CHAPTER 6

Analysis of Batch Service Retrial Queueing System with Server Failure, Threshold and Multiple Vacations

6.1 Introduction

Mathematical modelling of retrial queueing system with vacations is much useful in dealing with real life congestion problems like local area networks (LAN), communication networks, media access protocols, etc. In the modern technology, communication networks play a vital role in transmitting and accessing data from anywhere at any time. Retrial queueing system is characterized as an arrival of customers finds the server is busy, leaves the service area but after some random delay they request service again. If the customer finds the server is busy then he joins an orbit which is defined as a virtual queue formed by the customers after finding that the server is busy.

Analytical treatment of different models of retrial queues was extensively studied by Falin and Templeton(1997). A brief survey and an overview of retrial queues have been explained (Artalejo, 1999). Krishna Kumar and Arivudainambi (2002) non-Markovian retrial queue with Bernoulli schedules and general retrial times.

Madan and Choudhury (2005) discussed a single server queue with two phases of heterogeneous service under Bernoulli schedule and a general vacation time. Zhou Wenhui (2005) analysed single-server retrial queue with FCFS orbit and Bernoulli vacation. Mohamed Boualem et al. (2007) derived stochastic inequalities for M/G/1 retrial queues with vacations and constant retrial policy.

Retrial queueing system with vacations and breakdown has been analysed by many researchers which includes a study on retrial queue with constant retrial rate and server breakdown (Li and Zhao, 2005). Atencia, Bouza, and Moreno (2008) derived generating functions of system and orbit state of the bulk arrival retrial queue with server breakdown. Also they considered constant failure rate of the server. Chang and Ke (2009) used supplementary variable technique to derive some important results in batch arrival retrial queue with modified vacations. Shweta Upadhyaya (2010) examined operating characteristics of an $M^X/G/1$ retrial queueing system under Bernoulli vacation schedule with setup times. M/G/1 retrial queue with breakdown period and delay period was analysed by Choudhury and Ke (2014). They used Bernoulli schedule vacation and derived system size at a departure time epoch. Choudhury et al. (2010) analysed $M^X/G/1$ retrial queueing system with optional two phases of service and breakdown. In this chapter delay time is also introduced. Yang and Wu (2015) studied working vacation queueing model with threshold and server failure. In their work cost minimization was carried out.

In all the above queueing models, customers were served one by one. But in many real time applications it is essential to provide batch service too. Bulk arrival and batch service retrial queueing system has been analysed (Haridass et al., 2012). They used supplementary variable technique, derived some important performance measures and developed cost effective model for their proposed system. Dudin (2015) analysed single server retrial queue with group admission of customers

In this literature of bulk arrival retrial queueing models, only a few authors studied bulk arrival and batch service retrial queueing models. The bulk arrival and batch service retrial queueing model with multiple vacations were not considered yet. Once the server gets breakdown the service will be stopped in all the bulk arrival retrial queueing models with breakdown under consideration. But in this proposed model, though the server encounters failure service will not be stopped but will be continued for the current batch through some precaution in technical arrangements. Server will be repaired after the service completion, during renewal period of the server. The model so considered is peculiar because multiple vacations with threshold and server failure with non-disruptive service are used to model the proposed bulk arrival and batch service retrial queueing system.

6.2 Model Description

This chapter analyses bulk arrival and batch service queueing model with threshold, server failure, multiple vacations and constant retrial policy. Customers are entered into the system in bulk according to Poisson process with rate λ. Upon arrival, if the server is busy then entire customers choose to join the virtual queue called orbit. Customers in the orbit request service again after some time. On the contrary, if the customers find that the server is free then batch service will be provided with minimum of '1' and maximum of 'b' number of customers. Let ξ be the queue length. If $1 \leq \xi \leq b$ then entire batch will be served immediately. Additionally if $\xi > b$, then service will be provided for only 'b' customers. Remaining $\xi - b$ customers join the orbit. Since this proposed system follows constant retrial policy, customers in the orbit explore service one by one with constant retrial rate 'γ'. The server may encounter failure while serving customers. This chapter proposes a concept called server failure without service interruption. Though the server encounters failure the service will not be stopped, but will be continued for current batch by doing some technical precaution arrangements.

Proper maintenance of the server or repair of the server is defined as renewal of service station. When the server encounters failure with probability δ then the renewal of service station will be considered. After completing a renewal of service station or when there is no server failure with probability '$1 - \delta$' and the orbit size is zero, then the server leaves for vacation. On returning from a vacation if the orbit size is less than 'N' then the server leaves for another vacation. Likewise, the server continuously goes for vacation (multiple vacations) until the orbit size reaches the threshold value 'N' (N > b). At a vacation completion time, if the orbit size reaches the threshold value 'N', then the server becomes idle in the system to provide service for customers from primary source or orbit. The model under consideration is schematically represented in Fig. 6.1.

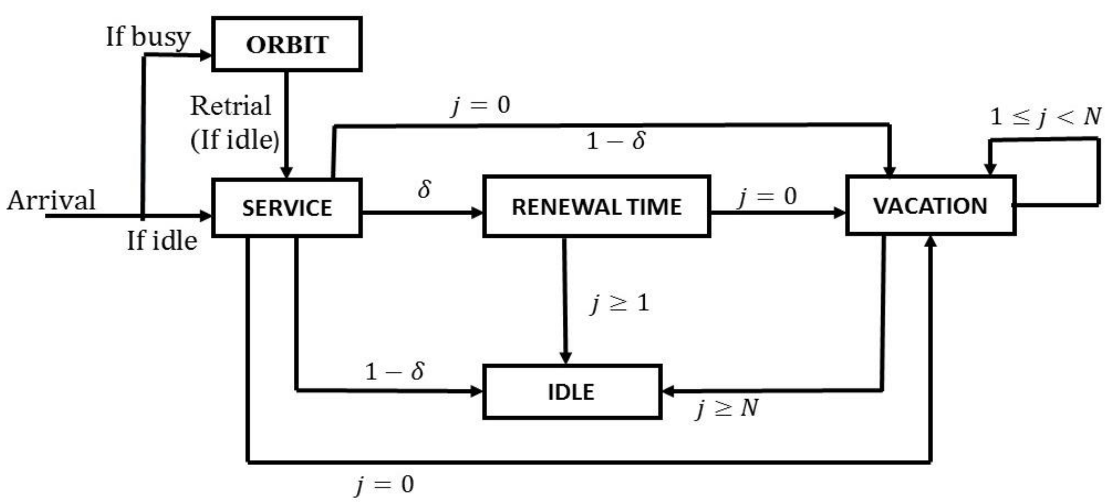

Fig. 6.1 Schematic Representation of the Queueing Model: j - Orbit Size

6.2.1 Motivation

Performance analysis of LAN executing under transmission protocol CSMA-CD (Carrier Sense Multiple Access with Collision Detection) is one of the application of our proposed queueing model. In order to remit data, any moderator on the segment of CSMA-CD is used to investigate whether the transmission channel (a bus) is free or not, to avoid collisions between the data. Moderator A transmits messages to other moderator through the transmission medium (server in our model). The messages are split into different packets (batch) in order to transmit to the destination station. First, moderator A checks whether the bus is free or not. If the transmission medium is free, then a group of packets is picked for transmission and the surplus is stored in a buffer (retrial group). On the contrary, if the bus is busy, then all the packets are stored in the buffer and the moderator A will retry the transmission later on. Sometimes while transmitting data, the server may be affected with virus (server failure), resulting in slow performance of the server. Though the server fails, service will not be interrupted, but will be continued for current batch of packets by including antivirus software. When the transmission medium fails, antivirus software will get stimulated immediately and helps in transmitting the data. The virus will be removed after the data transmission (renewal period). When the server is idle, maintenance activities (multiple vacations) such as temporary files can be cleaned to keep the server functioning well. This type of maintenance can be programmed to perform on a regular basis. This can be designed as bulk arrival and batch service queueing model with server failure, non-disruptive service and multiple vacations.

For the proposed model, the probability generating function (PGF) of the steady state orbit size distribution at an arbitrary time epoch is obtained by using supplementary variable technique. Various performance measures are derived. Cost effective model for the queueing system is developed. Some special cases are also discussed. Numerical solution for particular values of parameters is presented.

6.3 Mathematical Model

Let λ be the Poisson arrival rate, X be the group size random variable of the arrival, g_k be the probability that 'k' customers arrive in a batch, $X(z)$ be the probability generating function of X, N_q(t) be the number of customers waiting for service at time t, N_s(t) be the number of customers under the service at time t. $N(t)$ be the number of customers in the orbit at time t.

Let $'\gamma'$ be the retrial rate of the customer from the orbit and $'\delta'$ be the probability of server failure.

Table 6.1 Notations

	Cumulative Distribution Function	Probability Density Function	Laplace-Stieltjes Transform	Remaining Service Time
Service	S(x)	s(x)	$\tilde{S}(\theta)$	S^0(x)
Vacation	V(x)	v(x)	$\tilde{V}(\theta)$	V^0(x)
Renewal	B(x)	b(x)	$\tilde{B}(\theta)$	B^0(x)

Let G (t) denotes different states of the server at time t, and define

$$G(t) = \begin{cases} 0, \text{ if the server is busy with service} \\ 1, \text{ if the server is on vacation} \\ 2, \text{ if the server is on renewal} \\ 3, \text{ if the server is idle} \end{cases}$$

Let $C(t) = m$ be the server is on mth vacation

The state probabilities are defined to obtain governing equations:

$$A_{ij}(x,t)dt = Pr\{N_s(t) = i, N_q(t) = j, x \le S^0(t) \le x + dt, G(t) = 0\}$$

$$1 \le i \le b, j \ge 0$$

$$Q_{jn}(t)dt = Pr\{N(t) = n, x \le V^0(t) \le x + dt, G(t) = 1, C(t) = j\}, \qquad n \ge 0$$

$$B_n(t)dt = Pr\{N(t) = n, x \le B^0(t) \le x + dt, G(t) = 2\}, \qquad n \ge 0$$

$$D_n(t)dt = Pr\{N(t) = n, G(t) = 3\}$$

By using supplementary variable technique and using remaining service time as a supplementary variable the following equations are obtained.

$$D_j(t + \Delta t) = D_j(t)(1 - \lambda\Delta t - \gamma\Delta t) + (1 - \delta)\sum_{m=1}^{b} A_{mj}(0,t)\Delta t + B_j(0,t)\Delta t \qquad (6.1)$$

$$1 \le j \le N-1$$

$$D_j(t+\Delta t) = D_j(t)(1-\lambda\Delta t-\gamma\Delta t) + (1-\delta)\sum_{m=1}^{b} A_{mj}(0,t)\Delta t + B_j(0,t)\Delta t \qquad (6.2)$$
$$+\sum_{l=1}^{\infty} Q_{lj}(0,t)\Delta t \qquad j \ge N$$

$$A_{1j}(x-\Delta t, t+\Delta t) = A_{ij}(x,t)(1-\lambda\Delta t) + \gamma D_{j+1}(t)s(x)\Delta t + D_j(t)\lambda g_1 s(x)\Delta t \qquad (6.3)$$

$$j \ge 0$$

$$A_{i0}(x-\Delta t, t+\Delta t) = A_{i0}(x,t)(1-\lambda\Delta t) + D_0(t)\lambda g_i s(x)\Delta t \qquad 2 \le i \le b \qquad (6.4)$$

$$A_{ij}(x-\Delta t, t+\Delta t) = A_{i0}(x,t)(1-\lambda\Delta t) + D_j(t)\lambda g_i s(x)\Delta t \qquad (6.5)$$

$$+\sum_{k=1}^{j} A_{i\ j-k}(x,t)\,\lambda g_k s(x)\Delta t \qquad 2 \le i \le b-1,\ j \ge 1$$

$$A_{bj}(x-\Delta t, t+\Delta t) = A_{bj}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} A_{b\ j-k}(x,t)\,\lambda g_k \Delta t \qquad (6.6)$$

$$+\sum_{k=0}^{j} D_{j-k}(t)\,\lambda g_{b+k} s(x)\Delta \mathrm{t} \qquad j \ge 1$$

$$Q_{10}(x-\Delta t, t+\Delta t) = Q_{10}(x,t)(1-\lambda\Delta t) + (1-\delta)A_{i0}(0,t)v(x)\Delta t \qquad (6.7)$$

$$+B_0(0,t)v(x)\Delta t$$

$$Q_{1j}(x-\Delta t, t+\Delta t) = Q_{1j}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} Q_{1\ j-k}(x,t)\,\lambda g_k \Delta t \qquad (6.8)$$

$$Q_{l0}(x-\Delta t, t+\Delta t) = Q_{l0}(x,t)(1-\lambda\Delta t) + Q_{l-1\ 0}(0,t)v(x)\,\Delta t \qquad l \ge 2 \qquad (6.9)$$

$$Q_{lj}(x-\Delta t, t+\Delta t) = Q_{lj}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} Q_{l\ j-k}(x,t)\,\lambda g_k \Delta t \qquad (6.10)$$

$$+Q_{l-1\ \ j}(0,t)v(x)\,\Delta t \qquad j = 1,2,\ldots N-1$$

$$Q_{lj}(x-\Delta t, t+\Delta t) = Q_{lj}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} Q_{l\ j-k}(x,t)\,\lambda g_k \Delta t \qquad (6.11)$$

$$j \ge N \qquad l \ge 2$$

$$B_0(x-\Delta t, t+\Delta t) = B_0(x,t)(1-\lambda\Delta t) + \delta\sum_{m=1}^{b} A_{m0}(0,t)b(x)\Delta t \qquad (6.12)$$

$$B_n(x-\Delta t, t+\Delta t) = B_n(x,t)(1-\lambda\Delta t) + \delta\sum_{m=1}^{b} A_{\square n}(0,t)b(x)\Delta t \qquad (6.13)$$

$$+\sum_{k=1}^{n} B_{n-k}(x,t)\,\lambda g_k\Delta t \qquad n \geq 1$$

6.4 Steady State Orbit Size Distribution

Dividing the above equations by Δt and allowing $\Delta t \to 0$, the steady state orbit size equations are obtained as follows:

$$0 = -(\lambda+\gamma)D_J + B_n(0) + (1-\delta)\sum_{m=1}^{b} A_{mj}(0) \qquad 1 \leq j \leq N-1 \qquad (6.14)$$

$$0 = -(\lambda+\gamma)D_J + B_n(0) + (1-\delta)\sum_{m=1}^{b} A_{mj}(0) + \sum_{l=1}^{\infty} Q_{lj}(0) \quad j \geq N \quad (6.15)$$

$$-\frac{d}{dx}A_{1j}(x) = -\lambda A_{1j}(x) + \gamma D_{j+1}s(x) + D_j\lambda g_1 s(x) \qquad j \geq 0 \qquad (6.16)$$

$$-\frac{d}{dx}A_{i0}(x) = -\lambda A_{1j}(x) + D_0\lambda g_i s(x) \qquad 2 \leq i \leq b \qquad (6.17)$$

$$-\frac{d}{dx}A_{ij}(x) = -\lambda A_{ij}(x) + D_j\lambda g_i s(x) + \sum_{k=1}^{j} A_{i\,j-k}(x)\,\lambda g_k s(x) \qquad (6.18)$$

$$2 \leq i \leq b-1,\ j \geq 1$$

$$-\frac{d}{dx}A_{bj}(x) = -\lambda A_{bj}(x) + \sum_{k=1}^{j} A_{b\,j-k}(x)\,\lambda g_{\square} + \sum_{k=0}^{j} D_{j-k}\,\lambda g_{b+k}s(x) \quad j \geq 1 \quad (6.19)$$

$$-\frac{d}{dx}Q_{10}(x) = -\lambda Q_{10}(x) + (1-\delta)A_{i0}(0)v(x) + B_0(0)v(x) \qquad 1 \leq i \leq b \qquad (6.20)$$

$$-\frac{d}{dx}Q_{1j}(x) = -\lambda Q_{1j}(x) + \sum_{k=1}^{j} Q_{1\,j-k}(x)\,\lambda g_k \qquad j \geq 1 \qquad (6.21)$$

$$-\frac{d}{dx}Q_{l0}(x) = -\lambda Q_{l0}(x) + Q_{l-1\,0}(0)v(x) \qquad l \geq 2 \qquad (6.22)$$

$$-\frac{d}{dx}Q_{lj}(x) = -\lambda Q_{lj}(x) + \sum_{k=1}^{j} Q_{l\,j-k}(x)\,\lambda g_k + Q_{l-1\ j}(0)v(x) \qquad (6.23)$$

$$j = 1,2,\ldots N-1$$

$$-\frac{d}{dx}Q_{lj}(x) = -\lambda Q_{lj}(x) + \sum_{k=1}^{j} Q_{l\ j-k}(x)\,\lambda g_k \qquad j \geq N \quad l \geq 2 \quad (6.24)$$

$$-\frac{d}{dx}B_0(x) = -\lambda B_0(x) + \delta\sum_{m=1}^{b} A_{m0}(0)b(x) \qquad (6.25)$$

$$-\frac{d}{dx}B_n(x) = -\lambda B_n(x) + \delta\sum_{m=1}^{b} A_{mn}(0)b(x) + \sum_{k=1}^{n} B_{n-k}(x)\,\lambda g_k \qquad n \geq 1 \quad (6.26)$$

The Laplace – Stieltjes transform of $A_{in}(x)$, $Q_{jn}(x)$ and $B_n(x)$ are defined as

$\tilde{A}_{in}(\theta) = \int_0^\infty e^{-\theta x}\, A_{in}(x)dx$ $\qquad \tilde{Q}_{ln}(\theta) = \int_0^\infty e^{-\theta x}\, Q_{ln}(x)dx$ and

$\tilde{B}_n(\theta) = \int_0^\infty e^{-\theta x}\, B_n(x)dx$

Taking Laplace – Stieltjes transform on both sides of equations from Eqn. 6.14 to Eqn. 6.26, we get

$$(\theta-\lambda)\tilde{A}_{1j}(\theta) = A_{1j}(0) - \gamma D_{j+1}\tilde{S}(\theta) - D_j\lambda g_1\tilde{S}(\theta) \qquad j \geq 0 \qquad (6.27)$$

$$(\theta-\lambda)\tilde{A}_{i0}(\theta) = A_{i0}(0) - D_0\lambda g_i\tilde{S}(\theta) \qquad 2 \leq i \leq b \qquad (6.28)$$

$$(\theta-\lambda)\tilde{A}_{ij}(\theta) = A_{ij}(0) - D_j\lambda g_i\tilde{S}(\theta) - \sum_{k=1}^{j}\tilde{A}_{i\ j-k}(\theta)\,\lambda g_k \qquad (6.29)$$

$$2 \leq i \leq b-1,\ j \geq 1$$

$$(\theta-\lambda)\tilde{A}_{bj}(\theta) = A_{bj}(0) - \sum_{k=1}^{j}\tilde{A}_{b\ j-k}(\theta)\,\lambda g_k - \sum_{k=0}^{j} D_{j-k}\,\lambda g_{b+k}\tilde{S}(\theta) \qquad (6.30)$$

$$j \geq 1$$

$$(\theta-\lambda)\tilde{Q}_{10}(\theta) = Q_{10}(0) - (1-\delta)A_{i0}(0)\tilde{V}(\theta) - B_0(0)\tilde{V}(\theta) \qquad 1 \leq i \leq b \qquad (6.31)$$

$$(\theta-\lambda)\tilde{Q}_{1j}(\theta) = Q_{1j}(0) - \sum_{k=1}^{j}\tilde{Q}_{1\ j-k}(\theta)\,\lambda g_k \qquad j \geq 1 \qquad (6.32)$$

$$(\theta-\lambda)\tilde{Q}_{l0}(\theta) = Q_{l0}(0) - Q_{l-1\ 0}(0)\tilde{V}(\theta) \qquad l \geq 2 \qquad (6.33)$$

$$(\theta-\lambda)\tilde{Q}_{lj}(\theta) = Q_{lj}(0) - \sum_{k=1}^{j}\tilde{Q}_{l\ j-k}(\theta)\,\lambda g_k + Q_{l-1\ j}(0)\tilde{V}(\theta) \qquad (6.34)$$

$$j = 1,2,\ldots N-1$$

$$(\theta-\lambda)\tilde{Q}_{lj}(\theta) = \mathrm{Q_{lj}}(0) - \sum_{k=1}^{j} \tilde{Q}_{l\;j-k}(\theta)\,\lambda g_k \qquad j \geq N \quad l \geq 2 \quad (6.35)$$

$$(\theta-\lambda)\tilde{B}_0(\theta) = \mathrm{B_0}(0) - \delta \sum_{m=1}^{b} A_{m0}(0)\tilde{B}(\theta) \quad (6.36)$$

$$(\theta-\lambda)\tilde{B}_n(\theta) = \mathrm{B_n}(0) - \delta \sum_{m=1}^{b} A_{mn}(0)\tilde{B}(\theta) - \sum_{k=1}^{j} \tilde{B}_{n-k}(\theta)\,\lambda g_k \quad j \geq 1 \quad (6.37)$$

6.5 Probability Generating Function (PGF)

To obtain the PGF of an orbit size distribution at an arbitrary time epoch, the following generating functions are defined.

$$\tilde{A}_i(z,\theta) = \sum_{j=0}^{\infty} \tilde{A}_{ij}(\theta) z^j \qquad A_i(z,0) = \sum_{j=0}^{\infty} A_{ij}(0) z^j \qquad 2 \leq i \leq b$$

$$\tilde{Q}_j(z,\theta) = \sum_{l=1}^{\infty} \tilde{Q}_{1j}(0) z^j \qquad Q_j(z,0) = \sum_{l=1}^{\infty} Q_{lj}(0) z^j \qquad j \geq 1 \quad (6.38)$$

$$\tilde{\mathrm{B}}(z,\theta) = \sum_{\mathrm{n}=0}^{\infty} \tilde{B}_n(\theta)\,\mathrm{z^n} \qquad \mathrm{B(z,0)} = \sum_{\mathrm{n}=0}^{\infty} \mathrm{B_n}(0)\mathrm{z^n} \qquad D(z) = \sum_{j=0}^{\infty} D_j z^j$$

$$(\lambda+\gamma)D(z) = (1-\delta)\sum_{m=1}^{b} A_m(z,0) - (1-\delta)A_{m0}(0) + B(z,0) - B_0(0) + \sum_{j=N}^{\infty}\sum_{l=1}^{\infty} Q_{lj}(z,0) - \gamma D_0 \quad (6.39)$$

$$(\theta-\lambda)\tilde{A}_1(z,\theta) = A_1(z,0) - \gamma\tilde{S}(\theta)\frac{1}{z}(D(z)-D_0) - \lambda g_1 D(z)\tilde{S}(\theta) \quad (6.40)$$

$$(\theta-\lambda+\lambda x(z))\tilde{A}_i(z,\theta) = A_i(z,0) - \lambda g_i X(z)\tilde{S}(\theta) \quad 2 \leq i \leq b-1 \quad (6.41)$$

$$(\theta-\lambda+\lambda x(z))\tilde{A}_b(z,\theta) = A_b(z,0) - \lambda g_b D_0\tilde{S}(\theta) - \sum_{k=0}^{\infty} \lambda g_{b+k} z^k D(z)\tilde{S}(\theta) \quad (6.42)$$

$$(\theta-\lambda+\lambda x(z))\tilde{Q}_1(z,\theta) = Q_1(z,0) - \tilde{V}(\theta)\big((1-\delta)A_{i0}(0) + B_0(0)\big) \quad (6.43)$$

$$(\theta-\lambda+\lambda x(z))\tilde{Q}_l(z,\theta) = Q_l(z,0) - \tilde{V}(\theta)\sum_{j=0}^{N-1} Q_{l-1\;j}(0)\, z^j \quad (6.44)$$

$$(\theta-\lambda+\lambda x(z))\tilde{B}(z,\theta) = B(z,0) - \delta\sum_{m=1}^{b} A_m(z,0)\tilde{B}(\theta) \quad (6.45)$$

Substituting $\theta = \lambda$ in Eqn. 6.40 implies

$$A_1(z,0) = \tilde{S}(\lambda)D(z)\left(\frac{\gamma}{z} + \lambda g_1\right) - \frac{\gamma}{z} D_0 \tilde{S}(\lambda) \tag{6.46}$$

Substituting $\theta = \lambda - \lambda x(z)$ in Eqn. 6.41 to Eqn. 6.45 implies

$$A_i(z,0) = \lambda g_i D(z) \tilde{S}(\lambda - \lambda x(z)) \qquad 2 \le i \le b-1 \tag{6.47}$$

$$A_b(z,0) = \lambda g_b D_0 \tilde{S}(\lambda - \lambda x(z)) + \sum_{k=0}^{\infty} \lambda g_{b+k}\, z^k\, D(z) \tilde{S}(\lambda - \lambda x(z)) \tag{6.48}$$

$$Q_1(z,0) = \tilde{V}\big(\lambda - \lambda x(z)\big)\big((1-\delta)A_{i0}(0) + B_0(0)\big) \tag{6.49}$$

$$Q_l(z,0) = \tilde{V}\big(\lambda - \lambda x(z)\big) \sum_{j=0}^{N-1} Q_{l-1\ j}(0)\, z^j \tag{6.50}$$

$$B(z,0) = \delta \sum_{m=1}^{b} A_m(z,0) \tilde{B}\big(\lambda - \lambda x(z)\big) \tag{6.51}$$

By using Eqn. 6.40 and 6.46, we get

$$\tilde{A}_1(z,\theta) = \frac{(\tilde{S}(\lambda) - \tilde{S}(\theta))\left(D(z)\left(\frac{\gamma}{z} + \lambda g_1\right) - \frac{\gamma}{z} \square_0\right)}{\theta - \lambda} \tag{6.52}$$

By using Eqn. 6.41 and Eqn. 6.47, we get

$$\tilde{A}_i(z,\theta) = \frac{\big(\tilde{S}(\lambda - \lambda x(z)) - \tilde{S}(\theta)\big)\, \lambda g_i D(z)}{\theta - \lambda + \lambda x(z)} \qquad 2 \le i \le b-1 \tag{6.53}$$

By using Eqn. 6.40 and Eqn. 6.46, we get

$$\tilde{A}_b(z,\theta) = \frac{\big(\tilde{S}(\lambda - \lambda x(z)) - \tilde{S}(\theta)\big)\big(\lambda g_b D_0 + \sum_{k=0}^{\infty} \lambda g_{b+k} z^k\, D(z)\big)}{\theta - \lambda + \lambda x(z)} \tag{6.54}$$

By using Eqns. 6.43, 6.44, 6.49 and 6.50, we get

$$\sum_{l=1}^{\infty} \tilde{Q}_l(z,\theta) = \frac{\big(\tilde{V}(\lambda - \lambda x(z)) - \tilde{V}(\theta)\big)\Big((1-\delta)A_{i0}(0) + B_0(0) + \sum_{j=0}^{N-1} Q_{l-1\ j}(0) z^j\Big)}{\theta - \lambda + \lambda x(z)} \tag{6.55}$$

By using Eqn. 6.45 and 6.51, we get

$$\tilde{B}(z,\theta) = \frac{\big(\tilde{B}(\lambda - \lambda x(z)) - \tilde{B}(\theta)\big)\delta \sum_{m=1}^{b} A_m(z,0)}{\theta - \lambda + \lambda x(z)} \tag{6.56}$$

Substituting equations through Eqn. 6.52 to 6.56, 6.50 and 6.51 in Eqn. 6.37, we get

$$D(z)=\frac{D_0 z^b\begin{Bmatrix}\gamma z+\left((1-\delta)+\delta\tilde{B}\left(\lambda-\lambda x(z)\right)\right)\\ \times\left(\lambda z g_b\tilde{S}\left(\lambda-\lambda x(z)\right)-\gamma\tilde{S}(\lambda)\right)\end{Bmatrix}+z^{b+1}\left(\tilde{V}\left(\lambda-\lambda x(z)\right)\sum_{j=0}^{N-1}Q_{l-1\,j}(0)\,z^j\right)}{z^{b+1}(\lambda+\gamma)-\left((1-\delta)+\delta\tilde{B}\left(\lambda-\lambda x(z)\right)\right)\times\begin{pmatrix}\tilde{S}(\lambda)(\gamma+\lambda z g_1)z^b\\ +\lambda\tilde{S}\left(\lambda-\lambda x(z)\right)\left(z^{b+1}\sum_{i=2}^{b-1}g_i-z\left(X(z)-\sum_{j=1}^{b-1}g_j z^j\right)\right)\end{pmatrix}} \quad (6.57)$$

Probability generating function of the orbit size at an arbitrary time is given by,

$$P(z)=D(z)+\tilde{A}_1(z,0)+\sum_{i=2}^{b-1}\tilde{A}_i(z,0)+\tilde{A}_b(z,0)+\sum_{l=1}^{\infty}\tilde{Q}_l(z,0)+\tilde{B}(z,0) \quad (6.58)$$

Substituting $\theta=0$ from Eqn. (6.50) to Eqn. 6.54 and using Eqn. 6.55, the Eqn. 6.56 is simplified as,

$$P(z)=D_0\left[\frac{W_2(W_3+W_4+W_5)}{W_1(-\lambda+\lambda x(z))\lambda}+\frac{W_6+W_7}{(-\lambda+\lambda x(z))\lambda z}\right]+\frac{W_9}{(-\lambda+\lambda x(z))W_1} \quad (6.59)$$

where,

$$W_1=z^{b+1}(\lambda+\gamma)-\left((1-\delta)+\delta\tilde{B}\left(\lambda-\lambda x(z)\right)\right)\times\begin{pmatrix}\tilde{S}(\lambda)(\gamma+\lambda z g_1)z^b+\lambda\tilde{S}\left(\lambda-\lambda x(z)\right)\\ +\lambda\tilde{S}\left(\lambda-\lambda x(z)\right)M_1\end{pmatrix}$$

$$W_2=z^b\left(\gamma z+\left((1-\delta)+\delta\tilde{B}\left(\lambda-\lambda x(z)\right)\right)\times\left(\lambda z g_b\tilde{S}\left(\lambda-\lambda x(z)\right)-\gamma\tilde{S}(\lambda)\right)\right)$$

$$W_3=\left(-\lambda+\lambda x(z)\right)\left(1+\left(1-\tilde{S}(\lambda)\right)(\,z+\lambda g_1)\right)$$

$$M_1=z^{b+1}\sum_{i=2}^{b-1}g_i-z\left(X(z)-\sum_{i=1}^{b-1}g_i z^i\right)$$

$$W_4=\left(\tilde{S}\left(\lambda-\lambda x(z)\right)-1\right)\lambda\left(\sum_{i=2}^{b-1}g_i+\sum_{k=0}^{\infty}\lambda g_{b+k}\,z^k\,\right)$$

$$W_5 = \lambda\big(\tilde{B}\big(\lambda - \lambda x(z)\big) - 1\big)\,\delta \begin{pmatrix} \tilde{S}(\lambda)\left(\frac{\gamma}{z} + \lambda g_1\right) \\ +\tilde{S}\big(\lambda - \lambda x(z)\big)\left(\sum_{k=0}^{\infty} \lambda g_{b+k}\, z^k + \sum_{i=2}^{b-1} g_i\right) \end{pmatrix}$$

$$W_6 = \lambda\left(\big(\tilde{S}(\lambda) - 1\big)\,\gamma\big(-\lambda + \lambda x(z)\big) + \big(\tilde{S}\big(\lambda - \lambda x(z)\big) - 1\big)\,\lambda z g_b\right)$$

$$W_7 = \big(\tilde{B}\big(\lambda - \lambda x(z)\big) - 1\big)\delta g_b \tilde{S}\big(\lambda - \lambda x(z)\big)$$

$$W_8 = z^{b+1}\left(\tilde{V}\big(\lambda - \lambda x(z)\big)\right)\left(\sum_{j=0}^{N-1} q_j\, z^j\right)$$

$$W_9 = W_1\left[\big(\tilde{V}\big(\lambda - \lambda x(z)\big) - 1\big)\left(\sum_{j=0}^{N-1} q_j\, z^j\right)\right] + W_8\big(-\lambda + \lambda x(z)\big)$$

The Probability generating function has to satisfy the condition $P(1) = 1$. The steady state condition for the proposed model under consideration is simplified as,

$\rho = \lambda E(X)\big(E(S) + \delta E(B)\big) < 1$ and the unknown constant D_0 is obtained as

$$D_0 = \frac{\lambda E(X)(1 - E(V))\left((\lambda + \gamma) + \lambda M_1 - \tilde{S}(\lambda)(\gamma + \lambda g_1) + \lambda\right)}{\left(\gamma + \left(\lambda g_b - \gamma \tilde{S}(\lambda)\right)\right) + S1 + S2\left(\sum_{k=0}^{\infty} \lambda g_{b+k} + \sum_{i=2}^{b-1} g_i\right)}$$

where,

$$S1 = \lambda E(X)\left(1 + \left(1 - \tilde{S}(\lambda)\right)(1 + \lambda g_1)\right) + E(S)\lambda^3 E(X)\left(\sum_{i=2}^{b-1} g_i + \sum_{k=0}^{\infty} g_{b+k}\right)$$
$$+E(B)\lambda^2 E(X)\varphi\left(\tilde{S}(\lambda)(\gamma + \lambda g_1) + \lambda\left(\sum_{i=2}^{b-1} g_i + \sum_{k=0}^{\infty} g_{b+k}\right)\right)$$

$$S2 = \left(\gamma + E(B)\lambda E(X)\left(\lambda g_b - \gamma\tilde{S}(\lambda)\right) + (\lambda g_b + E(S)\lambda E(X))\right) + b(\gamma + \lambda g_b - \gamma \tilde{S}(\lambda))$$

T**heorem 6.1.** If β_n is the probability of 'n' customers arriving during a vacation then

$$q_0 = \frac{\beta_0 D_0}{(1 - \beta_0)}$$

$$q_n = \frac{\left(\beta_n D_0 + \sum_{j=0}^{N-1} q_j \beta_{n-j}\right)}{(1 - \beta_0)} \qquad n = 1, 2, \dots, N - 1$$

Proof. Using $\sum_{l=1}^{\infty} Q_{lj}(0) = q_j \;\; D_0 = (1 - \delta)A_{i0}(0) + B_0(0)$

And the Eqn. 6.55 simplifies to

$\sum_{l=1}^{\infty} Q_l(z,0) = \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} Q_{ln}(0) z^n$

$= \sum_{n=0}^{\infty} q_n z^n$

$= (1-\delta) A_{i0}(0) \tilde{V}\big(\lambda - \lambda x(z)\big) + B_0(0) \tilde{V}\big(\lambda - \lambda x(z)\big) + \tilde{V}\big(\lambda - \lambda x(z)\big) \sum_{j=0}^{N-1} q_j z^j$

$= \tilde{V}\big(\lambda - \lambda x(z)\big)\big(D_0 + \sum_{j=0}^{N-1} q_j z^j\big)$

$= \sum_{n=0}^{\infty} \beta_n z^n \left(D_0 + \sum_{j=0}^{N-1} q_j z^j\right)$

$= D_0 \sum_{n=0}^{\infty} \beta_n z^n + \sum_{n=0}^{N-1} \sum_{j=0}^{n} q_j \, \beta_{n-j} + \sum_{n=N}^{\infty} \sum_{j=0}^{N-1} q_j \beta_{n-j}$

Equating the coefficients of z^n, $n = 0,1,2, \ldots N-1$ on both sides of the above equation we have

$$q_0 = \frac{\beta_0 D_0}{(1-\beta_0)}$$

$$q_n = \frac{\left(\beta_n D_0 + \sum_{j=0}^{N-1} q_j \beta_{n-j}\right)}{(1-\beta_0)}$$

Hence the theorem.

6.6 Performance Measures

In this section, some important performance measures for the given queueing system are derived.

6.6.1 Expected Orbit Length (E (Q))

The mean orbit length can be obtained by differentiating $P(z)$ with respect to $'z'$ at 1

$$E(Q) = \lim_{z \to 1} (P'(z))$$

$$E(Q) = D_0 \left[\frac{\lambda(u_1{}'u_3{}'' - u_3{}'u_1{}'')}{2(u_1{}')^2 \lambda^2} + \frac{u_2{}'u_4{}'' - u_4{}'u_2{}''}{2(u_2{}')^2}\right] + \frac{u_1{}'W_9{}'' - W_9{}'u_1{}''}{2(u_1{}')^2}$$

where

$F_1 = \lambda E(S) E(X)$ $\qquad G_1 = \lambda E(S) X''(1)$ $\qquad H_1 = \lambda^2 E(S^2)(E(X))^2$

$F_2 = \lambda E(B)E(X)$ $\quad G_2 = \lambda E(B)X''(1)$ $\quad H_2 = \lambda^2 E(B^2)(E(X))^2$

$F_3 = \lambda E(V)E(X)$ $\quad G_3 = \lambda E(V)X''(1)$ $\quad H_1 = \lambda^2 E(V^2)(E(X))^2$

$u_1 = W_1(-\lambda + \lambda x(z))$ $\quad u_2 = (-\lambda + \lambda x(z))\lambda z$ $\quad u_1{}' = c_1\, \lambda E(X)$

$u_3 = W_2(W_3 + W_4 + W_5)$ $\quad u_4 = W_6 + W_7$

$c_1 = \tilde{S}(\lambda)(\gamma + \lambda g_1) + \lambda + \lambda M_1$ $\quad c_1{}' = \tilde{S}(\lambda)(\gamma + \lambda g_1)b + \tilde{S}(\lambda)(\gamma + \lambda g_1) + \lambda F_1 M_1 + \lambda M_1{}'$

$$c_1{}'' = \tilde{S}(\lambda)(\gamma + \lambda g_1)b(b-1) + 2\tilde{S}(\lambda)(\gamma + \lambda g_1)b + \lambda(G_1 + H_1) + \lambda(G_1 + H_1)M_1 + 2\lambda F_1 M_1{}' + \lambda M_1{}''$$

$$W_1{}' = (b+1)(\gamma + \lambda) - c_1{}' - \delta c_1 F_2$$

$$W_1{}'' = b(b+1)(\gamma + \lambda) - c_1{}'' - 2\,\delta c_1{}' F_2 - c_1 \delta(G_2 + H_2)$$

$W_2{}' = \gamma + E_2{}' + E_2 E_1{}' + b(\gamma + E_2)$ $\quad E_1{}' = \delta F_2$ $\quad E_1{}'' = \delta(G_2 + H_2)$

$$W_2{}'' = E_2{}'' + 2E_1{}'E_2{}' + E_2 E_1{}'' + b(\gamma + E_2{}' + E_2 E_1{}') + b(b-1)(\gamma + E_2)$$

$E_2{}' = \lambda g_b(F_1 + 1)$ $\quad E_2 = \lambda g_b - \tilde{S}(\lambda)\,\gamma$

$E_2{}'' = \lambda g_b(G_2 + H_2 + 2F_1)$ $\quad W_3{}' = \lambda E(X)(1 + \left(1 - \tilde{S}(\lambda)\right)(1 + \lambda g_1))$

$$W_3{}'' = \lambda X''(1) + \left(1 - \tilde{S}(\lambda)\right)\left(2\lambda^2 g_1 E(X) + (1 + \lambda g_1)(\lambda X''(1)\right)$$

$$M_2 = \left(\sum_{i=2}^{b-1} g_i + \sum_{k=0}^{\infty} \lambda g_{b+k}\right)$$

$W_4{}' = F_1 \lambda M_3$ $\quad W_4{}'' = \lambda(F_1 M_2 + (G_1 + H_1)M_2)$

$M_3 = \sum_{k=0}^{\infty} k\lambda g_{b+k}$ $\quad M_5 = \sum_{j=0}^{N-1} q_j + c_0$

$W_5{}' = F_2 \lambda\, \delta\left(\tilde{S}(\lambda)(\lambda g_1 + \gamma) + M_2\right)$ $\quad M_4 = \sum_{j=0}^{N-1} q_j - c_0$

$$W_5{}'' = 2F_2\lambda\delta\left(\tilde{S}(\lambda)(\lambda g_1 - \gamma) + M_3 + F_1\, M_2\right) + \lambda\delta(G_2 + H_2)\left(\tilde{S}(\lambda)(\lambda g_1 + \gamma) + M_2\right)$$

$$W_6{}' = \lambda\left(\left(\tilde{S}(\lambda)-1\right)\gamma\lambda E(X) + F_1\lambda g_b\right)$$

$$W_6{}'' = \lambda\left\{\left(\tilde{S}(\lambda)-1\right)\gamma\lambda X''(1) + 2F_1\lambda g_b + (G_1+H_1)\lambda g_b\right\}$$

$$W_7{}'' = F_2\lambda^2\delta E(X)\left(\lambda g_b - \gamma\tilde{S}(\lambda)\right) \qquad W_8{}' = \sum_{j=0}^{N-1} jq_j + (F_3+(b+1))\,M_4$$

$$W_8{}'' = \sum_{j=0}^{N-1} j(j-1)q_j + 2(F_3+(b+1))\sum_{j=0}^{N-1} jq_j$$
$$+((G_3+H_3)+2(b+1)F_3+b(b+1))M_4$$

$$W_9{}' = W_1F_3M_5 + W_8\lambda E(X)$$

$$W_9{}'' = 3W_1{}'F_3\sum_{j=0}^{N-1} jq_j + 2W_8{}'\,\lambda E(X) + (W_1(G_3+H_3)+W_1{}'F_3)M_5 + W_8\,\lambda X''(1)$$

6.6.2 Probability that the Server is Busy

$$\mathrm{P(B)} = \frac{(\tilde{S}(\lambda)-1)\left(\lambda g_1+\gamma(1-D_0)\right)}{-\lambda} + E(s)D(1)\left(\sum_{i=2}^{b-1} g_i + \lambda g_b D_0 + \sum_{k=0}^{\infty}\lambda g_{b+k}\right)$$

where

$$\mathrm{D(1)} = \frac{D_0\left(\gamma+\lambda g_b-\gamma\tilde{S}(\lambda)\right)+\left(\sum_{j=0}^{N-1} Q_{l-1\,j}(0)-\left((1-\delta)A_{i0}(0)+B_0(0)\right)\right)}{(\lambda+\gamma)-\tilde{S}(\lambda)(\gamma+\lambda g_1)\times\lambda\left(\sum_{i=2}^{b-1} g_i-\left(1-\sum_{j=1}^{b-1} g_j z^j\right)\right)}$$

6.6.3 Probability that the Server is in Renewal Process

$$\mathrm{P(R)} = \mathrm{E(B)}\left(\frac{\left(\tilde{S}(\lambda)-1\right)\left(\lambda g_1+\gamma(1-D_0)\right)}{-\lambda}\right)$$

6.6.4 Probability that the Server is Idle

$$\mathrm{P(I)} = \lim_{z\to 1} D(z)$$

$$\mathrm{P(I)} = \frac{D_0\left(\gamma+\lambda g_b-\gamma\tilde{S}(\lambda)\right)+\left(\sum_{j=0}^{N-1} Q_{l-1\,j}(0)-\left((1-\delta)A_{i0}(0)+B_0(0)\right)\right)}{(\lambda+\gamma)-\tilde{S}(\lambda)(\gamma+\lambda g_1)\times\lambda\left(\sum_{i=2}^{b-1} g_i-\left(1-\sum_{j=1}^{b-1} g_j z^j\right)\right)}$$

6.6.5 Mean Waiting Time in Retrial Queue

$$E(R) = \frac{E(Q)}{\lambda E(X)}$$

6.6.6 Mean Number of Customers in the System

$$L_s = E(Q) + \rho$$

6.6.7 Mean Waiting Time in the System

$$W_s = \frac{L_s}{\lambda E(X)}$$

6.6.8 Expected Length of Busy Period

By the theory of alternating renewal process, the expected length of busy period is derived as

$$E(B) = \frac{1}{\lambda E(X)}\left(\frac{1}{D_0} - 1\right)$$

6.6.9 Expected Length of Busy Cycle

The expression for expected length of busy cycle is obtained by the theory of alternating renewal process.

$$E(B_c) = \frac{1}{\lambda E(X)}\left(\frac{1}{D_0}\right)$$

6.7 Special Cases

The proposed model is developed with the assumption that the service time is arbitrary. However, to analyse real time systems, suitable distribution is required. This section presents some special cases of the system by indicating bulk service time as exponential distribution, hyper exponential distribution and Erlangian distribution.

Case. 1: Exponential bulk service time

The probability density function of exponential service time is $s(x) = e^{-\mu x}$, where μ is a parameter. Therefore,

$$\tilde{S}\big(\lambda - \lambda x(z)\big) = \left(\frac{\mu}{\mu + (\lambda - \lambda x(z))}\right)$$

The PGF of the orbit size for exponential service time is derived by substituting the expression for $\tilde{S}(\lambda-\lambda x(z))$ in Eqn. 6.59.

$$P(z)=D_0\left[\frac{W_2(W_3+W_4+W_5)}{W_1(-\lambda+\lambda x(z))\lambda}+\frac{W_6+W_7}{(-\lambda+\lambda x(z))\lambda z}\right]+\frac{W_9}{(-\lambda+\lambda x(z))W_1}$$

where

$$W_1=z^{b+1}(\lambda+\gamma)-\left((1-\delta)+\delta\tilde{B}(\lambda-\lambda x(z))\right)\times\left(\begin{array}{c}\left(\frac{\mu}{\mu+(\lambda-\lambda x(z))}\right)(\gamma+\lambda z g_1)z^b+\lambda\left(\frac{\mu}{\mu+\lambda}\right)\\+\lambda\left(\frac{\mu}{\mu+(\lambda-\lambda x(z))}\right)M_1\end{array}\right)$$

$$W_2=z^b\left(\gamma z+\left((1-\delta)+\delta\tilde{B}(\lambda-\lambda x(z))\right)\times\left(\lambda z g_b\left(\frac{\mu}{\mu+(\lambda-\lambda x(z))}\right)-\gamma\left(\frac{\mu}{\mu+\lambda}\right)\right)\right)$$

$$W_3=(-\lambda+\lambda x(z))\left(1+\left(1-\left(\frac{\mu}{\mu+\lambda}\right)\right)(z+\lambda g_1)\right)$$

$$M_1=z^{b+1}\sum_{i=2}^{b-1}g_i-z\left(X(z)-\sum_{i=1}^{b-1}g_i z^i\right)$$

$$W_4=\left(\frac{\mu}{\mu+(\lambda-\lambda x(z))}-1\right)\lambda\left(\sum_{i=2}^{b-1}g_i+\sum_{k=0}^{\infty}\lambda g_{b+k}z^k\right)\quad c_0=\left((1-\delta)A_{i0}(0)+B_0(0)\right)$$

$$W_5=\lambda\left(\tilde{B}(\lambda-\lambda x(z))-1\right)\delta\left(\left(\frac{\mu}{\mu+\lambda}\right)\left(\frac{\gamma}{z}+\lambda g_1\right)+\left(\frac{\mu}{\mu+(\lambda-\lambda x(z))}\right)\left(\begin{array}{c}\sum_{k=0}^{\infty}\lambda g_{b+k}z^k\\+\sum_{i=2}^{b-1}g_i\end{array}\right)\right)$$

$$W_6=\lambda\left(\left(\left(\frac{\mu}{\mu+\lambda}\right)-1\right)\gamma(-\lambda+\lambda x(z))+\left(\left(\frac{\mu}{\mu+(\lambda-\lambda x(z))}\right)-1\right)\lambda z g_b\right)$$

$$W_7=\left(\tilde{B}(\lambda-\lambda x(z))-1\right)\delta g_b\left(\frac{\mu}{\mu+(\lambda-\lambda x(z))}\right)$$

$$W_8=z^{b+1}\left(\tilde{V}(\lambda-\lambda x(z))\right)\left(\sum_{j=0}^{N-1}q_j z^j-c_0\right)$$

$$W_9=W_1\left[\left(\tilde{V}(\lambda-\lambda x(z))-1\right)\left(\sum_{j=0}^{N-1}q_j z^j+c_0\right)\right]+W_8(-\lambda+\lambda x(z))$$

Case. 2: Hyper exponential bulk service time

When the service time follows hyper exponential distribution with probability density function, then $s(x) = cde^{-dx} + (1-c)\, fe^{-fx}$, where d and f are parameters, then,

$$\tilde{S}(\lambda - \lambda x(z)) = \left(\frac{dc}{d + (\lambda - \lambda x(z))}\right) + \left(\frac{f(1-c)}{f + (\lambda - \lambda x(z))}\right)$$

The PGF of the orbit size for hyper exponential service time is derived by substituting the expression for $\tilde{S}(\lambda - \lambda x(z))$ in Eqn. 6.59.

Case. 3: K-Erlangian bulk service time

Let us consider that service time follows K - Erlang distribution with probability density function

$s(x) = \frac{(k\mu)^k x^{k-1} e^{-(k\mu x)}}{(k-1)!}$, k > 0; where μ is the parameter, then

$$\tilde{S}(\lambda - \lambda x(z)) = \left(\frac{k\mu}{k\mu + (\lambda - \lambda x(z))}\right)^k$$

The PGF of the orbit size K-Erlangian bulk service time can be derived by substituting the expression for $\tilde{S}(\lambda - \lambda x(z))$ in Eqn. 6.59.

Case. 4: When there is no server failure, multiple vacations and threshold

i. e ($\delta = 0$ and $\tilde{B}(\lambda - \lambda x(z)) = \tilde{V}(\lambda - \lambda x(z)) = 1$)

The PGF given in Eqn. 6.59 is reduced to

$$P(z) = D_0 \left[\frac{W_2(W_3 + W_4)}{W_1(-\lambda + \lambda x(z))\lambda} + \frac{W_6}{(-\lambda + \lambda x(z))\lambda z}\right]$$

where

$$W_1 = z^{b+1}(\lambda + \gamma) - \begin{pmatrix} \tilde{S}(\lambda)(\gamma + \lambda z g_1) z^b + \lambda \tilde{S}(\lambda - \lambda x(z)) \\ + \lambda \tilde{S}(\lambda - \lambda x(z)) M_1 \end{pmatrix}$$

$$W_2 = z^b \left(\gamma z + \times \left(\lambda z g_b \tilde{S}(\lambda - \lambda x(z)) - \gamma \tilde{S}(\lambda)\right)\right)$$

$$W_3 = (-\lambda + \lambda x(z))\left(1 + \left(1 - \tilde{S}(\lambda)\right)(z + \lambda g_1)\right)$$

$$M_1 = z^{b+1} \textstyle\sum_{i=2}^{b-1} g_i - z\left(X(z) - \sum_{i=1}^{b-1} g_i z^i\right)$$

$$W_4 = \left(\tilde{S}\left(\lambda - \lambda x(z)\right) - 1\right)\lambda\left(\sum_{i=2}^{b-1} g_i + \sum_{k=0}^{\infty} \lambda g_{b+k}\, z^k\right)$$

$$W_6 = \lambda\left(\left(\tilde{S}(\lambda) - 1\right)\gamma\left(-\lambda + \lambda x(z)\right) + \left(\tilde{S}\left(\lambda - \lambda x(z)\right) - 1\right)\lambda z g_b\right)$$

The above equation coincides with Haridass et al. (2012).

6.8 Cost Effective Model

Optimization techniques take part in minimizing total average cost of the queueing system in many practical situations. Constraints in cost analysis are start-up cost, operating cost, holding cost, set up cost and reward cost (if any). It is obvious that management of the system aims to minimize the total average cost. In this part, the cost analysis of the proposed queueing system is developed to obtain total average cost of the system with the following assumptions:

A_h: holding cost per customer

A_0: operating cost per unit time

A_s: startup cost per cycle

A_r: reward cost per cycle due to vacation

Total average cost = Holding cost of customers per unit time in the queue + Start-up cost per cycle +Operating cost +Reward cost

$$\text{Total average cost} = A_s \frac{1}{E(B_c)} + A_h E(Q) + A_0 \frac{E(B)}{E(B_c)}$$

Therefore the TAC is obtained as

$$\text{TAC} = A_s \lambda E(X) D_0 + A_h E(Q) + A_0 (1 - D_0)$$

6.9 Numerical Illustrations

In the performance evaluation of LAN executing under transmission protocol CSMA-CD (Carrier Sense Multiple Access with Collision Detection), data are entered into the system according to Poisson arrival rate λ. This section presents a numerical example of the proposed queueing system, which is used to the moderator of a CSMA/CD protocol to take decision in utilizing idle time effectively. All the numerical results are obtained with following assumptions.

:

Service time follows exponential distribution with parameter	μ
Batch size distribution of the arrival is geometric with mean	3
Retrial rate	γ
Vacation time follows exponential distribution with parameter	$\alpha = 5$
Renewal time follows exponential distribution with parameter	$\beta = 6$
Maximum service capacity	b
Threshold	$N = 10$

Start-up cost	Rs. 1.40
Holding cost per customer	Rs. 0.50
Operating cost per unit time	Rs. 2.00
Reward per unit time due to vacation	Rs. 1.00
Renewal cost per unit time	Rs. 0.40

6.9.1 Effects of Different Parameters on the Performance Measures

The effects of retrial rate and service rate with respect to mean orbit size are given in Table 6.2. It is observed that if the retrial rate increases, then mean orbit size decreases. Also when service rate increases, mean orbit size decreases.

6.9.2 Effects of Different Parameters on the Total Average Cost

Cost estimation is essential for the management of the system because there is a chance to change the maximum capacity value 'b' for service and service rate when arrival rate is large. The management can reduce total average cost by increasing either the service rate or batch size of the service.

Various comparisons with respect to total average cost are given in Table 6.3 and Table 6.4. It is clear that when retrial rate increases, the total average cost decreases. Also when service rate increases, the total average cost decreases.

Table 6.2 Retrial Rate versus Mean Orbit Size (Arrival Rate $\lambda = 2$)

Retrial rate	Expected Orbit Length(E(Q))							
	Service rate							
	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
1	3.4592	3.0624	2.5317	1.9316	1.5296	1.1762	0.7915	0.2178
2	3.1972	2.8219	2..3125	1.7161	1.3261	0.9361	0.5715	0.0961
3	2.9635	2.6314	2.1362	1.5422	1.1965	0.5369	0.3191	0.0511
4	2.6192	2.3724	1.9352	1.3256	0.9561	0.2850	0.1061	0.0319
5	2.3207	2.0251	1.6541	1.1509	0.7169	0.0965	0.0951	0.0193
6	2.1579	1.8512	1.2981	0.9618	0.4196	0.0711	0.0621	0.00743
7	1.8271	1.6379	1.0263	0.6315	0.2193	0.0562	0.0417	0.00419
8	1.5324	1.5162	0.9572	0.3259	0.0961	0.0379	0.01884	0.00165
9	1.2193	1.3292	0.5192	0.1538	0.0541	0.0099	0.00092	0.000863

Table 6.3 Retrial Rate versus Mean orbit Size and Total Average Cost

(Arrival Rate $\lambda = 3$, Service Rate $\mu = 2$)

Retrial rate	Threshold Value 'b'					
	b=3		b=4		b=5	
	E(Q)	TAC	E(Q)	TAC	E(Q)	TAC
2	2.7913	4.5623	2.4031	4.3028	2.3765	4.2193
3	2.2357	4.3261	2.2568	4.2968	2.2061	4.1562
4	1.8291	4.2392	2.1982	4.1192	2.0591	3.9843
5	1.7032	4.0569	1.9768	3.8369	1.8561	3.7521
6	1.5639	3.7894	1.6893	3.6528	1.7961	3.5361
7	1.4091	3.4063	1.4331	3.2391	1.5369	3.3549
8	1.1938	3.2569	1.2941	3.1965	1.3291	3.0391

Table 6.4 Retrial Rate versus Total Average Cost

(Arrival Rate $\lambda = 3$ and b=5)

Retrial rate	Total Average Cost		
	Arrival rate=2.0	Arrival rate=2.5	Arrival rate=3.0
2	5.5639	5.1861	4.8932
4	5.2962	4.8182	4.5362
6	4.8296	4.6365	4.3293
8	4.5192	4.3192	4.1368
10	4.3265	3.9370	4.8964
12	3.9493	3.7568	3.5256
14	3.6591	3.4293	3.2128

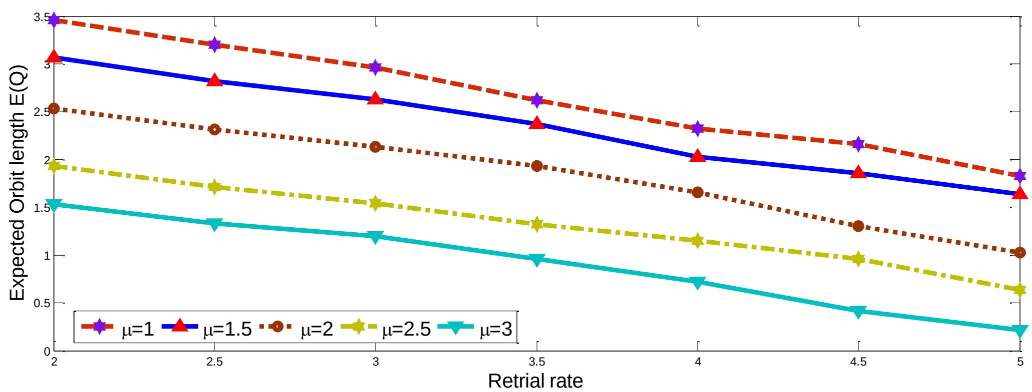

Fig. 6.2 Retrial Rate versus E(Q) for Different Service Rates

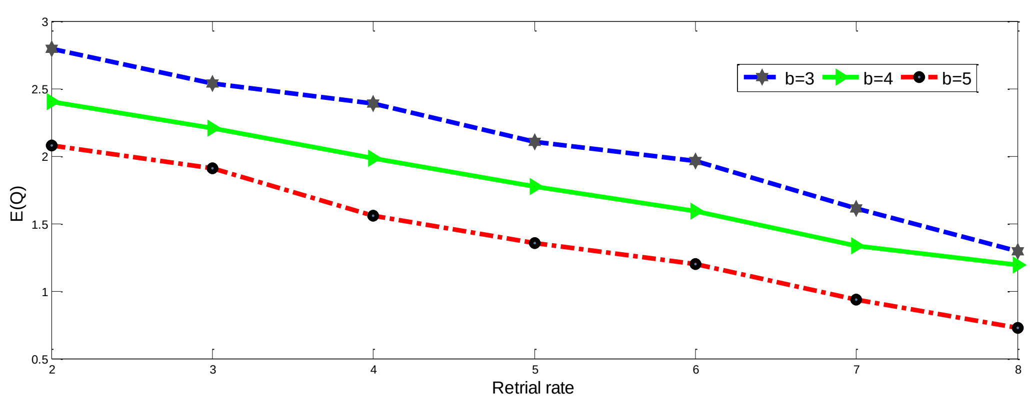

Fig. 6.3 Retrial Rate versus E(Q) for Different Values of *'b'*

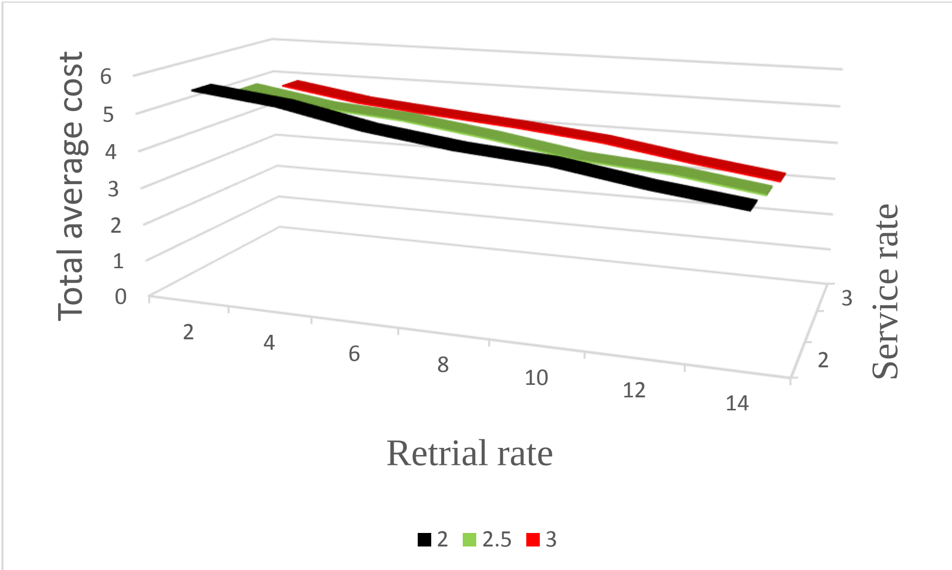

Fig. 6.4 Retrial Rate versus Total Average Cost for Different Service Rates

6.10 Conclusion

This chapter analysed bulk arrival and batch service retrial queueing system with server failure, threshold and multiple vacations are analysed. Probability generating function of the orbit size at an arbitrary time epoch was obtained by using supplementary variable technique. Various performance measures, particular case and special cases are also discussed. Additionally cost estimation analysis was also carried out with numerical example. All the obtained results will be useful in making decision to estimate overall cost and search for the best operating policy in a queueing system.