CHAPTER 5

Utilization of Idle time in $M^X/G(a,b)/1$ Queueing System with Secondary Service and Service Interruption

5.1 Introduction

Allowing servers to take vacations makes queueing model more realistic and flexible in the real world queueing systems. In the server vacation model, the server wishes to perform some useful internal processes during his idle time.

Neuts (1967) analysed general class of bulk queues with Poisson input, and provides general bulk service rule for service process. Stochastic modelling of queueing systems with vacations have been analysed by many of the researchers which includes Lee et al. (1994) analysed M^X/G/1 queueing system with 'N' policy and multiple vacations, using supplementary variable technique. Lee et al. (1994) derived probabilistic measures of the single vacation model with threshold. They developed procedure to optimize the long run average cost of the system. Only few authors have analysed about queue with server failure which includes Jain and Agrawal (2009) have analysed the optimal policy for bulk queue with multiple types of server breakdown.

Arumuganathan and Judeth Malliga (2006) derived various performance measures for bulk queueing system with repair of service station and setup times. Wang et al. (2009) obtained the various performance measures for the 'T' policy M/G/1 queue with server breakdowns and general start up times. Haridass and Arumuganathan (2012) introduced vacation interruption for bulk arrival and batch service queueing system and derived performance characteristics of the system. Wu (2014) has given the taxonomy of batch queueing models in manufacturing systems.

The proposed queueing model is unique because secondary service with service interruption is introduced for bulk arrival and batch service queueing system.

5.2 Model Description

In this chapter, single server bulk arrival queuing system with secondary service, server failure and vacation are considered. Server starts its work only if the queue length reaches the value ‘a’ (minimum capacity). Bulk service will be provided according to general bulk service rule (Neuts, 1967). After completing a batch of service, if the queue length is less than ‘a’ then the server remains idle. To utilize this idle time and provide service for ‘$a - 1$’ customers, the service mode will be changed from batch service to single service. That single service mode is called as secondary service. During secondary service, if the queue length reaches the threshold value ‘a’, then the server breaks the secondary service and switch over to batch service. During secondary service, if the queue length does not reach the value ‘a’ then single service will be provided continuously for ‘$a - 1$’ customers. The server may get failure during batch service. Though the server got failure, the service process will not be stopped and it will be continued for current batch by doing some technical arrangements, for example, Soft flow dyeing machine finding the server failure or proper maintenance of the server is called renewal time of the server. After completing a batch of service, if the server got failure with probability ‘β’ then the renewal of server is considered. Either after completion of renewal time of the server or there is no server failure with probability ’$(1 - \beta)$’, the server provides single service whenever the queue length is less than ‘a’. In the secondary service completion epoch, if the queue length is less than one then the server leaves for vacation. After vacation completion, if the queue length is still less than ‘a’ then the server remains idle (dormant) until the queue length reaches the value ‘a’. For the proposed model, probability generating function of queue size at an arbitrary time epoch is obtained. Various performance measures are also provided with suitable numerical illustrations.

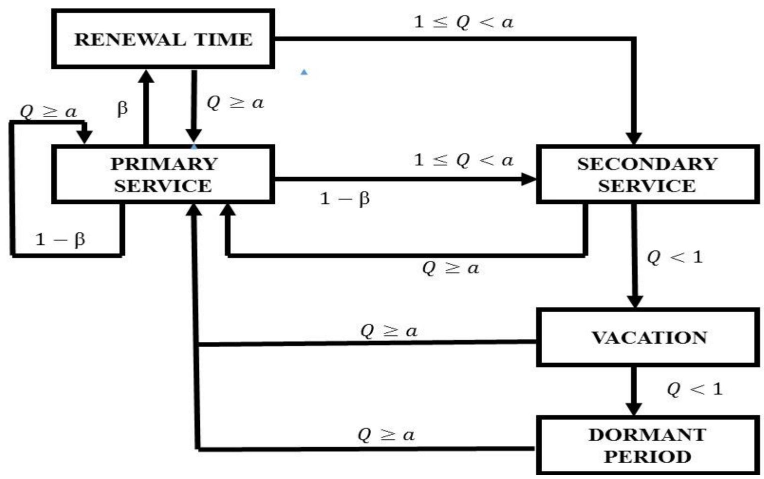

Fig. 5.1 Schematic Representation of the Queueing Model: Q-Queue Length

5.2.1 Motivation

The motivation of the proposed queueing system arises from a practical situation which occurs in machining casting in high speed lathe turning machine. In industry, castings are received in bulk for turning process and technician operates the turning unit with two high speed lathe turning machine one with single casting, which can pick up the castings one by one for turning and other with high capacity for bulk casting. The technician operates on any one of the machine depending upon the number of castings available. During bulk turning process a technician operates lathe machine with minimum of 200 castings to a maximum of 250 castings in a batch. However, the turning process will be initiated only if there is a minimum required number of castings available. During bulk casting if there are few castings in this batch which are found with run out (server failure), he is allowed to process the same with some additional service to remove the run out and complete the machining. After completion of service, if he finds that batch quantity is insufficient, then he changes the machine which can be able to process single casting (secondary service) at a time. The machine will undergo preventive maintenance works if no castings are available. After completing this vacation, if the required number of quantity in a batch is available then the service starts. This can be modelled as bulk arrival and batch service queueing system with secondary service server failure and vacation.

For the given queueing model probability generating function of queue size at an arbitrary time epoch is obtained. Various performance measures are derived with suitable numerical illustration. A real time application is also given.

5.3 Mathematical Model

Let λ be the Poisson arrival rate, X be the group size random variable of the arrival, g_k be the probability that 'k' customers arrive in a batch, X (z) be the probability generating function of X, N_q(t) be the number of customers waiting for service at time t, N_s(t) be the number of customers under the service at time t.

Table 5.1 Notations

	Cumulative distribution function	Probability density function	Laplace–Stieltjes transform	Remaining time
Primary Service	$S(x)$	$s(x)$	$\tilde{S}(\theta)$	S^0(x)
Secondary Service	$W(x)$	$w(x)$	$\widetilde{W}(\theta)$	W^0(x)
Vacation	$V(x)$	$v(x)$	$\tilde{V}(\theta)$	V^0(x)
Renewal Time	$R(x)$	$r(x)$	$\widetilde{R(\theta)}$	R^0(x)

$$C(t)=\begin{cases} 0, \text{when the server is busy with primary service} \\ 1, \text{ when the server is on secondary service} \\ 2, \text{ when the server is on vacation} \\ 3, \text{ when the server is on renewal} \\ 4, \text{ when the server is on dormant period} \end{cases}$$

The state probabilities are defined as follows

$$G_{ij}(x,t)dt = Pr\{N_s(t) = i, N_q(t) = j, x \leq S^0(t) \leq x + dt, C(t) = 0\}\ a \leq i \leq b; j \geq 1$$

$H_{1j}(x,t)dt = Pr\{N_s(t) = i, N_q(t) = j, x \le W^0(t) \le x + dt, C(t) = 1\}\ 1 \le j \le a-1$

$Q_n(x,t)dt = Pr\{N_q(t) = n, x \le V^0(t) \le x + dt, C(t) = 2\};\ n \ge 0$

$R_n(x,t)dt = Pr\{N_q(t) = n, x \le R^0(t) \le x + dt, C(t) = 3\};\ n \ge 0$

$T_n(t)dt \quad = Pr\{N_q(t) = n, C(t) = 4\};\ n \ge 0$

The following equations are obtained by using supplementary variable technique.

$$G_{i0}(x-\Delta t, t+\Delta t) = G_{i0}(x,t)(1-\lambda\Delta t) + (1-\delta)\sum_{m=a}^{b} G_{mi}(0,t)s(x)\Delta t \tag{5.1}$$
$$+\big(R_i(0,t) + Q_i(0,t)\big)s(x)\Delta t + \sum_{k=0}^{a-1} H_{1k}(x,t)\,\lambda g_{i-k}s(x)\Delta t$$
$$+\sum_{m=0}^{a-1} T_m(t)\,\lambda g_{i-m}s(x)\Delta t \qquad a \le i \le b$$

$$G_{ij}(x-\Delta t, t+\Delta t) = G_{ij}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} G_{ij-k}(x,t)\,\lambda g_k\Delta t \quad a \le i \le b-1 \tag{5.2}$$

$$G_{bj}(x-\Delta t, t+\Delta t) = G_{bj}(x,t)(1-\lambda\Delta t) + (1-\delta)\sum_{m=a}^{b} G_{mb+j}(0,t)s(x)\Delta t \tag{5.3}$$
$$+\sum_{k=1}^{j} G_{bj-k}(x,t)\,\lambda g_k\Delta t + \sum_{k=0}^{a-1} H_{1k}(x,t)\,\lambda g_{b+j-k}s(x)\Delta t$$
$$+\Big(R_{b+j}(0,t) + Q_{b+j}(0,t) + H_{1b+j}(0,t)\Big)\,s(x)\Delta t$$
$$+\sum_{m=0}^{a-1} T_m(t)\,\lambda g_{b+j-m}s(x)\Delta t \qquad j \ge 1$$

$$H_{1n}(x-\Delta t, t+\Delta t) = H_{1n}(x,t)(1-\lambda\Delta t) + (1-\delta)\sum_{m=a}^{b} G_{mn}(0,t)w(x)\Delta t \tag{5.4}$$
$$+\sum_{k=1}^{n} H_{1n-k}(x,t)\,\lambda g_k\Delta t + R_n(0,t)w(x)\Delta t \quad 1 \le n \le a-1$$

$$Q_0(x-\Delta t, t+\Delta t) = Q_0(x,t)(1-\lambda\Delta t) + H_{10}(0,t)v(x)\Delta t \tag{5.5}$$

$$Q_n(x-\Delta t, t+\Delta t) = Q_n(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{n} Q_{n-k}(x,t)\,\lambda g_k\Delta t \qquad n \ge 1 \tag{5.6}$$

$$R_0(x-\Delta t, t+\Delta t) = R_0(x,t)(1-\lambda\Delta t) + \delta\sum_{m=a}^{b} G_{m0}(0,t)r(x)\Delta t \tag{5.7}$$

$$R_n(x-\Delta t, t+\Delta t) = R_n(x,t)(1-\lambda\Delta t) + \delta\sum_{m=a}^{b} G_{mn}(0,t)r(x)\Delta t \tag{5.8}$$
$$+\sum_{k=1}^{n} R_{n-k}(x,t)\,\lambda g_k\Delta t \qquad n \ge 1$$

$$T_0(t+\Delta t) = T_0(t)(1-\lambda\Delta t) + Q_0(0,t)\Delta t \tag{5.9}$$

$$T_n(t+\Delta t) = T_n(t)(1-\lambda\Delta t) + Q_n(0,t)\Delta t + \sum_{k=1}^{n} T_{n-k}(t)\,\lambda g_k \Delta t \qquad (5.10)$$

$$1 \le n \le a-1$$

5.4 Steady State Queue Size Distribution

Dividing the above equations by Δt and allowing $\Delta t \to 0$, the steady state queue size equations are obtained as follows:

$$-\frac{\mathrm{d}}{\mathrm{dx}} G_{i0}(x) = -\lambda G_{i0}(x) + (1-\delta)\sum_{m=a}^{b} G_{mi}(0)s(x) + \sum_{m=0}^{a-1} T_m\,\lambda g_{i-m} s(x) \qquad (5.11)$$

$$+\big(R_i(0) + Q_i(0)\big)s(x) + \sum_{k=0}^{a-1} H_{1k}(x)\,\lambda g_{i-k} s(x) \qquad a \le i \le b$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} G_{ij}(x) = -\lambda G_{ij}(x) + \sum_{k=1}^{j} G_{ij-k}(x)\,\lambda g_k \qquad a \le i \le b-1 \qquad (5.12)$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} G_{bj}(x) = -\lambda G_{bj}(x) + (1-\delta)\sum_{m=a}^{b} G_{mb+j}(0)s(x) + \sum_{m=0}^{a-1} T_m\,\lambda g_{b+j-m} s(x) \qquad (5.13)$$

$$+\sum_{k=1}^{j} G_{bj-k}(x)\,\lambda g_k + \sum_{k=0}^{a-1} H_{1k}(x)\,\lambda g_{b+j-k} s(x)$$

$$+\Big(R_{b+j}(0) + Q_{b+j}(0) + H_{1b+j}(0)\Big)\,s(x) \qquad j \ge 1$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} H_{1n}(x) = -\lambda H_{1n}(x) + (1-\delta)\sum_{m=a}^{b} G_{mn}(0)w(x) + R_n(0)w(x) \qquad (5.14)$$

$$+\sum_{k=1}^{n} H_{1n-k}(x)\,\lambda g_k \qquad 1 \le n \le a-1$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} Q_0(x) = -\lambda Q_0(x) + H_{10}(0)v(x) \qquad (5.15)$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} Q_n(x) = -\lambda Q_n(x) + \sum_{k=1}^{n} Q_{n-k}(x)\,\lambda g_k \quad n \ge 1 \qquad (5.16)$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} R_0(x) = -\lambda R_0(x) + \delta\sum_{m=a}^{b} G_{m0}(0)r(x) \qquad (5.17)$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} R_n(x) = -\lambda R_n(x) + \delta\sum_{m=a}^{b} G_{mn}(0)r(x) + \sum_{k=1}^{n} R_{n-k}(x)\,\lambda g_k \qquad (5.18)$$

$$0 = -\lambda T_0 + Q_0(0) \qquad (5.19)$$

$$0 = -\lambda T_n + Q_n(0) + \sum_{k=1}^{n} T_{n-k}\,\lambda g_k \qquad 1 \le n \le a-1 \tag{5.20}$$

The Laplace – Stieltjes transform of $G_{kn}(x)$, $S_{1n}(x), V^1{}_n(x), V^2{}_n(x)$ and $W_n(x)$, are defined as,

$$\tilde{G}_{in}(\theta) = \int_0^\infty e^{-\theta x}\, G_{in}(x)dx \qquad \widetilde{H}_{1n}(\theta) = \int_0^\infty e^{-\theta x}\, H_{1n}(x)dx$$

$$\tilde{Q}_n(\theta) = \int_0^\infty e^{-\theta x}\, Q_n(x)dx \qquad \tilde{R}_n(\theta) = \int_0^\infty e^{-\theta x}\, R_n(x)dx$$

$$\theta\tilde{G}_{i0}(\theta) - G_{i0}(0) = \lambda\tilde{G}_{i0}(\theta) - ((1-\delta)\sum_{m=a}^{b} G_{mi}(0) + \sum_{m=0}^{a-1} T_m\,\lambda g_{i-m})\tilde{S}(\theta) \tag{5.21}$$
$$-\big(R_i(0) + Q_i(0)\big)\tilde{S}(\theta) - \sum_{k=0}^{a-1} \tilde{C}_k(\theta)\,\lambda g_{i-k} \qquad a \le i \le b$$

$$\theta\tilde{G}_{ij}(\theta) - G_{ij}(0) = \lambda\tilde{G}_{ij}(\theta) - \sum_{k=1}^{j} \tilde{G}_{ij-k}(\theta)\,\lambda g_k \quad a \le i \le b-1 \quad j \ge 1 \tag{5.22}$$

$$\theta\tilde{G}_{bj}(\theta) - G_{bj}(0) = \lambda\tilde{G}_{bj}(\theta) - \begin{pmatrix} (1-\delta)\sum_{m=a}^{b} G_{mb+j}(0) + H_{1b+j}(0) \\ R_{b+j}(0) + Q_{b+j}(0) + \sum_{m=0}^{a-1} T_m\,\lambda g_{b+j-m} \end{pmatrix}\tilde{S}(\theta) \tag{5.23}$$
$$-\sum_{k=1}^{j} \tilde{G}_{bj-k}(\theta)\,\lambda g_k - \sum_{k=0}^{a-1}\sum_{k=0}^{a-1} \tilde{C}_k(\theta)\,\lambda g_{b+j-k} \qquad j \ge 1$$

$$\theta\widetilde{H}_{1n}(\theta) - H_{1n}(0) = \lambda\widetilde{H}_{1n}(\theta) - [(1-\delta)\sum_{m=a}^{b} G_{mn}(0) + R_n(0)]\widetilde{W}(\theta) \tag{5.24}$$
$$-\sum_{k=1}^{n} \widetilde{H}_{1n-k}(\theta)\,\lambda g_k \qquad 1 \le n \le a-1$$

$$\theta\tilde{Q}_0(\theta) - Q_0(0) = \lambda\tilde{Q}_0(\theta) - H_{10}(0)\tilde{V}(\theta) \tag{5.25}$$

$$\theta\tilde{Q}_n(\theta) - Q_n(0) = \lambda\tilde{Q}_n(\theta) - \sum_{k=1}^{n} \tilde{Q}_{n-k}(\theta)\,\lambda g_k \qquad n \ge 1 \tag{5.26}$$

$$\theta\tilde{R}_0(\theta) - R_0(0) = \lambda\tilde{R}_0(\theta) - \delta\sum_{m=a}^{b} G_{m0}(0)\tilde{R}(\theta) \tag{5.27}$$

$$\theta\tilde{R}_n(\theta) - R_n(0) = \lambda\tilde{R}_n(\theta) - \delta\sum_{m=a}^{b} G_{mn}(0)\tilde{R}(\theta) - \sum_{k=1}^{n} \tilde{R}_{n-k}(\theta)\,\lambda g_k \quad n \ge 1 \tag{5.28}$$

5.5 Probability Generating Function

To derive the steady state probability generating function of an orbit size, the following probability generating functions are defined.

$$\tilde{G}_i(z,\theta) = \sum_{j=0}^{\infty} \tilde{G}_{ij}(\theta) z^j \qquad G_i(z,0) = \sum_{j=0}^{\infty} G_{ij}(0) z^j$$

$$\tilde{H}(z,\theta) = \sum_{j=1}^{a-1} \tilde{H}_{1n}(\theta) z^j \qquad H(z,0) = \sum_{j=1}^{a-1} H_{1j}(0) z^j \tag{5.29}$$

$$\tilde{Q}(z,\theta) = \sum_{n=0}^{\infty} \tilde{Q}_n(\theta) z^n \qquad Q(z,0) = \sum_{n=0}^{\infty} Q_n(0) z^n$$

$$\tilde{R}(z,\theta) = \sum_{n=0}^{\infty} \tilde{R}_n(\theta) z^n \qquad R(z,0) = \sum_{n=0}^{\infty} R_n(0) z^n \qquad T(z) = \sum_{j=0}^{a-1} T_j z^j$$

By multiplying the equations from Eqn. 5.21 to Eqn. 5.28 with suitable powers of z^n and summing over n, then by using PGF given above, we get

$$\left(\theta - \lambda + \lambda X(z)\right)\tilde{G}_i(z,\theta) = G_i(z,0) - \begin{pmatrix} (1-\delta)\sum_{m=a}^{b} G_{mi}(0) + \sum_{m=0}^{a-1} T_m \lambda g_{i-m} \\ \left(R_i(0) + Q_i(0)\right) \end{pmatrix} \tilde{S}(\theta)$$
$$- \sum_{k=0}^{a-1} \tilde{C}_k(\theta) \lambda g_{i-k} \qquad a \le i \le b-1 \tag{5.30}$$

$$\left(\theta - \lambda + \lambda X(z)\right)\tilde{G}_b(z,\theta) = G_b(z,0) - \frac{\lambda}{z^b}\left(\sum_{i=0}^{a-1} \tilde{C}_i(\theta) z^i \left(X(z) - \sum_{j=1}^{b-i-1} g_j z^j\right)\right) \tag{5.31}$$
$$- \frac{\tilde{S}(\theta)}{z^b}\begin{pmatrix} - \sum_{n=0}^{b-1}(H_n + R_n + Q_n + G_n) z^n \\ +\lambda\left(T(z)X(z) - \sum_{m=0}^{c-1}\left(T_m z^m \sum_{j=1}^{b-m-1} g_j z^j\right)\right) \\ +H(z,0) + R(z,0) + Q(z,0) + (1-\delta)\sum_{m=a}^{b} G_m(z,0) \end{pmatrix}$$

Let $G_i = \sum_{m=a}^{b} G_{mi}(0)$, $q_i = Q_i(0)$, $R_i = R_i(0)$, $H_n = H_{1n}(0)$

$$\left(\theta - \lambda + \lambda X(z)\right)\tilde{Q}(z,\theta) = Q(z,0) - \tilde{V}(\theta) H_{1n}(0) \qquad n < 1 \tag{5.32}$$

$$\left(\theta - \lambda + \lambda X(z)\right)\tilde{R}(z,\theta) = R(z,0) - \delta \sum_{m=a}^{b} G_{mn}(0) \tilde{R}(\theta) \tag{5.33}$$

$$\left(\theta - \lambda + \lambda \xi(z)\right)\tilde{H}(z,\theta) = H(z,0) - \sum_{n=1}^{a-1}\left((1-\delta)G_n + R_n\right) z^n \tilde{W}(\theta) \tag{5.34}$$

where $\xi(z) = \sum_{k=0}^{a-1} g_k z^k$

Let $P(z)$ be the probability generating function of the queue size at an arbitrary time epoch. Then,

$$P(z) = \sum_{m=a}^{b-1} \tilde{G}_i\,(z,0) + \tilde{G}_b(z,0) + \tilde{Q}(z,0) + \tilde{R}\,(z,0) + T\,(z) + \tilde{H}\,(z,0) \qquad (5.35)$$

Substituting $\theta = \lambda - \lambda\xi(z)$ in Eqn. 5.34 and $\theta = \lambda - \lambda\, X\,(z)$ in the equations from Eqn. 5.30 to Eqn. 5.33, after doing some algebra, the PGF of the queue size is defined in Eqn. 5.35 is simplified as,

$$P(z) = \frac{(-\lambda+\lambda\xi(z)\left(\begin{matrix}(\tilde{S}\,(\lambda-\lambda X(z)-1)\left(\begin{matrix}\sum_{i=a}^{b-1} d_i(z^b-z^i) \\ -\sum_{j=0}^{a-1} d_j z^j+\lambda M_4+(E_1+E_3)M_1\end{matrix}\right)+M_2-M_3 \\ +\sum_{n=0}^{a-1} f_n z^n\left[(\tilde{S}\,(\lambda-\lambda X(z)-1)\tilde{V}\,(\lambda-\lambda X(z)\tilde{W}\,(\lambda-\lambda X(z))+E_4\right]+M_5\end{matrix}\right) \\ (-\lambda+\lambda X(z))M_1(\tilde{W}\,(\lambda-\lambda X(z))-1)\sum_{n=0}^{a-1} f_n z^n+T(z)}{M_1(-\lambda+\lambda\xi(z))(-\lambda+\lambda X(z))} \qquad (5.36)$$

where

$$M_1 = [z^b - (1-\delta)\tilde{S}\,(\lambda-\lambda X(z)) - \delta\tilde{R}\,(\lambda-\lambda X(z))\tilde{S}(\lambda-\lambda X(z))]$$

$$M_2 = \sum_{i=0}^{a-1} z^i \left(\left(\tilde{C}_{1i}(\lambda-\lambda X(z)\right) - \tilde{C}_{1i}(0)\right) \times \left(\left(X(z) - \sum_{j=1}^{b-i-1} g_j z^j\right)\right.$$

$$M_3 = \left((1-\delta)+\delta\tilde{R}(\lambda-\lambda X(z))\right)\lambda\sum_{i=0}^{a-1} z^{i-b}\left(\begin{matrix}\left(\tilde{S}\,(\lambda-\lambda X(z))\right)\tilde{C}_{1i}(0) \\ -\left(\tilde{C}_{1i}(\lambda-\lambda X(z)\right)\end{matrix}\right)$$

$$\times\left(X(z) - \sum_{j=1}^{b-i-1} g_j z^j\right)$$

$$M_4 = T\,(z)\,X\,(z) - \sum_{m=0}^{a-1} T_m z^m \sum_{j=1}^{b-m-1} g_j z^j$$

$$M_5 = \delta(\tilde{R}\,(\lambda-\lambda X(z)-1)\left(\begin{matrix}\tilde{S}\,(\lambda-\lambda X(z))\left(\lambda M_4 - \sum_{j=0}^{a-1} d_j z^j + M_1\left(\sum_{i=a}^{b-1} d_j + E_3\right)\right) \\ -E_2 M_1 + \left((1-\delta)+\delta\tilde{R}(\lambda-\lambda X(z))\right) + M_2 \\ \times\left(\tilde{S}\,(\lambda-\lambda X(z))\right)\left(\sum_{i=a}^{b-1} d_j - E_2 + E_3\right)\end{matrix}\right)$$

$$E_1 = \sum_{i=a}^{b-1}\sum_{k=0}^{a-1}\left(\left(\tilde{C}_{1k}(\lambda-\lambda X(z)\right) - \tilde{C}_{1k}(0)\right)\lambda g_{i-k}$$

$$E_2 = \sum_{i=a}^{b-1}\sum_{k=0}^{a-1}\tilde{C}_{1k}(\lambda-\lambda X(z)\lambda g_{i-k}$$

$$E_3 = \sum_{i=a}^{b-1}\sum_{m=0}^{a-1} T_m\,\lambda g_{i-m} \qquad f_i = (1-\delta)G_i + R_i \qquad d_i = f_i + q_i + H_i$$

$$E_4 = \widetilde{W}(\lambda - \lambda X(z))\begin{pmatrix} (\widetilde{V}(\lambda - \lambda X(z) - 1))M_1 \\ +\widetilde{S}(\lambda - \lambda X(z)\widetilde{V}(\lambda - \lambda X(z)\delta(\widetilde{R}(\lambda - \lambda X(z) - 1) \end{pmatrix}$$

$$E_5 = M_1(-\lambda + \lambda X(z))T(z)\ (-\lambda + \lambda \xi(z))$$

In order to obtain steady state condition evaluate $\lim_{z \to 1} P(Z)$ and equating the expression to 1, the steady state condition for the given queueing model is derived as,

$$\rho = \frac{\lambda E(X)[E(S) + \delta E(R)]}{b} < 1$$

5.6 Computational Aspects of Unknown Probabilities

Eqn. 5.36 gives the probability generating function of the number of customers in the queue, which involves the unknowns T_i and $\widetilde{C_\iota}$. Using the following theorems T_i and $\widetilde{C_\iota}(\theta)$ are expressed in terms of f_i and the known function $\widetilde{W}(\lambda)$ respectively. To find the unknown constants, Rouche's theorem of complex variables can be used. It follows that the expression $z^b - \widetilde{S}(\lambda - \lambda X(z))(1 - \delta) - \delta\widetilde{R}(\lambda - \lambda X(z))\ \widetilde{S}(\lambda - \lambda X(z))$ has $b - 1$ zeros inside and one on the unit circle $|z| = 1$. Since *P(z)* is analytic within and on the unit circle, the numerator of $P(z)$ must vanish at these points, which gives 'b' equations and 'b' unknowns. These equations can be solved by suitable numerical techniques. MATLAB is used for programming.

Theorem 5.1. The unknown constants H_n are expressed in terms of f_n as, $H_n = \sum_{i=0}^{n} f_{n-i}\, \beta_i$ $n = 1, 2 \ldots a - 1$, where β_i is the probability that 'i' customers arrive during the secondary service.

Proof. From Eqn. 5.34, we have

$$H(z, 0) = \sum_{n=1}^{a-1} f_n z^n\, \widetilde{W}(\lambda - \lambda\xi(z))$$

$$H(z, 0) = \widetilde{W}(\lambda - \lambda\xi(z))\ \sum_{n=0}^{a-1} ((1 - \delta)G_n + R_n) z^n$$

$$\sum_{n=1}^{a-1} H_n z^n = (\sum_{n=1}^{\infty} \beta_n z^n)(\sum_{n=1}^{a-1} f_n\, z^n)$$

$$\sum_{n=1}^{a-1} H_n z^n = \sum_{n=1}^{a-1}(\sum_{i=1}^{n} f_{n-i}\beta_i)\, z^n + = \sum_{n=a}^{\infty}(\sum_{i=0}^{a-1} \beta_{n-i} f_i)\, z^n \qquad (5.37)$$

Equating the coefficient of z^n ; n = 1, 2, 3 ….*a-1*, on both sides of the Eqn. 5.37, we get

$$H_n = \sum_{i=1}^{n} f_{n-i}\, \beta_i. \tag{5.38}$$

Hence the proof.

Theorem 5.2. The unknown constants Q_n are expressed in terms of f_n a follows $Q_i = \sum_{k=0}^{a-1}\left(\sum_{n=a}^{i} \alpha_{i-n}\left(\gamma_{n-k} + \sum_{j=0}^{a-1-k} U_j\, \gamma_{n-j-k}\right)\right) f_k,\ i = 0,1,2 \ldots\ a-1$, where α_i is the probability that 'i' customers arrive during a vacation

Proof. From Eqn. 5.32, we have

$$Q(z,0) = \tilde{V}\left(\lambda - \lambda X(z)\right) \sum_{n=0}^{a-1} H_n z^n$$

$$\sum_{n=0}^{\infty} Q_n z^n = \left(\sum_{n=0}^{\infty} \alpha_n z^n\right)\left(\sum_{n=0}^{a-1} H_n z^n\right) \tag{5.39}$$

Equating the coefficients of z^n on both sides of Eqn. 5.39, we get

$$Q_n = \sum_{n=0}^{i} H_n \alpha_{i-n} \quad n \geq 1$$

Substitute for H_n from Eqn.5.38, we have

$$Q_i = \sum_{k=0}^{a-1}\left(\sum_{n=a}^{i} \alpha_{i-n}\left(\gamma_{n-k} + \sum_{j=0}^{a-1-k} U_j\, \gamma_{n-j-k}\right)\right) f_k$$

Hence the proof.

Theorem 5.3. Let B_j be the collection of set of positive integers (not necessarily distinct A, such that, sum of elements in A is j, then, $T_n = \frac{1}{\lambda}\sum_{j=0}^{n} q_{n-j} \sum_{j=1}^{n(B_j)} \prod_{l\in A} g_l$

Proof. From the Eqn. 5.19 and Eqn. 5.20, we have

$$\lambda T_0 = Q_0(0) = q_0$$

$$\lambda T_n = Q_n(0) + \lambda \sum_{k=1}^{n} T_{n-k}\, g_k;$$

when n = 1,

$$\lambda T_1 = Q_1(0) + \lambda T_0 g_1$$

$$= q_1 + q_0 g_1$$

when n = 2,

$$\lambda T_2 = Q_2(0) + \lambda \sum_{k=1}^{2} T_{2-k}\, g_k;$$

$$= Q_2(0) + \lambda T_1 g_1 + \lambda T_0 g_2$$

$$= q_2 + q_1 g_1 + q_0\left({g_1}^2 + g_2\right)$$

when n = 3

$$\lambda T_3 = Q_3(0) + \lambda \sum_{k=1}^{3} T_{3-k}\, g_k\ ;$$

$$= Q_3(0) + \lambda T_2 g_1 + \lambda T_1 g_2 + \lambda T_0 g_3$$

$$= q_3 + q_2 g_1 + q_1({g_1}^2 + g_2) + q_0({g_1}^3 + 2g_1 g_2 + g_3)$$

$$T_3 = \frac{1}{\lambda}\left(\sum_{j=0}^{3} q_{3-j} \sum_{j=1}^{n(B_j)} \prod_{l \in A} g_l\right)$$

where

$B_1 = \{\{1\}\}$, $B_2 = \{\{1,1\},\{2\}\}$, and $B_3 = \{\{3\},\{1,1,1\}\{1,2\},\{2,1\}\}$

By induction, we get

$$\mathrm{T}\,(\mathrm{z}) = \sum_{n=0}^{a-1} T_n z^n$$

$$= \frac{1}{\lambda}\left(\sum_{n=0}^{a-1}\left(\sum_{j=0}^{n} q_{n-j} \sum_{j=1}^{n(B_i)} \prod_{l \in A} g_l\right) \mathrm{z}^{\mathrm{n}}\right)\backslash$$

Therefore,

$$T_n = \frac{1}{\lambda}\left(\sum_{j=0}^{n} q_{n-j} \sum_{j=1}^{n(B_i)} \prod_{l \in A} g_l\right)$$

5.7 Performance Measures

In this section, some important performance measures like expected queue length (E(Q)), expected length of busy period (E(B)) and expected length of idle period (E(I)) are derived for the proposed model

5.7.1 Expected Queue Length

$$\mathrm{E}\,(\mathrm{Q}) = \lim_{\mathrm{z} \to 1} P'(\mathrm{z})$$

$$E(Q) = \frac{2\left({K_8}''(L_1+L_2+L_3+L_4)\right) - 2\left({K_8}'''N_1\right) - 3\left(({K_8}'^{v}T_1)\right)}{24(\mathrm{T13})^2} + \frac{\sum_{n=0}^{a-1} c_n(K_6 + K_7)}{(K_5)^2}$$

where

$u_1 = \lambda E(S)E(X)$

$y_1 = \lambda^2 E(S^2)(E(X))^2$

$b_1 = \lambda E(S)X'''(1)$

$c_1 = \lambda E(S)X''(1)$

$u_2 = \lambda E(W)E(X)$

$y_2 = \lambda^2 E(W^2)(E(X))^2$

$b_2 = \lambda E(W)X'''(1)$

$c_2 = \lambda E(W)X''(1)$

$u_3 = \lambda E(V)E(X)$

$y_3 = \lambda^2 E(V^2)(E(X))^2$

$b_3 = \lambda E(V)X'''(1)$ $\qquad$ $c_3 = \lambda E(V)X''(1)$

$u_4 = \lambda E(R)E(X)$ $\qquad$ $y_4 = \lambda^2 E(R^2)(E(X))^2$

$b_4 = \lambda E(R)X'''(1)$ $\qquad$ $c_4 = \lambda E(R)X''(1)$

$$L_1 = \sum_{n=0}^{a-1} nC_n \begin{pmatrix} 2u_4\delta(2(u_1+u_2+u_3)+y_4\delta \\ +6u_3{M_1}' + 4u_1(u_2+u_3)c_1 + 2y_1 + 2c_1 \end{pmatrix}$$

$$L_2 = \sum_{n=0}^{a-1} C_n \begin{bmatrix} c_2 + y_2 + 2u_2u_3 + 2u_2u_1 + 2u_1u_3 + (u_2+u_3+u_1) + 2c_4 + 2y_4 \\ +u_1(c_2+2y_2) + 4c_1E(W) + u_1y_2 + y_1(3u_2+2u_3) + y_3 \\ +c_3 + 3\lambda^2E(S^2)E(X)X''(1) + h_1 + u_3(3{M_1}'' + 6u_2{M_1}') \\ c_1 + y_1 + (2y_3 + \delta(h_4+b_4))(c_4(E(X) + (u_2+u_3+u_1)))) \\ +6(u_2u_3u_1) + {M_1}'(y_3+c_3) + c_1E(V) + u_1(2c_3+3y_3) + c_1 \end{bmatrix}$$

$$L_3 = \sum_{j=0}^{b-1} d_j\, \delta \begin{pmatrix} u_4c_1 + 2\delta u_1u_4 + 4\lambda^2E(R)E(S)X''(1) + b_4 + c_4 \\ +3\lambda^2E(R^2)E(X)X''(1) + h_4 + u_1(c_4+3y_4) + y_4 \end{pmatrix} \times F_1(\lambda)$$

$$+\sum_{j=0}^{a-1} jd_j\, \delta(6u_1u_4 + 2(y_4+c_4+u_4) + 3y_1 + 3c_1) + 2u_4\sum_{j=0}^{b-1} j(j-1)d_j\, \delta$$

$$+\sum_{n=0}^{a-1} n(n-1)c_n\,(u_1 + -(3{E_2}''u_4 + {E_2}'(2c_4+3y_4) + E_2b_4 + 3y_4 + h_4)$$

$$L_4 = A_1\delta \begin{pmatrix} 3u_4((1-\delta)(y_1+c_1) + \delta(2u_1u_4 + y_1 \\ +(9u_4u_1 + 9c_4){M_1}' + c_1 + y_4 + c_4 + 5u_4{M_1}' \end{pmatrix} - \sum_{i=a}^{b-1} i(i-1)K_1$$

$$+2(C_4+y_4)\Big((2u_4+c_4)\big((1-\delta)u_1 + \delta(u_1+u_4)\big)\Big) + M_2(b_4+y_4+h_4)$$

$$+A_2\,\delta(2u_4{M_1}'' + 2{M_1}' + (y_4+c_4){M_1}') + 2(u_1(b(b-1)(c_1K_1 + y_1)(b-i))$$

$$+\delta(3u_4{M_2}'' + {M_2}'(2y_4+6c_4) - {M_2}''' + {M_3}''' + 3u_4{M_3}'' + 3{M_3}'(y_4+c_4)$$

$$+K_2 \begin{pmatrix} b_1 + \lambda^2E(S^2)E(X)X''(1) + 3bc_1 \\ +2b\lambda^2E(S^2)X''(1) + h_1 + 4by_1 + 3b(b-1) \end{pmatrix}$$

$$N_1 = \sum_{i=a}^{b-1} 2\,u_1(b-i)K_1 + K_2(C_1 + y_1 + 2bu_1) + \sum_{j=0}^{a-1} d_j\,(y_1+c_1) + 2\sum_{j=0}^{a-1} jd_j\,u_1 +$$

$$+\sum_{n=0}^{a-1} C_n\,(u_1(2u_2+2u_3) + C_1 + y_1 + \delta(3u_4{M_2}' + (y_4+2c_4)M_2 + y_4M_3 + y_4)$$

$$+2u_3{M_1}' + (u_4+y_4)\delta + c_4 + 2(u_2+u_3+u_1) - {M_2}'' - \delta(E_2(c_4+y_4)$$

$$+\sum_{n=0}^{a-1} nC_n(2\,u_1 + u_4) + \delta A_1u_4(2{M_1}' + \big((1-\delta)u_1 + \delta(u_1+u_4)\big))$$

$+\delta u_4 M_1{}' A_2 + \sum_{j=0}^{b-1} d_j \left((y_4 + c_4)\delta + 2\delta u_1 u_4\right) + 2\delta u_4 \sum_{j=0}^{b-1} j d_j + 2u_4 \delta E_2{}')$

$T_1 = u_1\left(2\sum_{i=a}^{b-1}(b-i)d_i + K_2 + \sum_{j=0}^{a-1} d_j + \sum_{n=0}^{a-1} c_n\right)$

$+u_4\delta\left(M_2 + A_1 - \sum_{j=0}^{b-1} d_j + \sum_{n=0}^{a-1} c_n - E_2 - M_3\right)$

$K_1 = \sum_{i=a}^{b-1} d_i \qquad K_2 = \sum_{i=a}^{b-1} \sum_{m=0}^{a-1} T_m \lambda g_{i-m} \quad A_1 = K_1 + K_2 \quad A_2 = E_2$

$K_3 = \lambda E(X)(b - u_1 - u_4\delta) \qquad F_1(\lambda) = \sum_{j=0}^{a-1} j(j-1) d_j (3\, u_1)$

$K_5 = (-\lambda + \lambda\xi\,(1));\ \xi\,(1) = \sum_{k=1}^{a-1} g_k\,; \quad \xi'(1) = \sum_{k=1}^{a-1} k g_k$

$K_6 = [\,\widetilde{w}(\lambda - \lambda\xi\,(1)) - 1]\,[n\,(K_5) - \lambda\,\xi'(1)] \quad K_7 = (K_5)\,[-\lambda\xi'(1)\,\widetilde{w}'(\lambda - \lambda\xi(1))]$

$K_8 = M_1(-\lambda + \lambda X(z))$

5.7.2 Expected Waiting Time in the Queue

$$E\,(W) = \frac{E(Q)}{\lambda E(X)}$$

5.7.3 Expected Length of Busy Period

Let B be the busy period random variable, T be the residence time that the server is rendering primary service or under renewal process and E(A) be the expected secondary service time. Then

$$E\,(T) = E(S) + \delta E(R)$$

where $E(S)$ is the expected primary service time

$E(R)$ is the expected renewal time

Define a random variable J as,

$$J = \begin{cases} 0, & \text{if the server finds less than } 'a-1' \text{ customers after secondary service} \\ 1, & \text{if the server finds atleast } 'a-1' \text{customers after secondary service} \\ 2, & \text{if the server finds less than } 'a' \text{customers after primary service} \\ 3, & \text{if the server finds atleast } 'a' \text{customers after primary service} \end{cases}$$

Then expected length of busy period is given by,

$$E(B) = E(B/J=0)P(J=0) + E(B/J=1)P(J=1) + E(B/J=2)P(J=2) + E(B/J=3)P(J=3)$$

$$= E(T)P(J=0) + \big(E(T)+E(B)\big)P(J=1) + E(A)P(J=2)$$
$$+(E(A)+E(B))\,P(J=3)$$
$$E(B)\big(1-P(J=1)-P(J=3)\big) = E(T)\big(P(J=0)+P(J=1)\big)$$
$$+E(A)\big(P(J=2\quad)+P(J=3)\big)$$
$$= \big(E(T)-E(A)\big)(P(J=0)+P(J=1)) + E(A)$$
$$\text{E (B)} = \big(E(T)-E(A)\big) + \frac{E(A)}{\big(P(J=0)+P(J=2)\big)}$$
$$\text{E (B)} = \big(E(T)-E(A)\big) + \frac{E(A)}{W_{1\,a-1}+\sum_{i=0}^{a-1}(1-\delta)\mathrm{p_i}+R_{\mathrm{i}}},$$

5.7.4 Expected Length of Idle Period

An idle time period is defined as time period between vacation initiation epoch and the busy period initiation epoch. Let A be the random variable for 'idle period'.

$$\text{Let } A_j = \begin{cases} 1, & \text{if the system states becomes j during idle period} \\ & 0, \text{other wise} \end{cases}$$

Let $P\big(A_j = 1\big) = Y_j, j = 0,1,2,3 \ldots a-1$

Define ${Y_j}^1, {Y_j}^2$ as the probability that there are 'j' customers in the queue at primary service completion and secondary service completion epochs respectively.
Conditioning on the queue size at service completion epochs, for primary service completion epochs

$${Y_j}^1 = \alpha_j + \sum_{k=0}^{j} \alpha_k\ P\big(A'_{j-k} = 1\big), \quad j = 1,2,3 \ldots a-1,$$

where $P\big(A'_j = 1\big)$ is the probability that the system state becomes 'j' when the server in idle period and $\alpha_j = p_j$ is the probability that 'j' customers are in the queue at primary service completion epoch or secondary service completion epoch, ${Y_j}^2 = \pi_j$, $j = 0$, where π_j is the probability that no customers present in the queue at secondary service completion epoch

Let $P\big(A'_j = 1\big) = \varphi_j$ and $\varphi_0 = 1$

$$\varphi_j = \sum_{k-1}^{j-1} g_k \varphi_{j-k}, \quad j = 1,2,3 \ldots$$

Thus $Y_j = {Y_j}^1, \;\; j = 0,1,2, \dots a-2$

Thus $Y_{a-1} = Y_{(a-1)^1} + Y_{(a-1)^2}$

$Y_j(j = 0,1,2, \dots a-1)$ can be computed using $\alpha_j\big(= g_j\big),\ \pi_0(= H_0)$ from the Eqns. 5.32 From the above statement, $\sum_{j-0}^{a-1} A_j$, j is the number of states visited during idle period, thus

$$E\left(\sum_{j=0}^{a-1} A_j\right) = \sum_{j=0}^{a-1} E\big(A_j\big) = \sum_{j=0}^{a-1} P\big(A_j = 1\big) = \sum_{j=0}^{a-1} Y_j$$

Since the average staying time in any state during an idle period is $\frac{1}{\lambda}$,

the expected length of idle period $E(A) = \frac{1}{\lambda}\sum_{j=0}^{a-1} Y_j$

5.8 Numerical Illustrations

This section presents some numerical examples to justify the obtained theoretical results. To study the effect of arrival rate on performance measures the following notations are used and some assumptions are made.

Primary service time distribution is 4-Erlang with parameter	μ
Secondary service time distribution is 2-Erlang with parameter	μ^1
Batch size distribution of the arrival is geometric with mean	2
Vacation time is exponential with parameter	ε̡
Renovation time is exponential with parameter	η

5.8.1 Effect of Arrival Rates and Service on Performance Measures

In Table 5.2, Table 5.3 and Fig. 5.2 an effect of arrival rate and service rate on performance measures are presented with the following parameters $a = 2, b = 4$, ε̡ $= 10, \eta = 8$, $\delta = 0.2$. The following points are observed from the tables and figure.

- When arrival rate increases the expected queue length, expected waiting time in the queue and expected length of busy period are increases, whereas expected length of idle period decreases.

5.8.2 Effect of Breakdown Probability on Performance Measures

An effect of breakdown probability on performance measures is given in Table 5.3 and Fig. 5.4. It is observed that when probability of server breakdown increases, expected queue length, expected length of busy period and expected waiting time in the queue increases, whereas expected length of idle period decreases.

Table 5.2 Effect of Arrival Rates and Service Rates on Performance Measures

λ	$\mu=3$, $\mu^1=2$			
	E (Q)	E (W)	E (B)	E (I)
3.0	1.7291	1.1274	2.5432	0.9265
3.3	2.1065	1.1351	2.70962	0.8592
3.6	2.3629	1.1568	2.8383	0.8042
3.9	2.7146	1.1926	3.3742	0.7327
4.2	3.2724	1.2386	3.6160	0.6834
4.5	3.6927	1.2791	3.8641	0.6426

Table 5.3 Effect of Arrival Rates and Service on Performance Measures

λ	$\mu = 3.5$, $\mu^1 = 2.5$			
	E (Q)	E (W)	E (B)	E (I)
3.0	1.1384	0.9421	1.5380	1.1042
3.3	1.3459	1.1148	1.7463	0.9374
3.6	1.7941	1.3678	1.8415	0.9071
3.9	2.0483	1.4792	2.0362	0.8820
4.2	2.4762	1.7481	2.3768	0.8537
4.5	2.8395	1.8862	2.5489	0.8245

Table 5.4 Breakdown Probability versus Performance Measures

δ	$E(Q)$	$E(B)$	$E(I)$	$E(W)$
0.1	3.4361	2.4436	0.1292	1.5327
0.2	4.5468	2.4724	0.1145	1.7263
0.3	6.5982	2.5193	0.0945	1.8354
0.4	7.9430	2.5682	0.0826	2.0345
0.5	9.0341	2.6192	0.0523	2.2764

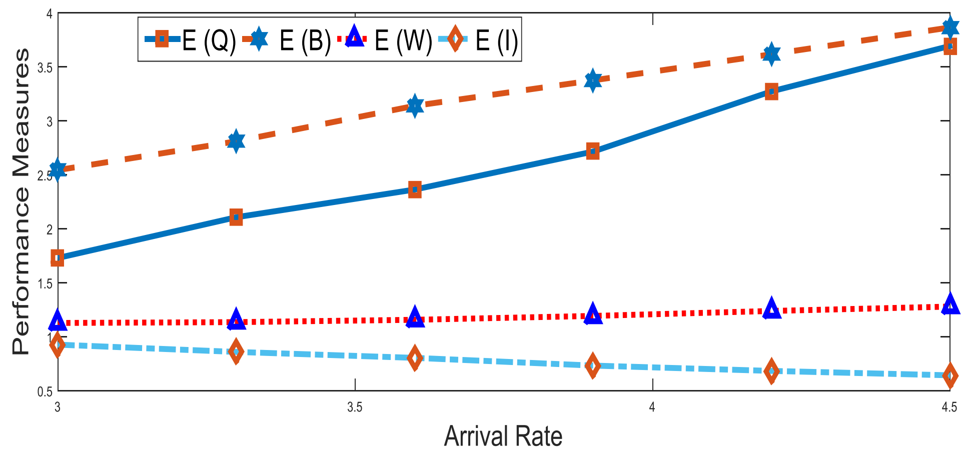

Fig. 5.2 Arrival Rate versus Performance Measures

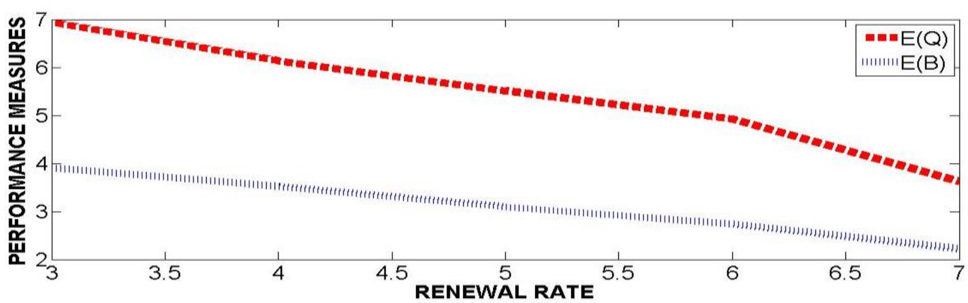

Fig. 5.3 Renewal Rate versus Performance Measures

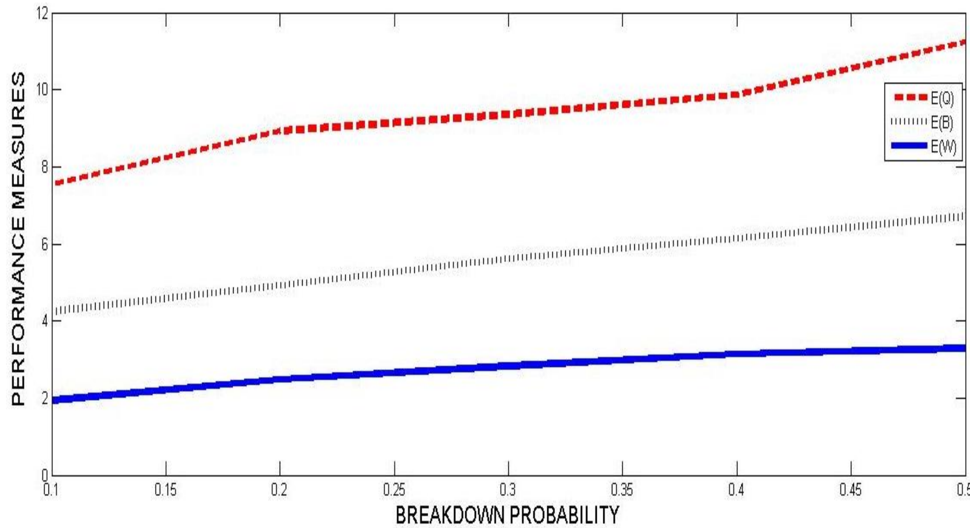

Fig. 5.4 Breakdown Probability versus Performance Measures

5.9 Conclusion

In this chapter, single server bulk arrival and batch service queueing system with secondary service, server failure and vacation is analysed. The model so considered is unique in the sense that secondary service and service interruption are introduced for $M^X/G(a,b)/1$ queueing system. For the given model, probability generating function of the queue size at an arbitrary time is obtained by using supplementary variable technique. Various performance measures are also derived with appropriate numerical illustrations.