threshold are considered. Customers are arriving into the system in bulk with rate λ_a when the server is idle and with rate λ_b when the server is busy or in vacation. These type of assumptions will give motivation in analysing real time applications. "An arrival of customers find the server is free then customers are served in batches with minimum of one and maximum of 'b' number of customers according to general bulk service rule". If $1 \leq \varepsilon \leq b$ then entire batch will get service, where $'\varepsilon'$ is the queue length. Similarly, if $\varepsilon > b$ then service is possible only for 'b' customers, and then the remaining '$\varepsilon - b'$ customers will join to orbit. On the other hand, if an arrival of customers finds the server is busy or on vacation then entire customers will join to orbit to get service later. Customers from the orbit will request service one by one with constant retrial rate 'γ'. An orbit is a virtual queue formed by customers upon finding that the server is busy. On service completion epoch, the leaving customers may either select for additional service as a feedback with probability δ or leave the system with a probability of '$1-\delta$'. Customers who need feedback will be taken for service immediately. After service completion, if the orbit size is empty then the server goes for vacation (secondary job). The vacation period will be continued until the orbit size reaches the threshold value 'N' ($N > b$). When the server finds 'N' customers waiting in the orbit during vacation completion then the server will switch on to serve customers either from orbit or from the primary pool. The pictorial representation of the proposed model is depicted below.

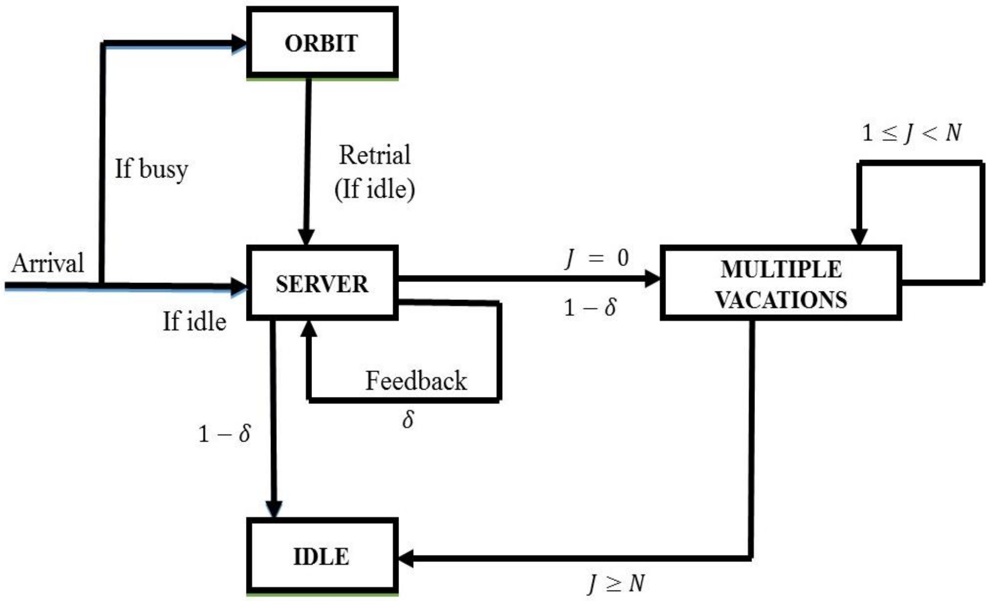

Fig. 7.5 Schematic Representation of the Queueing System: J-Orbit Length

For the proposed model, the probability generating function (PGF) of the steady state orbit size distribution at an arbitrary time epoch is obtained by using supplementary variable technique. Some important performance measures are derived. Special cases are also discussed. Numerical solution for particular values of parameters is presented.

7.3.2 Mathematical Model

Let λ_a be the Poisson arrival rate, when the server is in idle state and λ_b be the Poisson arrival rate, when the server is busy or in vacation.

Let X be the group size random variable of the arrival, g_k be the probability that 'k' customers arrive in a batch, $X(z)$ be the probability generating function (PGF) of X.

N_q(t) be the number of customers waiting for service at time t.

N_s(t) be the number of customers under the service at time t.

$N(t)$ be the number of customers in the orbit at time t.

Let 'γ' be the retrial rate (constant) of the customer from the orbit.

Let A(x) (a(x)) {$\tilde{A}(\theta)$} [A^0(x)] be the cumulative distribution function (probability density function) {Laplace-Stieltjes transform}[remaining primary service time] of service.

Let B(x) (b(x) {($\tilde{B}(\theta)$} [B^0(x)] be the cumulative distribution function (probability density function) {Laplace-Stieltjes-transform}[remaining vacation time] of vacation.

Let C (t) denotes different states of the server at time t.

$$C(t) = \begin{cases} 0, \text{ if the server is busy with service} \\ 1, \text{ if the server is on vacation} \\ 2, \text{ if the server is idle} \end{cases}$$

$Y(t) = j$, if the server is on jth vacation

The state probabilities are defined to obtain governing equations:

$$R_{ij}(x,t)dt = Pr\{N_s(t) = i, N_q(t) = j, x \leq A^0(t) \leq x + dt, C(t) = 0\}$$

$$1 \leq i \leq b, j \geq 0$$

$$S_{jn}(t)dt = Pr\{N(t) = n, x \leq B^0(t) \leq x + dt, C(t) = 1, Y(t) = j\}, n \geq 0$$

$$I_n(t)dt = Pr\{N(t) = n, C(t) = 2\},\ n \geq 0$$

The following equations are obtained by using supplementary variable technique.

$$I_j(t + \Delta t) = I_j(t)(1 - \lambda_a \Delta t - \gamma \Delta t) + (1 - \delta) \sum_{m=1}^{b} R_{mj}(0,t)\Delta t \quad (7.47)$$

$$1 \leq j \leq N - 1$$

$$I_j(t+\Delta t) = I_j(t)(1-\lambda_a\Delta t-\gamma\Delta t)+(1-\delta)\textstyle\sum_{m=1}^{b} R_{mj}(0,t)\Delta t \tag{7.48}$$

$$+\textstyle\sum_{l=1}^{\infty} S_{lj}(0,t)\Delta t \qquad j\geq N$$

$$R_{1j}(x-\Delta t,t+\Delta t) = R_{ij}(x,t)(1-\lambda_b\Delta t)+\gamma I_{j+1}(t)a(x)\Delta t \tag{7.49}$$

$$+\left(I_j(t)\lambda_a g_1+\delta R_{ij}(0,t)\right)a(x)\Delta t \qquad j\geq 0$$

$$R_{i0}(x-\Delta t,t+\Delta t) = R_{i0}(x,t)(1-\lambda_b\Delta t)+I_0(t)\lambda_a g_i a(x)\Delta t \tag{7.50}$$

$$+\delta R_{i0}(0,t)a(x)\Delta t \qquad 2\leq i\leq b$$

$$R_{ij}(x-\Delta t,t+\Delta t) = R_{ij}(x,t)(1-\lambda_b\Delta t)+I_j(t)\lambda_a g_i a(x)\Delta t \tag{7.51}$$

$$+\textstyle\sum_{k=1}^{j} R_{i\ j-k}(x,t)\,\lambda_b g_k a(x)\Delta t+\delta R_{ij}(0,t)a(x)\Delta t$$

$$2\leq i\leq b-1,\ \ j\geq 1$$

$$R_{bj}(x-\Delta t,t+\Delta t) = R_{bj}(x,t)(1-\lambda_b\Delta t)+\textstyle\sum_{k=1}^{j} R_{b\ j-k}(x,t)\,\lambda_b g_k\Delta t \tag{7.52}$$

$$+\textstyle\sum_{k=0}^{j} I_{j-k}(t)\,\lambda_a g_{b+k}s(x)\Delta t+\delta R_{bj}(0,t)a(x)\Delta t \quad j\geq 1$$

$$S_{10}(x-\Delta t,t+\Delta t) = S_{10}(x,t)(1-\lambda_b\Delta t)+(1-\alpha)R_{i0}(0,t)b(x)\Delta t \tag{7.53}$$

$$S_{1j}(x-\Delta t,t+\Delta t) = S_{1j}(x,t)(1-\lambda_b\Delta t)+\textstyle\sum_{k=1}^{j} S_{1\ j-k}(x,t)\,\lambda_b g_k\Delta t \tag{7.54}$$

$$S_{l0}(x-\Delta t,t+\Delta t) = S_{l0}(x,t)(1-\lambda_b\Delta t)+S_{l-1\ 0}(0,t)b(x)\,\Delta t \qquad l\geq 2 \tag{7.55}$$

$$S_{lj}(x-\Delta t,t+\Delta t) = S_{lj}(x,t)(1-\lambda_b\Delta t)+\textstyle\sum_{k=1}^{j} S_{l\ j-k}(x,t)\,\lambda_b g_k\Delta t \tag{7.56}$$

$$+S_{l-1\ j}(0,t)b(x)\,\Delta t \qquad j=1,2,\ldots N-1$$

$$S_{lj}(x-\Delta t,t+\Delta t) = S_{lj}(x,t)(1-\lambda_b\Delta t)+\textstyle\sum_{k=1}^{j} S_{l\ j-k}(x,t)\ \lambda_b g_k\Delta t \tag{7.57}$$

$$j\geq N \quad l\geq 2$$

7.3.3 Steady State Orbit Size Distribution

Dividing the above equations by Δt and allowing $\Delta t \to 0$, the steady state orbit size equations are obtained as follows:

$$0 = -(\lambda_a + \gamma)I_J + (1-\delta)\sum_{m=1}^{b} R_{mj}(0) \qquad 1 \le j \le N-1 \quad (7.58)$$

$$0 = -(\lambda_a + \gamma)I_J + (1-\delta)\sum_{m=1}^{b} R_{mj}(0) + \sum_{l=1}^{\infty} S_{lj}(0) \qquad j \ge N \quad (7.59)$$

$$-\frac{d}{dx}R_{1j}(x) = -\lambda_b R_{1j}(x) + \gamma I_{j+1}(t)a(x)\Delta t + I_j\lambda_a g_1 a(x) + \delta R_{ij}(0)a(x) \qquad j \ge 0 \quad (7.60)$$

$$-\frac{d}{dx}R_{i0}(x) = -\lambda_b R_{i0}(x) + I_0\lambda_a g_i a(x) + \delta R_{i0}(0)a(x) \qquad 2 \le i \le b \quad (7.61)$$

$$-\frac{d}{dx}R_{ij}(x) = -\lambda_b R_{ij}(x) + \delta R_{ij}(0)a(x) + \sum_{k=1}^{j} R_{i\ j-k}(x)\,\lambda_b g_k a(x) \qquad 2 \le i \le b-1,\ j \ge 1 \quad (7.62)$$

$$-\frac{d}{dx}R_{bj}(x) = -\lambda_b R_{bj}(x) + \sum_{k=1}^{j} R_{b\ j-k}(x)\,\lambda_b g_k + \sum_{k=0}^{j} I_{j-k}(t)\,\lambda_a g_{b+k}a(x) + \delta R_{bj}(0)a(x) \qquad j \ge 1 \quad (7.63)$$

$$-\frac{d}{dx}S_{10}(x) = -\lambda_b S_{10}(x) + (1-\delta)R_{i0}(0)b(x) \quad (7.64)$$

$$-\frac{d}{dx}S_{1j}(x) = -\lambda_b S_{1j}(x) + \sum_{k=1}^{j} S_{1\ j-k}(x)\,\lambda_b g_k \qquad j \ge 1 \quad (7.65)$$

$$-\frac{d}{dx}S_{l0}(x) = -\lambda_b S_{l0}(x) + S_{l-1\ 0}(0)b(x) \qquad l \ge 2 \quad (7.66)$$

$$-\frac{d}{dx}S_{lj}(x) = -\lambda_b S_{lj}(x) + \sum_{k=1}^{j} S_{l\ j-k}(x)\,\lambda_b g_k + S_{l-1\ j}(0)b(x) \qquad j = 1,2,\dots N-1 \quad (7.67)$$

$$-\frac{d}{dx}S_{lj}(x) = -\lambda_b S_{lj}(x) + \sum_{k=1}^{j} S_{l\ j-k}(x)\,\lambda_b g_k \qquad j \ge N \quad l \ge 2 \quad (7.68)$$

The Laplace – Stieltjes transform of $R_{in}(x)$ and $S_{jn}(x)$ are defined as

$$\tilde{R}_{in}(\theta) = \int_0^{\infty} e^{-\theta x} R_{in}(x)dx \qquad \tilde{S}_{ln}(\theta) = \int_0^{\infty} e^{-\theta x} S_{ln}(x)dx$$

$$(\theta-\lambda_b)\tilde{R}_{1j}(\theta) = R_{1\mathrm{j}}(0) - \left(\gamma I_{j+1} + I_j\lambda_a g_1 - \delta R_{ij}(0)\right)\tilde{A}(\theta) \qquad j \geq 0 \quad (7.69)$$

$$(\theta-\lambda_b)\tilde{R}_{i0}(\theta) = \mathrm{R_{i0}}(0) - I_0\lambda_a g_i\tilde{A}(\theta) - \delta R_{i0}(0)\tilde{A}(\theta) \qquad 2 \leq i \leq b \qquad (7.70)$$

$$(\theta-\lambda_b)\tilde{R}_{ij}(\theta) = \mathrm{R_{ij}}(0) - \delta R_{ij}(0)\tilde{A}(\theta) - \sum_{k=1}^{j}\tilde{R}_{i\ j-k}(\theta)\,\lambda_b g_k \qquad (7.71)$$

$$2 \leq i \leq b-1,\ j \geq 1$$

$$(\theta-\lambda_b)\tilde{R}_{bj}(\theta) = \mathrm{R_{bj}}(0) - \sum_{k=1}^{j}\tilde{R}_{b\ j-k}(\theta)\,\lambda_b g_k \qquad (7.72)$$

$$-\delta R_{bj}(0)\tilde{A}(\theta) - \sum_{k=0}^{j} I_{j-k}\,\lambda_a g_{b+k}\tilde{A}(\theta) \qquad j \geq 1$$

$$(\theta-\lambda_b)\tilde{S}_{10}(\theta) = \mathrm{S}_{10}(0) - (1-\delta)R_{i0}(0)\tilde{B}(\theta) \qquad 1 \leq i \leq b \qquad (7.73)$$

$$(\theta-\lambda_b)\tilde{S}_{1j}(\theta) = \mathrm{S_{1j}}(0) - \sum_{k=1}^{j}\tilde{S}_{1\ j-k}(\theta)\,\lambda g_k \qquad j \geq 1 \qquad (7.74)$$

$$(\theta-\lambda_b)\tilde{S}_{l0}(\theta) = \mathrm{S_{l0}}(0) - S_{l-1\ 0}(0)\tilde{B}(\theta) \qquad l \geq 2 \qquad (7.75)$$

$$(\theta-\lambda_b)\tilde{S}_{lj}(\theta) = \mathrm{S_{lj}}(0) - \sum_{k=1}^{j}\tilde{S}_{l\ j-k}(\theta)\,\lambda g_k - S_{l-1\ j}(0)\tilde{B}(\theta) \qquad (7.76)$$

$$j = 1,2,\ldots N-1$$

$$(\theta-\lambda_b)\tilde{S}_{lj}(\theta) = \mathrm{S_{lj}}(0) - \sum_{k=1}^{j}\tilde{S}_{l\ j-k}(\theta)\,\lambda g_k \qquad j \geq N \quad l \geq 2 \qquad (7.77)$$

7.3.4 Probability Generating Function

To obtain the PGF of an orbit size distribution at an arbitrary time epoch, the following generating functions are defined.

$$\tilde{R}_i(z,\theta) = \sum_{j=0}^{\infty}\tilde{R}_{ij}(\theta)z^j \qquad R_i(z,0) = \sum_{j=0}^{\infty}R_{ij}(0)z^j \qquad 1 \leq i \leq b$$

$$\tilde{S}_j(z,\theta) = \sum_{l=1}^{\infty}\tilde{Q}_{1j}(0)z^j \qquad S_j(z,0) = \sum_{l=1}^{\infty}Q_{lj}(0)z^j \qquad j \geq 1 \qquad (7.78)$$

$$(\lambda_a+\gamma)I(z) - \gamma I_0 = (1-\delta)\left(\sum_{m=1}^{b}R_m(z,0) - R_{m0}(0)\right) \qquad (7.79)$$

$$+\sum_{l=1}^{\infty}\left(S_l(z,0) - \sum_{j=0}^{N-1}S_{lj}(0)z^j\right)$$

$$(\theta-\lambda_b)\tilde{R}_1(z,\theta) = R_1(z,0) - \left(\gamma\frac{1}{z}(I(z)-I_0) + \lambda_a g_1 I(z) + \delta R_1(z,0)\right)\tilde{A}(\theta) \ (7.80)$$

$$(\theta - \lambda_b + \lambda_b x(z))\tilde{R}_i(z,\theta) = R_i(z,0) - \lambda_a g_i I(z)\tilde{A}(\theta) - \delta R_i(z,0) \quad (7.81)$$

$$2 \le i \le b-1$$

$$\big(\theta - \lambda_b + \lambda_b x(z)\big)\tilde{R}_b(z,\theta) = R_b(z,0) - \lambda_a g_b I_0 \tilde{A}(\theta) \quad (7.82)$$

$$- \textstyle\sum_{k=0}^{\infty} \lambda_a g_{b+k}\, z^k\, I(z)\tilde{A}(\theta) - \delta R_b(z,0)\tilde{A}(\theta)$$

$$(\theta - \lambda_b + \lambda_b x(z))\tilde{S}_1(z,\theta) = S_1(z,0) - \tilde{B}(\theta)(1-\delta)R_{i0}(0) \quad (7.83)$$

$$(\theta - \lambda_b + \lambda_b x(z))\tilde{S}_l(z,\theta) = S_l(z,0) - \tilde{B}(\theta)\textstyle\sum_{j=0}^{N-1} Q_{l-1\,j}(0)\, z^j \quad (7.84)$$

Substituting $\theta = \lambda_b$ in Eqn. 7.80

$$R_1(z,0) = \frac{\tilde{A}(\lambda_b)\Big(\gamma\frac{1}{z}(I(z)-I_0)+\lambda_a g_1 I(z)\Big)}{1-\delta\tilde{A}(\lambda_b)} \quad (7.85)$$

Substituting $\theta = \lambda_b - \lambda_b x(z)$ in equations from Eqn. 7.81 to Eqn. 7.83

$$R_i(z,0) = \frac{\tilde{A}(\lambda_b-\lambda_b x(z))\lambda_a g_i I(z)}{1-\delta\tilde{A}(\lambda_b-\lambda_b x(z))} \qquad 2 \le i \le b-1 \quad (7.86)$$

$$R_b(z,0) = \frac{\tilde{A}(\lambda_b-\lambda_b x(z))\Big(\lambda_a g_b I_0+\sum_{k=0}^{\infty}\lambda_a g_{b+k} z^k\, I(z)\Big)}{1-\delta\tilde{A}(\lambda_b-\lambda_b x(z))} \quad (7.87)$$

$$S_1(z,0) = \tilde{B}(\lambda_b - \lambda_b x(z))(1-\delta)R_{i0}(0) \quad (7.88)$$

$$S_l(z,0) = \tilde{B}(\lambda_b - \lambda_b x(z))\textstyle\sum_{j=0}^{N-1} S_{l-1\,j}(0)\, z^j \quad (7.89)$$

By using Eqn. 8.34 and Eqn. 8.39

$$\tilde{R}_1(z,\theta) = \frac{\big(\tilde{A}(\lambda_b)-\tilde{A}(\theta)\big)\Big(\gamma\frac{1}{z}(I(z)-I_0)+\lambda_a g_1 I(z)\Big)}{(\theta-\lambda_b)(1-\delta\tilde{A}(\lambda_b))} \quad (7.90)$$

By using Eqn. 7.80 and Eqn. 7.85

$$\tilde{R}_i(z,\theta) = \frac{\big(\tilde{A}(\lambda_b-\lambda_b x(z))-\tilde{A}(\theta)\big)\lambda_a g_i I(z)}{(\theta-\lambda_b+\lambda_b x(z))(1-\delta\tilde{A}(\lambda_b-\lambda_b x(z)))} \quad (7.91)$$

By using Eqn. 7.82 and Eqn. 7.87

$$\tilde{R}_b(z,\theta) = \frac{\big(\tilde{A}(\lambda_b-\lambda_b x(z))-\tilde{A}(\theta)\big)\Big(\lambda_a g_b I_0+\sum_{k=0}^{\infty}\lambda_a g_{b+k} z^k\, I(z)\Big)}{(\theta-\lambda_b+\lambda_b x(z))(1-\delta\tilde{A}(\lambda_b-\lambda_b x(z)))} \quad (7.92)$$

By using Eqn. 7.83, Eqn. 7.84, Eqn. 7.88 and Eqn. 7.89, we get

$$\sum_{l=1}^{\infty}\tilde{S}_l(z,\theta)=\frac{(\tilde{B}(\lambda_b-\lambda_b x(z))-\tilde{B}(\theta))\left((1-\delta)R_{i0}(0)+\sum_{j=0}^{N-1}S_{l-1\,j}(0)z^j\right)}{\theta-\lambda+\lambda x(z)} \tag{7.93}$$

Substituting Eqn. 7.85, Eqn. 7.86, Eqn. 7.87 and Eqn. 7.89 in Eqn. 7.79, we get

$$I(z)=\frac{z^bI_0\begin{pmatrix}z\gamma\left(1-\delta\tilde{A}(\lambda_b)\right)\left(1-\delta\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)\right)\\-(1-\delta)\left(1-\delta\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)\right)\tilde{A}(\lambda_b)\\+\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)\lambda_a g_b\left(1-\delta\tilde{A}(\lambda_b)\right)z\end{pmatrix}+z^{b+1}(\tilde{B}(\lambda_b-\lambda_b x(z))-1)\sum_{j=0}^{N-1}S_jz^j}{\begin{matrix}z^{b+1}(\lambda_a+\gamma)\left(1-\delta\tilde{A}(\lambda_b)\right)\left(1-\delta\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)\right)\\-z^b(1-\delta)\tilde{A}(\lambda_b)(\gamma+\lambda_a z g_1)\left(1-\delta\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)\right)\\+\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)\left(1-\delta\tilde{A}(\lambda_b)\right)\lambda_a\left(z^{b+1}\sum_{i=2}^{b-1}g_i-z\left(X(z)-\sum_{j=1}^{b-1}g_jz^j\right)\right)\end{matrix}} \tag{7.94}$$

The PGF of the orbit size at an arbitrary time is given by

$$P(z)=I(z)+\tilde{R}_1(z,0)+\sum_{i=2}^{b-1}\tilde{R}_i(z,0)+\tilde{R}_b(z,0)+\sum_{l=1}^{\infty}\tilde{S}_l(z,0) \tag{7.95}$$

Substituting $\theta=0$ in equations from Eqn. 7.90 to Eqn. 7.93 the Eqn. 7.94 is simplified as

$$P(z)=I_0\left[\frac{G(z)H_1+R_1K(z)}{F(z,\lambda_b)K(z)}\right]+\frac{H_1M_1+M_2S_1(-\lambda_b)(1-\delta\tilde{A}(\lambda_b))\left(1-\delta\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)\right)}{F(z,\lambda_b)K(z)} \tag{7.96}$$

where

$$R_1=\left(\tilde{A}(\lambda_b)-1\right)\left(-\frac{\gamma}{z}\right)\left(-\lambda_b+\lambda_b x(z)\right)\left(1-\delta\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)\right)$$
$$+\left(\tilde{A}(\lambda_b-\lambda_b x(z))-1\right)\lambda_a g_b(-\lambda_b)\left(1-\delta\tilde{A}(\lambda_b)\right)$$

$$H_1=F(z,\lambda_b)+\left(\tilde{A}(\lambda_b)-1\right)\left(\frac{\gamma}{z}+\lambda_a z g_1\right)\left(-\lambda_b+\lambda_b x(z)\right)\left(1-\delta\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)\right)$$
$$+\left(\tilde{A}(\lambda_b-\lambda_b x(z))-1\right)\left(\sum_{k=0}^{\infty}\lambda_a g_{b+k}z^k+\sum_{i=2}^{b-1}\lambda_a g_i\right)(-\lambda_b)\left(1-\delta\tilde{A}(\lambda_b)\right)$$

$$G(z)=\begin{pmatrix}z\gamma\left(1-\delta\tilde{A}(\lambda_b)\right)\left(1-\delta\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)\right)\\-(1-\delta)\left(1-\delta\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)\right)\tilde{A}(\lambda_b)\\+\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)\lambda_a g_b\left(1-\delta\tilde{A}(\lambda_b)\right)z\end{pmatrix}z^b$$

$$K(z) = z^{b+1}(\lambda_a + \gamma)\left(1 - \delta\tilde{A}(\lambda_b)\right)\left(1 - \delta\tilde{A}(\lambda_b - \lambda_b x(z))\right)$$

$$-z^b(1-\delta)\tilde{A}(\lambda_b)(\gamma + \lambda_a z g_1)\left(1 - \delta\tilde{A}(\lambda_b - \lambda_b x(z))\right)$$

$$+\tilde{A}(\lambda_b - \lambda_b x(z))\left(1 - \delta\tilde{A}(\lambda_b)\right)\lambda_a\left(z^{b+1}\sum_{i=2}^{b-1} g_i - z\left(X(z) - \sum_{j=1}^{b-1} g_j z^j\right)\right)$$

$$F(z,\lambda_b) = (-\lambda_b)\left(1 - \delta\tilde{A}(\lambda_b)\right)\left(1 - \delta\tilde{A}(\lambda_b - \lambda_b x(z))\right)\left(-\lambda_b + \lambda_b x(z)\right)$$

$$M_1 = z^{b+1}\left(\tilde{B}(\lambda_b - \lambda_b x(z)) - 1\right)\sum_{j=0}^{N-1} S_j\, z^j$$

$$M_2 = \left(\tilde{B}(\lambda_b - \lambda_b x(z)) - 1\right)\left((1-\delta)R_0 + \sum_{j=0}^{N-1} S_j\, z^j\right)$$

The steady state condition for the proposed model can be obtained from the above expression

$$\lim_{z\to 1} P(z) = 1,$$

Therefore the steady state condition is derived as $\rho = \lambda_b E(A)E(X) < 1$

$$I_0 = \frac{-{\lambda_b}^2 E(X)(1-\delta)V_4 U_1}{G_1 {H_1}' + U_1 {R_1}'}$$

where

$$U_1 = (\lambda_a + \gamma)V_4(1-\delta) - (1-\delta)^2\tilde{A}(\lambda_b)(\gamma + \lambda_a g_1)$$

$$+\lambda_a\left(\sum_{i=2}^{b-1} g_i - \left(1 - \sum_{j=1}^{b-1} g_j\right)\right)$$

$${H_1}' = F'(z,\lambda_a) + \left(\tilde{A}(\lambda_b) - 1\right)(\gamma + \lambda_a g_1)\lambda_b E(X)(1-\delta) + S_1 d_1(-\lambda_b)V_4$$

$${R_1}' = -\left(\tilde{A}(\lambda_b) - 1\right)\gamma\lambda_b E(X)(1-\delta)$$

$$G_1 = \gamma V_4(1-\delta) - (1-\delta)^2\tilde{A}(\lambda_b) + \lambda_a g_b V_4$$

Theorem 7.2. If β_n is the probability of 'n' customers arriving during a vacation then

$$S_0 = \frac{\beta_0 I_0}{(1-\beta_0)}$$

$$q_n = \frac{\left(\beta_n I_0 + \sum_{j=0}^{N-1} S_j\, \beta_{n-j}\right)}{(1-\beta_0)}$$

$$\text{n} = 1, 2, \ldots, \text{N} - 1$$

Proof. Using $\sum_{l=1}^{\infty} S_{lj}(0) = S_j \;\; I_0 = (1-\delta)R_{i0}(0) + S_0(0)$

and the Eqn. 7.89 simplifies to

$\sum_{l=1}^{\infty} S_l(z,0) = \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} S_{ln}(0) z^n$

$\qquad = \sum_{n=0}^{\infty} S_n z^n$

$= (1-\delta)S_{i0}(0)\tilde{B}\big(\lambda - \lambda x(z)\big) + B_0(0)\tilde{B}\big(\lambda - \lambda x(z)\big) + \tilde{B}\big(\lambda - \lambda x(z)\big)\sum_{j=0}^{N-1} S_j z^j$

$= \tilde{B}\big(\lambda - \lambda x(z)\big)\big(I_0 + \sum_{j=0}^{N-1} S_j z^j\big)$

$= \sum_{n=0}^{\infty} \beta_n z^n \left(I_0 + \sum_{j=0}^{N-1} S_j z^j\right)$

$= I_0 \sum_{n=0}^{\infty} \beta_n z^n + \sum_{n=0}^{N-1} \sum_{j=0}^{n} S_j\, \beta_{n-j} + \sum_{n=N}^{\infty} \sum_{j=0}^{N-1} S_j \beta_{n-j}$

Equating the coefficients of z^n, $n = 0,1,2, \ldots N-1$ on both sides of the above equation we have

$$S_0 = \frac{\beta_0 I_0}{(1-\beta_0)}$$

$$S_n = \frac{\left(\beta_n I_0 + \sum_{j=0}^{N-1} S_j \beta_{n-j}\right)}{(1-\beta_0)}$$

Hence the theorem.

7.3.5 Performance Measures

In this section some important performance characteristics are derived from the steady-state probability distribution function given in Eqn. 7.96.

7.3.5.1 Expected Orbit Length

$$E(Q) = \lim_{z \to 1} P'(z)$$

$$E(Q) = I_0 \left[\frac{M_1 K_2 - M_2 K_1}{2{M_1}^2}\right] + \frac{M_1 L_2}{2{M_1}^2}$$

where

$$S_1 = E(A)\,\lambda_b E(X) \quad S_2 = E(A)\,\lambda_a E(X) \quad S_1 = E(B)\,\lambda_b E(X)$$

$$V_1 = E(A)\lambda_b X''(1) + E(A^2){\lambda_b}^2(E(X))^2$$

$$V_2 = E(B)\lambda_b X''(1) + E(B^2){\lambda_b}^2(E(X))^2$$

$$V_3 = E(A)\lambda_a X''(1) + E(A^2){\lambda_a}^2(E(X))^2 \quad V_4 = \left(1 - \delta\tilde{A}(\lambda_b)\right)$$

$$K_1 = G_1 {H_1}' + {R_1}' U_1$$

$$G_1 = \gamma V_4(1-\delta) - (1-\delta)^2\tilde{A}(\lambda_b) + \lambda_a g_b V_4$$

$${H_1}' = F'(z,\lambda_a) + \left(\tilde{A}(\lambda_b) - 1\right)(\gamma + \lambda_a g_1)\lambda_b E(X)(1-\delta) + S_1 d_1(-\lambda_b)V_4$$

$$G_2 = \gamma V_4 - \delta\tilde{A}'(\lambda_b) - (1-\delta)\left((1-\delta)\tilde{A}'(\lambda_b) - \tilde{A}(\lambda_b)\delta S_1\right)$$

$$+\lambda_a g_b\left(V_4 - \delta\tilde{A}'(\lambda_b) + S_1 V_4\right) + b\left(V_4 - (1-\delta)^2\tilde{A}(\lambda_b) + \lambda_a g_b(1-\delta)\right)$$

$${R_1}' = -\left(\tilde{A}(\lambda_b) - 1\right)\gamma\lambda_b E(X)(1-\delta)$$

$$U_1 = (\lambda_a + \gamma)V_4(1-\delta) - (1-\delta)^2\tilde{A}(\lambda_b)(\gamma + \lambda_a g_1)$$

$$+\lambda_a\left(\sum_{i=2}^{b-1} g_i - \left(1 - \sum_{j=1}^{b-1} g_j\right)\right)$$

$$K_2 = G_1 {H_1}'' + 2G_2 {H_1}' + 2{R_1}' U_2 + {R_1}'' U_1$$

$${H_1}'' = F''(z,\lambda_a) + \left(\tilde{A}(\lambda_b) - 1\right)\begin{pmatrix} 2(\gamma + \lambda_a g_1)\lambda_b E(X)(-\delta)S_1 \\ +(-\gamma + \lambda_a g_1)(\lambda_b(X''(1) + 2E(X))(1-\delta) \end{pmatrix}$$

$$+2\tilde{A}'(\lambda_b)(\gamma + \lambda_a g_1)\lambda_b E(X)(1-\delta) - V_4 d_1 \lambda_b V_1$$

$$+S_1(2d_1 + 2{d_1}')(\lambda_b)((-\delta)\tilde{A}'(\lambda_b) + V_4)$$

$$U_2 = (\gamma + \lambda_a)\left((1-\delta)(-\delta)\tilde{A}'(\lambda_b) - \delta V_4 S_1\right) + (b+1)(\gamma + \lambda_a)V_4(1-\delta)$$

$$+(1-\delta)\tilde{A}(\lambda_b)(\gamma + \lambda_a g_1)\delta S_1 - (1-\delta)\tilde{A}(\lambda_b)\lambda_a g_1(1-\delta)) + S_1 V_4 d_2 \lambda_a$$

$$-(1-\delta)(\tilde{A}'(\lambda_b) + b\tilde{A}(\lambda_b))(\gamma + \lambda_a g_1)(1-\delta) + (V_4 - \delta\tilde{A}'(\lambda_b))\lambda_a({d_2}' + d_2$$

$${R_1}'' = \left(\tilde{A}(\lambda_b) - 1\right)\gamma\left(2\delta\lambda_b E(X)S_1 + \lambda_b X''(1)(1-\delta) + 2\lambda_b E(X)(1-\delta)\right)$$

$$-2\gamma\tilde{A}'(\lambda_b)\lambda_b E(X)(1-\delta)$$

$$M_1 = -\lambda_b{}^2 E(X)(1-\delta)V_4 U_1 \qquad d_1 = \sum_{k=0}^{\infty} \lambda_a g_{b+k} + \sum_{i=2}^{b-1} \lambda_a g_i$$

$$d_1{}' = \sum_{k=0}^{\infty} k\lambda_a g_{b+k} \qquad d_1{}'' = \sum_{k=0}^{\infty} k(k-1)\lambda_a g_{b+k}$$

$$d_2 = \sum_{i=2}^{b-1} g_i - \left(1 - \sum_{j=1}^{b-1} g_j\right)$$

$$d_2{}' = (b+1)\sum_{i=2}^{b-1} g_i - \left(1 - \sum_{j=1}^{b-1} g_j\right) - \left(E(X) - \sum_{j=1}^{b-1} j g_j\right)$$

$$M_2 = \left(-\lambda_b{}^2\left(2E(X)\left(\delta^2(1-\delta)\tilde{A}'(\lambda_b) + V_4 S_1\right) + X''(1)(1-\delta)V_4\right)\right)U_1$$

$$+2(-\lambda_b{}^2 E(X)(1-\delta)V_4)U_2$$

$$L_2 = S_1\left(T_1 \sum_{j=0}^{N-1} S_j + T_2\right) + (1-\delta)V_4 \begin{pmatrix} (1-\delta)R_0 \\ +\sum_{j=0}^{N-1} S_j \end{pmatrix} (-\lambda_b{}^2 E(X)(1-\delta)V_4)$$

$$T_1 = -\lambda_b{}^2 E(X)(1-\delta)V_4 U_1 + \left(\tilde{A}(\lambda_b) - 1\right)(\gamma + \lambda_a g_1)\lambda_b E(X)(1-\delta) - S_1 d_1 \lambda_b V_4$$

$$T_2 = \left((1-\delta)R_0 + \sum_{j=0}^{N-1} S_j\right)\left(-\lambda_b{}^2 E(X)(1-\delta)V_4\right)\left(-(1-\delta)\delta\tilde{A}'(\lambda_b) - V_4 S_1 \delta\right)$$

7.3.5.2 Probability that the Server is Busy

$$\mathrm{P(B)} = \lim_{z\to 1} \sum_{m=1}^{b} \tilde{A}_m(z,0)$$

$$\mathrm{P(B)} = \frac{\left(\tilde{S}(\lambda_b) - 1\right)\left(\lambda_b g_1 + \gamma(1 - I_0)\right)}{-\lambda_b} + E(A)I(1)\left(\sum_{i=2}^{b-1} g_i + \lambda_a g_b I_0 + \sum_{k=0}^{\infty} \lambda_b g_{b+k}\right)$$

where

$$I(1) = \frac{I_0\left(\left(1-\delta\tilde{A}(\lambda_{\square})\right)(\gamma(1-\delta)+\lambda_a g_b) - (1-\delta)^2\tilde{A}(\lambda_b)\right)}{\left(1-\delta\tilde{A}(\lambda_b)\right)\begin{pmatrix} (\lambda_a+\gamma)(1-\delta) \\ +\lambda_a\left(\sum_{i=2}^{b-1} g_i - \left(1-\sum_{j=1}^{b-1} g_j\right)\right)\end{pmatrix} - (1-\delta)^2\tilde{A}(\lambda_b)(\gamma+\lambda_a g_1)}$$

7.3.5.3 Probability that the Server is Idle

$$\mathrm{P(I)} = \lim_{z\to 1} I(z)$$

$$\mathrm{P(I)} = \frac{I_0\left(\left(1-\delta\tilde{A}(\lambda_b)\right)(\gamma(1-\delta)+\lambda_a g_b) - (1-\delta)^2\tilde{A}(\lambda_b)\right)}{\left(1-\delta\tilde{A}(\lambda_b)\right)\begin{pmatrix} (\lambda_a+\gamma)(1-\delta) \\ +\lambda_a\left(\sum_{i=2}^{b-1} g_i - \left(1-\sum_{j=1}^{b-1} g_j\right)\right)\end{pmatrix} - (1-\delta)^2\tilde{A}(\lambda_b)(\gamma+\lambda_a g_1)}$$

7.3.5.4 Probability that the Server is on Vacation

$$\mathrm{P(V)} = \lim_{z\to 1} \sum_{l=1}^{\infty} \tilde{S}_l(z,\theta)$$

$$P(\mathrm{V}) = E(B)\lambda_b \left((1-\delta)R_0 + \sum_{j=0}^{N-1} S_j\right)$$

7.3.6 Special Cases

The proposed model is developed with the assumption that the service time is arbitrary. However to analyse real time system, proper distribution is mandatory. This segment presents some distinct cases of the proposed system via indicating service time as exponential distribution, hyper exponential distribution, Erlangian distribution.

Case. 1: Exponential bulk service time

The PDF of exponential distribution is defined as

$$A(x) = e^{-\mu x}, \text{ where } \mu \text{ is parameter}$$

$$\tilde{A}\big(\lambda_b - \lambda_b x(z)\big) = \left(\frac{\mu}{\mu + (\lambda_b - \lambda_b x(z))}\right)$$

The PGF of the orbit size for exponential service time is derived by substituting the expression for $\tilde{A}\big(\lambda_b - \lambda_b x(z)\big)$ in Eqn. 7.96.

$$P(z) = I_0\left[\frac{G(z)H_1 + R_1K(z)}{F(z,\lambda_b)K(z)}\right] + \frac{H_1M_1 + M_2S_1(-\lambda_b)(1-\delta\tilde{A}(\lambda_b))\left(1-\delta\left(\frac{\mu}{\mu+(\lambda_b-\lambda_b x(Z))}\right)\right)}{F(z,\lambda_b)K(z)}$$

where

$$R_1 = \left(\left(\frac{\mu}{\mu+\lambda_b}\right) - 1\right)\left(-\frac{\gamma}{z}\right)\left(-\lambda_b + \lambda_b x(z)\right)\left(1 - \delta\left(\frac{\mu}{\mu+(\lambda_b-\lambda_b x(z))}\right)\right)$$
$$+\left(\left(\frac{\mu}{\mu+(\lambda_b-\lambda_b x(z))}\right) - 1\right)\lambda_a g_b(-\lambda_b)\left(1-\delta\left(\frac{\mu}{\mu+\lambda_b}\right)\right)$$

$$H_1 = F(z,\lambda_b) + \left(\left(\frac{\mu}{\mu+\lambda_b}\right) - 1\right)\left(\frac{\gamma}{z} + \lambda_a z g_1\right)\left(-\lambda_b + \lambda_b x(z)\right)$$
$$\times\left(1 - \delta\left(\frac{\mu}{\mu+\big(\lambda_b-\lambda_b x(z)\big)}\right)\right) + \left(\left(\frac{\mu}{\mu+\big(\lambda_b-\lambda_b x(z)\big)}\right) - 1\right)$$
$$\times\left(\sum_{k=0}^{\infty}\lambda_a g_{b+k} z^k + \sum_{i=2}^{b-1}\lambda_a g_i\right)(-\lambda_b)\left(1-\delta\left(\frac{\mu}{\mu+\lambda_b}\right)\right)$$

$$G(z)=\begin{pmatrix} z\gamma\left(1-\delta\left(\frac{\mu}{\mu+\lambda_b}\right)\right)\left(1-\delta\left(\frac{\mu}{\mu+(\lambda_b-\lambda_b x(z))}\right)\right) \\ -(1-\delta)\left(1-\delta\left(\frac{\mu}{\mu+(\lambda_b-\lambda_b x(z))}\right)\right)\left(\frac{\mu}{\mu+\lambda_b}\right) \\ +\left(\frac{\mu}{\mu+(\lambda_b-\lambda_b x(z))}\right)\lambda_a g_b\left(1-\delta\left(\frac{\mu}{\mu+\lambda_b}\right)\right)z \end{pmatrix} z^b$$

$$K(z)=z^{b+1}(\lambda_a+\gamma)\left(1-\delta\tilde{A}(\lambda_b)\right)\left(1-\delta\left(\frac{\mu}{\mu+(\lambda_b-\lambda_b x(z))}\right)\right)$$

$$-z^b(1-\delta)\left(\frac{\mu}{\mu+\lambda_b}\right)(\gamma+\lambda_a z g_1)\left(1-\delta\left(\frac{\mu}{\mu+(\lambda_b-\lambda_b x(z))}\right)\right)$$

$$+\left(\frac{\mu}{\mu+(\lambda_b-\lambda_b x(z))}\right)\left(1-\delta\left(\frac{\mu}{\mu+\lambda_b}\right)\right)\lambda_a\left(z^{b+1}\sum_{i=2}^{b-1} g_i\right)$$

$$-z\left(X(z)-\sum_{j=1}^{b-1} g_j z^j\right)$$

$$F(z,\lambda_b)=(-\lambda_b)\left(1-\delta\left(\frac{\mu}{\mu+\lambda_b}\right)\right)\left(1-\delta\left(\frac{\mu}{\mu+(\lambda_b-\lambda_b x(z))}\right)\right)\left(-\lambda_b+\lambda_b x(z)\right)$$

$$M_1=z^{b+1}\left(\tilde{B}(\lambda_b-\lambda_b x(z))-1\right)\sum_{j=0}^{N-1} S_j z^j$$

$$M_2=\left(\tilde{B}(\lambda_b-\lambda_b x(z))-1\right)\left((1-\delta)R_0+\sum_{j=0}^{N-1} S_j z^j\right)$$

Case. 2: Hyper exponential bulk service time

When the service time follows hyper exponential distribution with probability density function, then $a(x)=cde^{-dx}+(1-c)\,fe^{-fx}$, where d and f are parameters, then,

$$\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)=\left(\frac{dc}{d+\left(\lambda_b-\lambda_b x(z)\right)}\right)+\left(\frac{f(1-c)}{f+\left(\lambda_b-\lambda_b x(z)\right)}\right)$$

The PGF of the orbit size for hyper exponential service time is derived by substituting the expression for $\tilde{A}\left(\lambda_b-\lambda_b x(z)\right)$ in Eqn. 7.96, then

$$P(z) = I_0 \left[\frac{G(z)H_1 + R_1 K(z)}{F(z,\lambda_b)K(z)}\right] + \frac{H_1 M_1 + M_2 S_1(-\lambda_b) M_3 M_4}{F(z,\lambda_b)K(z)}$$

where

$$R_1 = \left(\left(\frac{dc}{d+\lambda_b} + \frac{f(1-c)}{f+\lambda_b}\right) - 1\right)\left(-\frac{\gamma}{z}\right)\left(-\lambda_b + \lambda_b x(z)\right)M_4$$

$$+\left(\left(\left(\frac{dc}{d+\left(\lambda_b - \lambda_b x(z)\right)}\right) + \left(\frac{f(1-c)}{f+\left(\lambda_b - \lambda_b x(z)\right)}\right)\right) - 1\right)\lambda_a g_b(-\lambda_b)M_3$$

$$H_1 = F(z,\lambda_b) + \left(\tilde{A}(\lambda_b) - 1\right)\left(\frac{\gamma}{z} + \lambda_a z g_1\right)\left(-\lambda_b + \lambda_b x(z)\right)M_4$$

$$+\left(\left(\left(\frac{dc}{d+\left(\lambda_b - \lambda_b x(z)\right)}\right) + \left(\frac{f(1-c)}{f+\left(\lambda_b - \lambda_b x(z)\right)}\right)\right) - 1\right)$$

$$\times \left(\sum_{k=0}^{\infty} \lambda_a g_{b+k} z^k + \sum_{i=2}^{b-1} \lambda_a g_i\right)(-\lambda_b)M_3$$

$$G(z) = \begin{pmatrix} z\gamma M_3 M_4 \\ -(1-\delta)M_4\left(\frac{dc}{d+\lambda_b} + \frac{f(1-c)}{f+\lambda_b}\right) \\ +\left(\left(\frac{dc}{d+\left(\lambda_b - \lambda_{\square} x(z)\right)}\right) + \left(\frac{f(1-c)}{f+\left(\lambda_b - \lambda_b x(z)\right)}\right)\right)\lambda_a g_b M_3 z \end{pmatrix} z^b$$

$$K(z) = z^{b+1}(\lambda_a + \gamma)M_3 M_4 - z^b(1-\delta)\left(\frac{dc}{d+\lambda_b} + \frac{f(1-c)}{f+\lambda_b}\right)(\gamma + \lambda_a z g_1)M_4$$

$$+\left(\left(\frac{dc}{d+\left(\lambda_b - \lambda_b x(z)\right)}\right) + \left(\frac{f(1-c)}{f+\left(\lambda_b - \lambda_b x(z)\right)}\right)\right)M_3 \lambda_a$$

$$\times \left(z^{b+1} \sum_{i=2}^{b-1} g_i z\left(X(z) - \sum_{j=1}^{b-1} g_j z^j\right)\right)$$

$$F(z,\lambda_b) = (-\lambda_b)M_3 M_4\left(-\lambda_b + \lambda_b x(z)\right)$$

$$M_1 = z^{b+1}\left(\tilde{B}(\lambda_b - \lambda_b x(z)) - 1\right)\sum_{j=0}^{N-1} S_j z^j \quad M_3 = \left(1 - \delta\left(\frac{dc}{d+\lambda_b} + \frac{f(1-c)}{f+\lambda_b}\right)\right)$$

$$M_4 = \left(1 - \delta\left(\frac{dc}{d+\left(\lambda_b - \lambda_b x(z)\right)} + \frac{f(1-c)}{f+\left(\lambda_b - \lambda_b x(z)\right)}\right)\right)$$

$$M_2 = \left(\tilde{B}(\lambda_b - \lambda_{\square} x(z)) - 1\right)\left((1-\delta)R_0 + \sum_{j=0}^{N-1} S_j z^j\right)$$

Case. 3: K - Erlangian bulk service time

Let us consider that service time follows K - Erlang distribution with probability density function

$$a(x) = \frac{(k\mu)^k x^{k-1} e^{-(k\mu x)}}{(k-1)!}, \text{ k > 0; where } \mu \text{ is the parameter, then}$$

$$\tilde{A}(\lambda_b - \lambda_b x(z)) = \left(\frac{k\mu}{k\mu + (\lambda_b - \lambda_b x(z))}\right)^k$$

The PGF of the orbit size K-Erlangian bulk service time is derived by substituting the expression for $\tilde{A}(\lambda_b - \lambda_b x(z))$ in Eqn. 7.96.

7.3.7 Numerical Illustrations

In this section, obtained theoretical results are validated with suitable numerical example. Numerical results are derived with the following assumptions.

Mean arrival rate when the system is idle	λ_a
Mean arrival rate when the system is busy or on vacation	λ_b
Service time follows exponential distribution with parameter	μ
Batch size of the arrival follows geometric distribution with mean	3
Retrial rate	γ
Vacation time follows exponential distribution with parameter	η
Maximum server capacity	b

Effects of arrival rates with respect to expected orbit length is given in Table 7.9 and Table 7.10 with parameters $\eta = 2$, μ=3, b=4, $\delta = 0.4$. From tables it is observed that mean orbit size increases when the arrival rate increases and mean orbit size decreases when retrial rate increases.

Table 7.11 and Fig. 7.9 shows the way in which the orbit size changes for different values of arrival rate λ_a (when the server is idle). Considering the service times as exponential, Erlang-2 and hyper exponential with parameters $\lambda_b = 2$, $\gamma = 5$, $\eta = 2$, μ=3, b=4, N=7, $\delta = 0.4$, it can be observed that the mean orbit size increases when the arrival rate λ_a increases.

Table 7.12 and Fig. 7.10 show the way in which the orbit size changes for different values of arrival rate λ_b (when the server is busy or in vacation). Considering the service times as exponential, Erlang-2 and hyper exponential with parameters

$\lambda_a = 2, \gamma = 5, \eta = 2, \mu = 3, b = 4, N = 7, \ \delta = 0.4$, it can be observed that the mean orbit size increases when the arrival rate λ_b increases.

Table 7.9 Retrial Rate versus Mean Orbit Size (Arrival Rate $\lambda_b = 2$)

Retrial rate γ	Expected orbit length E(Q)				
	Arrival rates(λ_a)				
	2	**3**	**4**	**5**	**6**
1	3.9261	4.7963	5.8321	6.5920	8.1928
2	3.2351	4.1641	5.2193	6.0821	7.6291
3	2.7936	3.9365	4.9287	5.9768	7.4362
4	2.3261	3.7590	4.7981	5.7989	7.3190
5	2.0982	3.6963	4.6362	5.6923	7.1982
6	1.5391	3.3872	4.4902	5.4329	6.9360
7	1.1972	3.2015	4.3198	5.0763	6.2769

Table 7.10 Retrial Rate versus Mean Orbit Size (Arrival Rate $\lambda_a = 2$)

Retrial rate γ	Expected orbit length E(Q)					
	Arrival rates(λ_b)					
	1	**2**	**3**	**4**	**5**	**6**
1	4.6321	5.7962	6.0361	6.5328	7.7912	8.3792
2	4.4982	5.5728	5.9253	6.3291	7.6324	8.0523
3	4.3026	5.3264	5.7421	6.0328	7.4561	7.8392
4	4.1982	5.1920	5.6378	5.8391	7.2793	7.5091
5	3.9751	4.9324	5.4231	5.5923	6.9261	7.2592
6	3.7782	4.8396	5.3438	5.3792	6.7938	6.9132
7	3.5329	4.6523	5.0632	5.1902	6.4982	6.7981

Table 7.11 Arrival Rate (λ_a) versus Expected Orbit Length

λ_a	**Expected orbit length**		
	Exponential	Erlang	Hyper-exponential
0.2	0.0572	0.0612	0.0644
0.4	0.0831	0.0874	0.0892
0.6	0.1357	0.2461	0.3195
0.8	0.3492	0.3964	0.4263
1.0	0.5763	0.6297	0.6921
1.2	0.8324	0.8921	0.9035
1.4	1.2497	1.3592	1.4257

Table 7.12 Arrival Rate (λ_b) versus Expected Orbit Length

λ_b	Expected orbit length		
	Exponential	Erlang	Hyper-exponential
0.2	0.0652	0.0735	0.0943
0.4	0.0839	0.0962	0.2361
0.6	0.1368	0.2437	0.4495
0.8	0.2764	0.3984	0.5327
1.0	0.4985	0.5321	0.6792
1.2	0.5362	0.7092	0.8321
1.4	0.6952	0.8361	0.9067

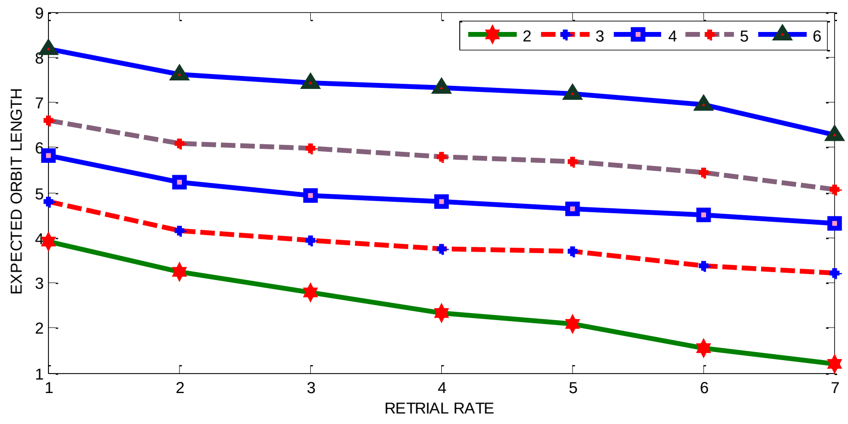

Fig. 7.6 Retrial Rate versus Expected Orbit Length (Arrival Rate $\lambda_b = 2$)

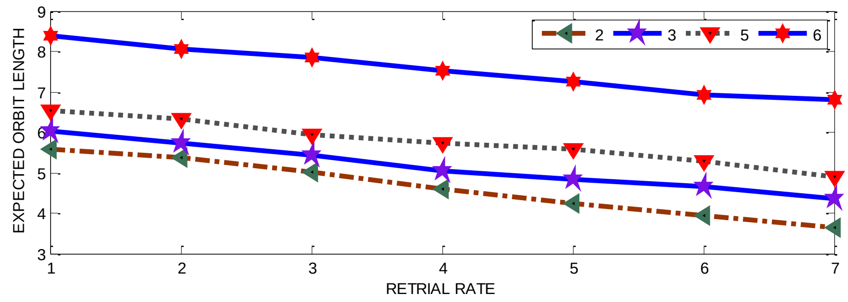

Fig. 7.7 Retrial Rate versus Expected Orbit Length (Arrival Rate $\lambda_a = 2$)

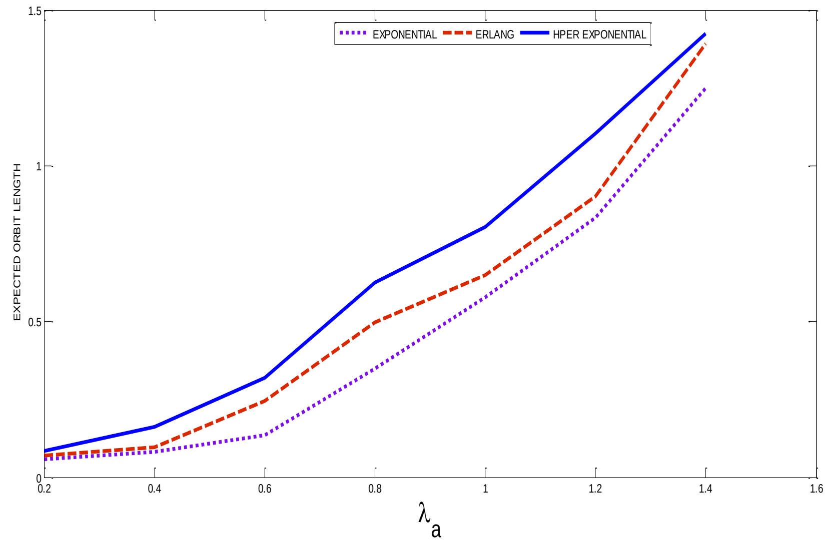

Fig. 7.8 Arrival Rate(λ_a) versus Expected Orbit Length

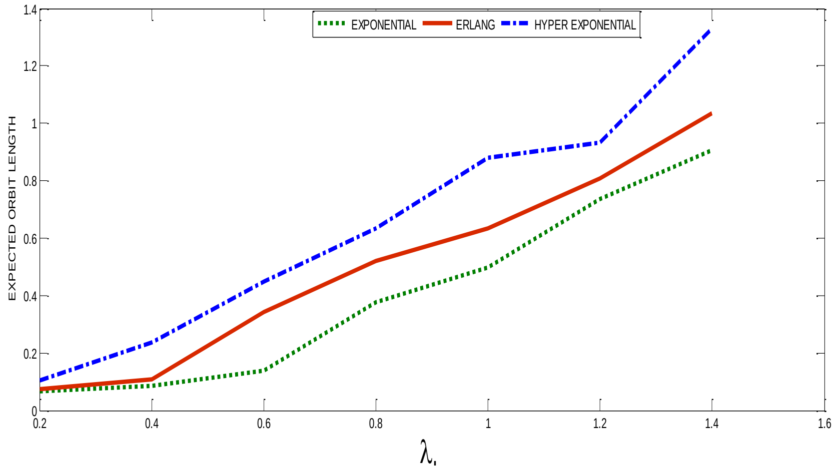

Fig. 7.9 Arrival rate(λ_b) versus Expected Orbit Length

7.3.8 Conclusion

In this chapter, two retrial queueing models are discussed. In model 1, working vacation, non-working vacation and server failure are introduced for bulk arrival and batch service retrial queueing system, whereas in model 2, state dependent arrival and active Bernoulli feedback are introduced for bulk arrival and batch service retrial queueing system with multiple vacations and threshold. For the modelled queueing systems, the PGF describing the orbit size is obtained by using supplementary variable technique. Various performance characteristics are also presented with appropriate numerical illustrations.