CHAPTER 4

Analysis of $M^X/G/1$ Queueing System with Batch Size Dependent Service and Two Patterns of Working Vacation

4.1 Introduction

Utilization of idle time in queueing systems for supplementary works is much more flexible and realistic while analysing real time systems like manufacturing industries, communications networks, inventory management and production line systems.

Krishnamoorthy and Ushakumari (2000) discussed M/M/C queueing system with accessible service batches. They computed the system size probabilities in transient as well as in steady states, also they considered arrival and departure as single whereas service is bulk. Wang et al (2011) discussed discrete time Geo/G/1 queue with randomized vacations and at most J vacations.

Bulk arrival queueing system has been taken into consideration by many of the researchers. Haridass and Arumuganathan (2012) studied bulk arrival and batch service queueing system with vacation interruption. They have considered general bulk service rule for varying batch sizes. Queueing system with fixed batch size has been analysed (Lee et al., 1995). They have used decomposition method to derive probability generating function of the queue size for both single and multiple vacation.

In some real time systems, server provides service in two service modes like single service and batch service. M/G/1 queueing model with two service model has been analysed by Nishimura and Jiang (1994). They derived result for the number of customers in the system. Changing the service mode based on the queue length in bulk arrival queueing systems has been studied by Rein and Timjs(1999).

Queueing system with working vacation has been studied by many authors in recent times. In particular, Zhang and Hou (2010) derived steady state results of a service status and queue length distribution of a non-Markovian queuing system with working vacation and vacation interruption. Gao and Liu (2013) adapted Bernoulli schedule control to interrupt vacation in M/G/1 single working vacation queueing system. Yang and Wu (2015) have used matrix-geometric method to derive queue length distribution M/M/1 queue system with working vacation and N policy. Also they have extended the model with breakdown and cost optimization. Recently Jeyakumar and Senthilnathan (2016) discussed $M^X/G(a,b)/1$ queueing system with working vacation. In their work they have derived queue length distribution and various performance measures.

In the above literature, working vacation queueing models were discussed only in continuous time systems, whereas Shweta Upadhyaya (2015) studied bulk arrival discrete time retrial queueing system with working vacation which concentrates on joint optimal values of vacation returning rate and service rate of the server during working vacation by using direct search method based on heuristic approach.

In all the above queueing models, working vacation is not considered for state dependent service queueing systems. This gives motivation to model a queueing system called bulk arrival queueing system with batch size dependent service and two patterns of working vacation.

4.2 Model Description

Customers are arriving into the system in bulk according to Poisson process with rate λ. Service process is split into two service modes as single service and fixed batch service. Depending upon the batch size the server provides single service or batch service. During single service, arriving customers are served one by one whereas in fixed batch service customers are served in batches with fixed batch size 'k'. If the queue length is at least 'a' after the service completion then the server provides single service. On the other hand, if the queue length reaches the threshold value 'k' ($k > a$) then the server provides batch service with fixed batch size 'k'. In service completion (single service or batch service) if the queue length say ξ is greater than 'k' then the server provides service for only 'k' customers, remaining $\xi - k$ customers have to wait in the queue for next batch of service. Transformation of service mode is possible only at the service initiation time. If the queue length is less than 'a' after the service completion then the server leaves for working vacation. During working vacation, arriving customers are served with lower service rate than the regular service rate. In the working vacation period, if the queue length say ζ is such that $k > \zeta \geq a$, then single service will be provided. But if $\zeta \geq k$ during the working vacation, then fixed batch service will be provided in lower service rate. Change of service mode is possible in working vacation period too. As switch over time is basically very small, it is not taken into consideration. If the slow service ends prior to the working vacation period then the server remains idle until the working vacation period ends. On the other hand, if slow service exceeds the working vacation period, then the slow service rate will be changed into regular service rate. On completion of working vacation if the queue size is still less than 'a' then the server remains idle (dormant period) until the queue length reaches the value 'a'. The schematic representation of the proposed queueing model is pictured below.

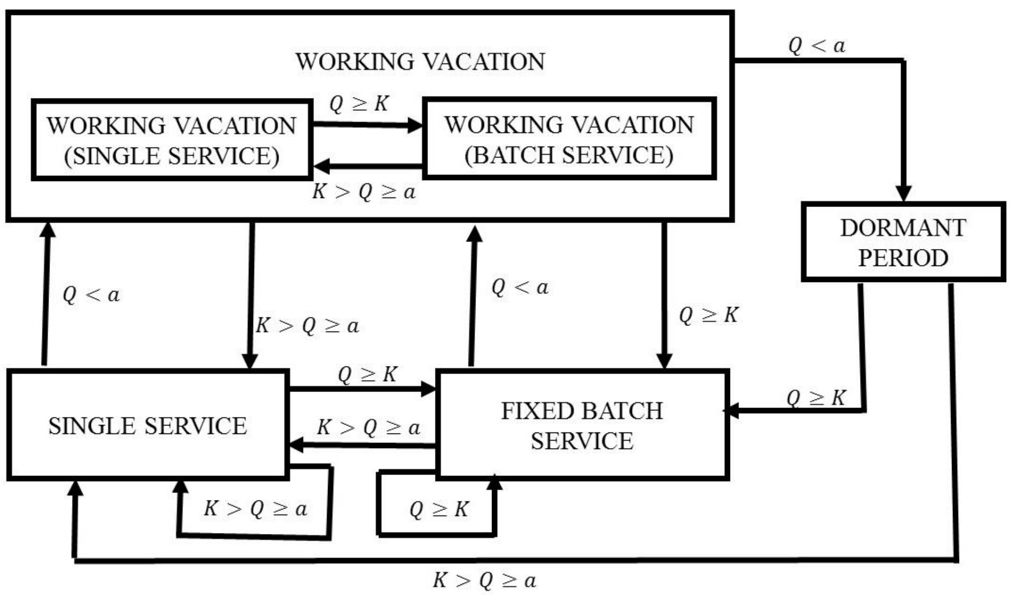

Fig 4.1 Schematic Representation of the Queueing System: Q - Queue Length

For the proposed model, probability generating function of queue size at an arbitrary time epoch is obtained. Various performance measures are derived with suitable numerical illustrations. Some special cases are also discussed.

4.3 Mathematical Model

Let X be the group size random variable of the arrival, λ be the Poisson arrival rate, g_k be the probability that 'k' customers arrive in a batch. Service rate during batch service is

$\mu^1{}_b$ (when the server is not in working vacation) and the service rate during batch service is $\mu^1{}_v(\mu^1{}_v < \mu^1{}_b)$ (when the server is in working vacation). Service rate during single service is $\mu^2{}_b$ (when the server is not in working vacation) and the service rate during single service is $\mu^2{}_v(\mu^1{}_v < \mu^1{}_b)$ (when the server is in working vacation). Working vacation duration follows exponential distribution with parameter η.

Table 4.1 Notations

	Cumulative distribution function	Probability density Function	Laplace-Stieltjes transform	Remaining time
Single service	$A_b(x)$	$a_b(x)$	$\widetilde{A_b}(\theta)$	$P^0(x)$
Fixed batch service	$P_b(x)$	$p_b(x)$	$\widetilde{P_b}(\theta)$	${P_b}^0(x)$
Single service during working vacation	$A_v(x)$	$a_v(x)$	$\widetilde{A_v}(\theta)$	${A_v}^0(x)$
Batch service during working vacation	$P_v(x)$	$p_v(x)$	$\widetilde{P_v}(\theta)$	${P_v}^0(x)$
No service during working vacation	$W(x)$	$w(x)$	$\widetilde{W}(\theta)$	$W^0(x)$

$$Y(t) = \begin{cases} 0, \text{ When the server is busy with regular single service} \\ 1, \text{ when the server is busy with regular batch service} \\ 2, \text{ when the server is on working vacation with single service} \\ 3, \text{ when the server is on working vacation with batch service} \\ 4, \text{ when the server is idle during working vacation} \\ 5, \text{ when the server is on dormant period} \end{cases}$$

The state probabilities are defined as follows :

$$G_{kj}(x,t)dt = P_r\{N_s(t)=k,\ \ N_q\ (t)=j,\ x \le A_b{}^0(t) \le\ x+dt,\ Y(t)=0\}$$

$$S_{1n}(x,t)dt = P_r\{\ N_s(t)=1, N_q\ (t)=n,\ x \le P_b{}^0(t) \le\ x+dt,\ Y(t)=1\}$$

$$n \ge 1$$

$$V^1{}_{kn}(x,t)dt = P_r\{\ N_s(t)=k, N_q\ (t)=n,\ x \le A_v{}^0(t) \le\ x+dt,\ Y(t)=2\};$$

$$1 \le n \le a-1$$

$$V^2{}_{1n}(x,t)dt = P_r\{\ N_s(t)=1, N_q\ (t)=n,\ x \le P_v(t) \le\ x+dt,\ Y(t)=3\};$$

$$1 \le n \le a-1$$

$$W_n(t)dt = Pr\{N_q = n, C(t)=4\}1 \le n \le a-1$$

$$I_n(t) = P_r\{N_q(t)=n,\ \ C(t)=4\} \qquad 0 \le n \le a-1$$

The following equations are obtained by using supplementary variable technique.

$$S_{1a-1}(x-\Delta t, t+\Delta t) = S_{1a-1}(x,t)(1 - \lambda\Delta t) + \left(S_{1a}(0,t) + G_{ka}(0,t)\right)p_b(x)\Lambda t \quad (4.1)$$

$$+(\textstyle\sum_{m=1}^{a-1} I_m(\mathrm{t})\lambda g_{a+1-m} + \sum_{m=0}^{a-1} V^1{}_{1m}(x,t)\lambda g_{a+1-m})p_b(x)\Lambda t$$

$$+(W_a(0,t) + \textstyle\sum_{m=0}^{a-1} V^2{}_{km}(x,t)\lambda g_{a+1-m})p_b(x)\Lambda t$$

$$S_{1n}(x-\Delta t, t+\Delta t) = S_{1n}(x,t)(1-\lambda\Delta t) + \textstyle\sum_{m=0}^{a-1} I_m \lambda g_{n+1-m} p_b(x)\,\Lambda t \quad (4.2)$$

$$+\left(S_{1n+1}(0,t) + G_{kn+1}(0,t) + W_{n+1}(0,t)\right)p_b(x)\Lambda t$$

$$+\textstyle\sum_{k=1}^{n-a} S_{1n-k}(x)\lambda g_k \Lambda t\ + \sum_{m=0}^{a-1} V^2{}_{km}(x,t)\lambda g_{a+n-m}\, p_b(x)\Lambda t$$

$$+\textstyle\sum_{m=0}^{a-1} V^1{}_{1m}(x,t)\lambda g_{n+1-m}\, p_b(x)\Lambda t \qquad a \le n \le k-2$$

$$S_{1n}(x-\Delta t, t+\Delta t) = S_{1n}(x,t)(1-\lambda\Delta t) + \textstyle\sum_{m=0}^{a-1} I_m(\mathrm{t})\lambda g_{n+1-m} p_b(x)\Lambda t \quad (4.3)$$

$$+\textstyle\sum_{k=1}^{n-a} S_{1n-k}(x,t)\lambda g_k\,\Lambda t\ + \sum_{m=0}^{a-1} V^1{}_{1m}(x,t)\lambda g_{n+1-m}\, p_b(x)\Lambda t$$

$$+\textstyle\sum_{m=0}^{a-1} V^2{}_{km}(x,t)\lambda g_{a+n-m}\, p_b(x)\Lambda t \qquad n = k-1$$

$$S_{1n}(x-\Delta t, t+\Delta t) = S_{1n}(x,t)(1-\lambda\Delta t) + \textstyle\sum_{k=1}^{n-a} S_{1n-k}(x,t)\lambda g_k\,\Delta t \quad n \ge k \quad (4.4)$$

$$G_{k0}(x-\Delta t, t+\Delta t) = G_{k0}(x,t)(1-\lambda\Delta t) + \big(S_{1k}(0) + G_{kk}(0,t) + W_k(0,t)\big)a_b(x)\Delta t$$

$$+(\textstyle\sum_{m=0}^{a-1} V^1{}_{1m}(x,t)\lambda g_{k-m} + \sum_{m=0}^{a-1} V^2{}_{km}(x,t)\lambda g_{k-m})a_b(x)\Delta t$$

$$+\textstyle\sum_{m=0}^{a-1} I_m(\mathrm{t})\lambda g_{k-m} a_b(x)\Delta t \qquad (4.5)$$

$$G_{kn}(x-\Delta t, t+\Delta t) = G_{kn}(x,t)(1-\lambda\Delta t) + \textstyle\sum_{k=1}^{n} G_{kn-k}(x,t)\lambda g_k \Delta t \qquad (4.6)$$

$$+\big(S_{1k+n}(0,t) + G_{kk+n}(0,t) + W_{k+n}(0,t)\big)a_b(x)\Delta t$$

$$+(\textstyle\sum_{m=0}^{a-1} V^2{}_{km}(x,t)\lambda g_{k+n-m} + \sum_{m=0}^{a-1} I_m(\mathrm{t})\lambda g_{k+n-m})a_b(x)\Delta t$$

$$+\textstyle\sum_{m=0}^{a-1} V^1{}_{1m}(x,t)\lambda g_{k+n-m} a_b(x)\,\Delta t$$

$$V^1{}_{1n}(x-\Delta t, t+\Delta t) = V^1{}_{1n}(x,t)(1-\lambda\Delta t) + (W_n(\mathrm{t})\lambda g_a\Delta t \quad + W_{n+a}(t))p_v(x)\Delta t$$

$$+\textstyle\sum_{k=1}^{n} V^1{}_{1n-k}(x,t)\lambda g_k\,\Delta t \qquad (4.7)$$

$$V^2{}_{kn}(x-\Delta t, t+\Delta t) = V^2{}_{kn}(x,t)(1-\lambda\Delta t) + W_n(t)\lambda g_k a_v(x)\Delta t + W_{n+k}(t)a_v(x)\Delta t$$

$$+\textstyle\sum_{k=1}^{n} V^2{}_{kn-k}(x,t)\lambda g_k\,\Delta t \qquad (4.8)$$

$$I_0(t+\Delta t) \;=\; I_0(1-\lambda\Delta t) + V^1{}_{10}(0,t)\Delta t + V^2{}_{k0}(0,t)\Delta t + W_0(t)\Delta t \qquad (4.9)$$

$$I_n(t+\Delta t) \;=\; I_0(1-\lambda\Delta t) + V^1{}_{1n}(0,t)\Delta t + V^2{}_{kn}(0,t)\Delta t + W_n(t)\Delta t \qquad (4.10)$$

$$+\;\sum_{k=1}^{n} I_{n-k}(t)\lambda g_k\,\Delta t \qquad\qquad 1 \;\le\; n \;\le\; a-1$$

$$W_0(t+\Delta t) = W_0(1-\lambda\Delta t) + \big(V^1{}_{10}(0,t) + V^2{}_{k0}(0,t) + G_{k0}(0.t)\big)\Delta t \qquad (4.11)$$

$$+S_{10}(0,t)\Delta t$$

$$W_n(t+\Delta t) = W_n(1-\lambda\Delta t) + V^1{}_{1n}(0,t)\Delta t + V^2{}_{kn}(0,t)\Delta t \qquad (4.12)$$

$$1 \;\le\; n \;\le\; a-1$$

4.4 Steady State Queue Size Distribution

Dividing the above equations by Δt and allowing $\Delta t \to 0$, the steady state queue size equations are obtained as follows:

$$-\frac{\mathrm{d}}{\mathrm{dx}} S_{1a-1}(x) = -\lambda S_{1a-1}(x) + \big(S_{1a}(0) + G_{ka}(0) + W_a(0)\big)p_b(x) \tag{4.13}$$

$$+ \sum_{m=1}^{a-1} I_m \lambda g_{a+1-m} p_b(x) + \sum_{m=0}^{a-1} V^1{}_{1m}(x)\lambda g_{a+1-m} p_b(x)$$

$$+ \sum_{m=0}^{a-1} V^2{}_{km}(x)\lambda g_{a+1-m} p_b(x)$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} S_{1n}(x) = -\lambda S_{1n}(x) + \big(S_{1n+1}(0) + G_{kn+1}(0) + W_{n+1}(0)\big)p_b(x) \tag{4.14}$$

$$+ \sum_{m=0}^{a-1} I_m \lambda g_{n+1-m} p_b(x) + \sum_{k=1}^{n-a} S_{1n-k}(x)\lambda g_k \qquad a \le n \le k-2$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} S_{1n}(x) = -\lambda S_{1n}(x) + \sum_{k=1}^{n-a} S_{1n-k}(x)\lambda g_k + \sum_{m=0}^{a-1} I_m \lambda g_{n+1-m} p_b(x) \tag{4.15}$$

$$+\Big(\sum_{m=0}^{a-1} V^1{}_{1m}(x)\lambda g_{n+1-m} + \sum_{m=0}^{a-1} V^2{}_{km}(x)\lambda g_{a+n-m}\Big)p_b(x) \quad n = k-1$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} S_{1n}(x) = -\lambda S_{1n}(x) + \sum_{k=1}^{n-a} S_{1n-k}(x)\lambda g_k \qquad n \ge k \tag{4.16}$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} G_{k0}(x) = -\lambda G_{k0}(x) + \big(S_{1k}(0) + G_{kk}(0) + W_k(0)\big)a_b(x) + \sum_{m=0}^{a-1} I_m \lambda g_{k-m} a_b(x)$$

$$+\Big(\sum_{m=0}^{a-1} V^1{}_{1m}(x)\lambda g_{k-m} + \sum_{m=0}^{a-1} V^2{}_{km}(x)\lambda g_{k-m}\Big)a_b(x) \tag{4.17}$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} G_{kn}(x) = -\lambda G_{kn}(x) + \big(S_{1k+n}(0) + G_{kk+n}(0) + W_{k+n}(0)\big)a_b(x) \tag{4.18}$$

$$+ \sum_{m=0}^{a-1} V^2{}_{km}(x)\lambda g_{k+n-m} a_b(x) + \sum_{m=0}^{a-1} I_m \lambda g_{k+n-m} a_b(x)$$

$$+ \sum_{m=0}^{a-1} I_m \lambda g_{k+n-m} a_b(x) + \sum_{m=0}^{a-1} V^1{}_{1m}(x)\lambda g_{k+n-m} a_b(x)$$

$$+ \sum_{k=1}^{n} G_{kn-k}(x)\lambda g_k$$

$$-\frac{\mathrm{d}}{\mathrm{dx}} V^1{}_{1n}(x) = -\lambda V^1{}_{1n}(x) + W_n \lambda g_a p_v(x) + W_{n+a} p_v(x) + \sum_{k=1}^{n} V^1{}_{1n-k}(x)\lambda g_k) \tag{4.19}$$

$$n \ge 1$$

$$-\frac{\mathrm{d}}{\mathrm{dx}}V^2{}_{kn}(x) = -\lambda V^2{}_{kn}(x) + W_n\delta g_k a_v(x) + W_{n+k}a_v(x) + \sum_{k=1}^{n} V^2{}_{kn-k}(x)\delta g_k \quad (4.20)$$

$$n \geq 1$$

$$0 \;=\; -\lambda I_0 + V^1{}_{10}(0) + V^2{}_{k0}(0) + W_0 \quad (4.21)$$

$$0 \;=\; -\lambda I_n + V^1{}_{1n}(0) + V^2{}_{kn}(0) + W_n + \sum_{k=1}^{n} I_{n-k}\lambda g_k \qquad 1 \leq n \leq a-1 \quad (4.22)$$

$$0 = -\lambda W_0(x) + V^1{}_{10}(0) + V^2{}_{k0}(0) + G_{k0}(0) + S_{10}(0) \quad (4.23)$$

$$0 = -\lambda W_n(x) + V^1{}_{1n}(0) + V^2{}_{kn}(0) \qquad 1 \leq n \leq a-1 \quad (4.24)$$

The Laplace – Stieltjes transform of $\mathrm{G_{kn}}(x)$, $\mathrm{S_{1n}}(x), V^1{}_n(x), V^2{}_n(x)$ and $\mathrm{W_n(x)}$ are defined as

$$\tilde{G}_{kn}(\theta) = \int_0^\infty \mathrm{e}^{-\theta \mathrm{x}}\; \mathrm{G_{kn}(x)}dx \qquad \widetilde{V^1}_{1n}(\theta) = \int_0^\infty \mathrm{e}^{-\theta \mathrm{x}}\, V^1{}_{1\mathrm{n}}(x)dx$$

$$\widetilde{W}_n(\theta) = \int_0^\infty \mathrm{e}^{-\theta\mathrm{x}}\, \mathrm{W_n(x)dx} \qquad \widetilde{V^2}_{kn}(\theta) = \int_0^\infty \mathrm{e}^{-\theta \mathrm{x}}\, V^2{}_{\mathrm{kn}}(x)dx$$

$$\tilde{S}_{1n}(\theta) = \int_0^\infty \mathrm{e}^{-\theta \mathrm{x}}\; \mathrm{S_{1n}(x)}dx \qquad \tilde{C}_{1n}(\theta) = \int_0^\infty \mathrm{e}^{-\theta \mathrm{x}}\, V^1{}_{1\mathrm{n}}(x)P_b(x)dx$$

$$\widetilde{D}_{kn}(\theta) = \int_0^\infty \mathrm{e}^{-\theta \mathrm{x}}\, V^2{}_{\mathrm{kn}}(x)P_b(x)dx \qquad \tilde{E}_{1n}(\theta) = \int_0^\infty \mathrm{e}^{-\theta \mathrm{x}}\, V^1{}_{1\mathrm{n}}(x)A_b(x)dx$$

$$\tilde{F}_{kn}(\theta) = \int_0^\infty \mathrm{e}^{-\theta \mathrm{x}}\, V^2{}_{\mathrm{kn}}(x)A_b(x)dx$$

Taking Laplace – Stieltjes transform on both sides of equations from Eqn. 4.13 to Eqn. 4.24, the following equations are obtained.

$$\theta\tilde{\mathrm{S}}_{\mathrm{ia}-1}(\theta) - S_{1a-1}(0) = \lambda\tilde{S}_{1a-1}(\theta) - \big(S_{1a}(0) + G_{ka}(0) + W_a(0)\big)\widetilde{P_b}(\theta) \quad (4.25)$$

$$- \sum_{m=1}^{a-1} I_m\lambda g_{a+1-m}\, \widetilde{P_b}(\theta) - \sum_{m=0}^{a-1} \tilde{C}_{1m}(\theta)\lambda g_{a+1-m}$$

$$- \sum_{m=0}^{a-1} \widetilde{D}_{km}(\theta)\lambda g_{a+1-m}$$

$$\theta\tilde{\mathrm{S}}_{\mathrm{in}}(\theta) - S_{1n}(0) = \lambda\tilde{S}_{1n}(\theta) - \big(S_{1n+1}(0) + G_{kn+1}(0) + W_{n+1}(0)\big)\widetilde{P_b}(\theta) \quad (4.26)$$

$$-\sum_{m=0}^{a-1} I_m\lambda g_{n+1-m}\widetilde{P_b}(\theta) - \sum_{k=1}^{n-a}\tilde{S}_{1n-k}(\theta)\lambda g_k$$
$$-\sum_{m=0}^{a-1}\tilde{C}_{1m}(\theta)\lambda g_{n+1-m} - \sum_{m=0}^{a-1}\widetilde{D}_{km}(\theta)\lambda g_{a+n-m}$$
$$a \le n \le k-2$$

$$\theta\tilde{S}_{in}(\theta) - S_{1n}(0) = \lambda\tilde{S}_{1n}(\theta) - \sum_{k=1}^{n-a}\tilde{S}_{1n-k}(\theta)\lambda g_k + \sum_{m=0}^{a-1} I_m\lambda g_{n+1-m}\widetilde{P_b}(\theta) \quad (4.27)$$
$$-\sum_{m=0}^{a-1}\tilde{C}_{1m}(\theta)\lambda g_{n+1-m} + \sum_{m=0}^{a-1}\widetilde{D}_{km}(\theta)\lambda g_{a+n-m}$$
$$n = k-1$$

$$\theta\tilde{S}_{in}(\theta) - S_{1n}(0) = \lambda\tilde{S}_{1n}(\theta) - \sum_{k=1}^{n-a}\tilde{S}_{1n-k}(\theta)\lambda g_k \qquad n \ge k \quad (4.28)$$

$$\theta\tilde{G}_{k0}(\theta) - G_{k0}(0) = \lambda\tilde{G}_{k0}(\theta) - \left(S_{1k}(0) + G_{kk}(0) + W_k(0)\right)\widetilde{A_b}(\theta) \quad (4.29)$$
$$-\left(\sum_{m=0}^{a-1}\tilde{E}_{1m}(\theta)\lambda g_{k-m} + \sum_{m=0}^{a-1}\tilde{F}_{km}(\theta)\lambda g_{k-m}\right)$$
$$-\sum_{m=0}^{a-1} I_m\lambda g_{k-m}\widetilde{A_b}(\theta)$$

$$\theta\tilde{G}_{kn}(\theta) - G_{kn}(0) = \lambda\tilde{G}_{kn}(\theta) - \left(S_{1k+n}(0) + G_{kk+n}(0) + W_{k+n}(0)\right)\widetilde{A_b}(\theta) \quad (4.30)$$
$$-\sum_{m=0}^{a-1} I_m\lambda g_{k+n-m}\widetilde{A_b}(\theta) - \sum_{m=0}^{a-1}\tilde{E}_{1m}(\theta)\lambda g_{k+n-m}$$
$$-\sum_{k=1}^{n} G_{kn-k}(x)\lambda g_k - \sum_{m=0}^{a-1}\tilde{F}_{km}(\theta)\lambda g_{k+n-m} \qquad n \ge 1$$

$$\theta\widetilde{V^1}_{1n}(\theta) - V^1{}_{1n}(0) = \lambda\widetilde{V^1}_{1n}(\theta) - (W_{n+a} + W_n\delta g_a)\widetilde{P_v}(\theta) \quad (4.31)$$
$$-\sum_{k=1}^{n}\widetilde{V^1}_{1n-k}(\theta)\lambda g_k$$

$$\theta\widetilde{V^2}_{kn}(\theta) - V^2{}_{kn}(0) = \lambda\widetilde{V^2}_{kn}(\theta) - (W_n\delta g_k + W_{n+k})\widetilde{A_v}(\theta) \quad (4.32)$$
$$-\sum_{k=1}^{n}\widetilde{V^2}_{1n-k}(\theta)\lambda g_k$$

4.5 Probability Generating Function (PGF)

To derive the steady state probability generating function of an orbit size, the following probability generating functions are defined.

$$\tilde{G}_k(z,\theta) = \sum_{j=0}^{\infty}\tilde{G}_{kj}(\theta)z^j \qquad G_k(z,0) = \sum_{j=0}^{\infty} G_{kj}(0)z^j$$

$$\tilde{B}_j(z,\theta) = \sum_{j=0}^{\infty} \tilde{B}_j(0) z^j \qquad B_j(z,0) = \sum_{j=0}^{\infty} Q_j(0) z^j$$

$$\tilde{S}(z,\theta) = \sum_{n=0}^{\infty} \tilde{S}_{1n}(\theta)\, z^n \qquad S(z,0) = \sum_{n=0}^{\infty} S_{1n}(0) z^n$$

$$I(z) = \sum_{j=0}^{\infty} I_j\, z^j \qquad \widetilde{V^1}(z,\theta) = \sum_{n=0}^{\infty} \tilde{V}^1{}_{1n}(0) z^n$$

$$V^1(z,0) = \sum_{n=0}^{\infty} V^1{}_{1n}(0) z^n \qquad \widetilde{V^2}(z,\theta) = \sum_{n=0}^{\infty} \tilde{V}^2{}_{kn}(0) z^n$$

$$V^2(z,0) = \sum_{n=0}^{\infty} V^2{}_{kn}(0) z^n \qquad W(z) = \sum_{n=0}^{\infty} W_n\, z^n$$

By multiplying the equations from Eqn. 4.25 to Eqn. 4.32 with suitable powers of z^n and summing over n, then by using PGF given above, we get

$$\big(\theta - \lambda + \lambda X(z)\big)\tilde{S}(z,\theta) = S(z,0) - \sum_{m=0}^{a-1} \tilde{C}_{1m}(\theta)\lambda g_{a+1-m} - \sum_{m=0}^{a-1} \widetilde{D}_{km}(\theta)\lambda g_{a+1-m}$$
$$-\widetilde{P_b}(\theta)\left\{\sum_{n=a}^{k-1}\begin{pmatrix} S_{1n}(0) + G_{kn}(0) + W_n(0) \\ +\sum_{m=1}^{a-1} I_m \lambda g_{n-m}\end{pmatrix} z^{n-1}\right\} \tag{4.33}$$

$$\big(\theta - \lambda + \lambda X(z)\big)\tilde{G}_k(z,\theta) = G_k(z,0) \tag{4.34}$$
$$-\lambda\Big(X(z)\sum_{m=0}^{a-1} \tilde{E}_{1m}(\theta) z^m - \sum_{i=0}^{a-1}\big(\tilde{E}_{1i}(\theta) z^i \sum_{j=1}^{a-i-1} g_j\, z^j\big)\Big)$$
$$-\lambda\Big(X(z)\sum_{m=0}^{k-1} \tilde{F}_{km}(\theta) z^m - \sum_{i=0}^{k-1}\big(\tilde{F}_{ki}(\theta) z^i \sum_{j=1}^{k-i-1} g_j\, z^j\big)\Big)$$
$$-\frac{\widetilde{A_b}(\theta)}{z^k}\left\{\begin{matrix} G_k(z,0) - \sum_{j=0}^{k-1} G_{kj}(0) z^j \\ +W(z) - \sum_{n=0}^{k-1} W_n\, z^n \\ +\lambda\big(I(z)X(z) - \sum_{m=0}^{k-1}\big(I_m z^m \sum_{j=1}^{b-m-1} g_j\, z^j\big)\big) \\ +S(z,0) - \sum_{n=0}^{k-1} S_{1n}(0) z^n \end{matrix}\right\}$$

$$\big(\theta - \lambda + \lambda X(z)\big)\widetilde{V^1}(z,\theta) = V^1(z,0) - \frac{1}{z^{n+a}}\big(W(z) - W_0\big)\widetilde{P_v}(\theta) \tag{4.35}$$
$$-\lambda g_a W(z)\widetilde{P_v}(\theta)$$

$$\big(\theta - \lambda + \lambda X(z)\big)\widetilde{V^2}(z,\theta) = V^2(z,0) - \frac{1}{z^{n+k}}\big(W(z) - W_0\big)\widetilde{A_v}(\theta) \tag{4.36}$$
$$-\lambda g_k W(z)\widetilde{A_v}(\theta)$$

$$\lambda W(z) = \lambda W_0 + G_0 + S_0 + V^1(z,0) + V^2(z,0) \tag{4.37}$$

$$W(z) = \frac{W_0\left(\lambda z^{n+a+k} - \widetilde{P_v}(\lambda-\lambda X(z))z^{n+k} - \widetilde{A_v}(\lambda-\lambda X(z))z^{n+a}\right)}{\lambda z^{n+a+k} - \widetilde{P_v}(\lambda-\lambda X(z))\left(z^{n+k} - \lambda g_a z^{n+a+k}\right) - \widetilde{A_v}(\lambda-\lambda X(z))\left(z^{n+a} - \lambda g_k z^{n+a+k}\right)} \tag{4.38}$$

Let $P(z)$ be the PGF of the queue size at an arbitrary time, then

$$\mathrm{P(z)} = \widetilde{\mathrm{G}}_{\mathrm{k}}(\mathrm{z},0) + \tilde{\mathrm{S}}(\mathrm{z},0) + \widetilde{V^1}(\mathrm{z},0) + \widetilde{V^2}(\mathrm{z},0) + \mathrm{I}(\mathrm{z}) + \mathrm{W(z)} \tag{4.39}$$

Substituting $\theta = \lambda - \lambda \mathrm{X}(\mathrm{z})$ in the equations from Eqn. 4.33 to Eqn. 4.36, after doing some algebra, the PGF of the queue size is defined in Eqn. 4.39 is simplified as

$$P(z) = \frac{\begin{array}{c}\left(\widetilde{P_b}(\lambda-\lambda X(z))-1\right)h(P_b,\ \lambda)F_1 + \left(\widetilde{A_b}(\lambda-\lambda X(z))-1\right)\left(\widetilde{P_b}(\lambda-\lambda X(z))h(P_b,\lambda) - \sum_{j=0}^{k-1} C_j z^j + F_2\right) \\ +\Psi(P_b,\lambda)F_1\left\{\left(-\lambda+\lambda X(z)\right)+g(P_b,\lambda)\right\} + F_1\left(-\lambda+\lambda X(z)\right)\left\{M(P_b,\lambda)W_0 + I(z)\right\} + F_3\end{array}}{\left(-\lambda+\lambda X(z)\right)F_1} \tag{4.40}$$

where

$$h(P_b,\lambda) = \sum_{n=a}^{k-1}\left(C_n + W_n + \sum_{m=1}^{a-1} I_m \lambda g_{n-m}\right)z^{n-1}$$

$$+\sum_{m=0}^{a-1} \tilde{C}_{1m}\left(\lambda - \lambda X(z)\right)\lambda g_{a+1-m} - \sum_{m=0}^{a-1} \tilde{C}_{1m}(\theta)\lambda g_{a+1-m}$$

$$+\sum_{m=0}^{a-1} \widetilde{D}_{km}\left(\lambda - \lambda X(z)\right)\lambda g_{a+1-m} - \sum_{m=0}^{a-1} \widetilde{D}_{km}(\theta)\lambda g_{a+1-m}$$

$$g(P_b,\lambda) = \frac{F_1\left(\widetilde{P_v}\left(\left(\lambda - \lambda X(z)\right) - 1\right)(z^k - \lambda g_a) + \left(\widetilde{A_v}\left(\lambda - \lambda X(z)\right) - 1\right)(z^n - \lambda g_k)\right)}{z^{a+n+k}}$$

$$\Psi(P_b,\lambda) = \frac{W_0\left(\lambda z^{n+a+k} - \widetilde{P_v}(\lambda-\lambda X(z))z^{n+k} - \widetilde{A_v}(\lambda-\lambda X(z))z^{n+a}\right)}{\lambda z^{n+a+k} - \widetilde{P_v}(\lambda-\lambda X(z))\left(z^{n+k} - \lambda z^{n+a+k}\right) - \widetilde{A_v}(\lambda-\lambda X(z))\left(z^{n+a} - \lambda g_k z^{n+a+k}\right)}$$

$$M(P_b,\lambda) = \frac{F_1\left(\widetilde{P_v}\left((\lambda-\lambda X(z))-1\right)z^k + \left(\widetilde{A_v}(\lambda-\lambda X(z))-1\right)z^n\right)}{z^{a+n+k}} \quad F_1 = \left(z^k - \widetilde{A_b}\left(\lambda - \lambda X(z)\right)\right)$$

$$F_2 = \begin{pmatrix} I(z)X(z) - \lambda \sum_{m=0}^{a-1} I_m z^m \sum_{j=0}^{k-m-1} g_j z^j \\ +\lambda \left(\sum_{i=0}^{a-1} z^i \left(\tilde{E}_{1m}\big(\lambda - \lambda X(z)\big) - \tilde{E}_{1m}(\theta) \right) \left(X(z) - \sum_{j=0}^{k-m-1} g_j z^j \right) \right) \\ -\lambda \left(\sum_{i=0}^{a-1} z^{i-k} \left(\tilde{C}_{1m}\big(\lambda - \lambda X(z)\big) \widetilde{P_b}(\theta) - \tilde{C}_{1m}(\theta) \widetilde{P_b}\big(\lambda - \lambda X(z)\big) \right) \right) \end{pmatrix} -$$

$$F_3 = \begin{pmatrix} \lambda \left(\sum_{i=0}^{a-1} z^i \left(\tilde{F}_{km}\big(\lambda - \lambda X(z)\big) - \tilde{F}_{km}(\theta) \right) \left(X(z) - \sum_{j=0}^{k-m-1} g_j z^j \right) \right) \\ -\lambda \left(\sum_{i=0}^{a-1} z^{i-k} \left(\widetilde{D}_{km}\big(\lambda - \lambda X(z)\big) \widetilde{P_b}(\theta) - \widetilde{D}_{km}(\theta) \widetilde{P_b}\big(\lambda - \lambda X(z)\big) \right) \right) \end{pmatrix}$$

$$C_j = G_j + S_j$$

4.6 Computational Aspects of Unknown Probabilities

Eqn. 4.40 gives the probability generating function of the number of customers in the queue, which involves the unknowns C_j. To find the unknown constants, Rouche's theorem of complex variables can be used. It follows that the expression $z^k - \widetilde{A_b}\big(\lambda - \lambda X(z)\big)$ has $k - 1$ zeros inside and one on the unit circle $|z| = 1$. Since $P(z)$ is analytic within and on the unit circle, the numerator of $P(z)$ must vanish at these points, which gives 'k' equations and 'k' unknowns. These equations can be solved by suitable numerical techniques. MATLAB is used for programming.

4.7 Performance Measures

In a waiting line, it is customary to access the mean number of waiting units and mean waiting time. In this section, measures of effectiveness of the system are derived from the steady-state probability distribution function given in Eqn. 4.40, which are useful to find the total average cost of the system.

4.7.1 Expected Queue Length

$$E(Q) = \lim_{z \to 1} P'(z)$$

$$E(Q) = \frac{4\lambda^2(E(X))^2\left(F'_1\right)\Omega(\mathrm{X},\lambda) - 2\left(3\lambda E(X)F''_1 + (\lambda X'''(1) + 3\lambda X''(1))F'_1\right)\psi(\mathrm{X},\lambda) - 3\left(6\lambda X''(1)F''_1 + 3\lambda X'''(1)(F'_1 + F''_1 + 3\lambda X'^{v}(1)(F'_1) + 3\lambda E(X)F''_1\right)\Phi(\mathrm{X},\lambda)}{24\lambda E(X)(k - E(X)E(P_b))^2}$$

where

$$P_1 = \lambda E(P_b)E(X) \qquad P_2 = E(C)\lambda X''(1) + \lambda^2 E((P_b)^2)\left(E(X)\right)^2$$

$$B_1 = \lambda E(P_V)E(X) \qquad B_2 = E(P_V)\lambda X''(1) + \lambda^2 E((P_V)^2)\left(E(X)\right)^2$$

$$C_1 = \lambda E(A_V)E(X) \qquad C_2 = E(A_V)\lambda X''(1) + \delta^2 E((A_V)^2)\left(E(X)\right)^2$$

$$A_1 = \lambda E(A_b)E(X) \qquad A_2 = E(A_b)\lambda X''(1) + \lambda^2 E((A_b)^2)\left(E(X)\right)^2$$

$$A_3 = E(A_b)\lambda X'''(1) + 3\,\lambda^2 E((A_b)^2)E(X)X''(1) + \lambda^3 E((A_b)^3)(E(X))^3$$

$$B_3 = E(P_V)\delta X'''(1) + 3\,\delta^2 E((P_V)^2)E(X)X''(1) + \delta^3 E((P_V)^3)(E(X))^3$$

$$C_3 = E(A_V)\lambda X'''(1) + 3\,\lambda^2 E((A_V)^2)E(X)X''(1) + \lambda^3 E((A_V)^3)(E(X))^3$$

$$P_3 = E(P_b)\lambda X'''(1) + 3\,\lambda^2 E((P_b)^2)E(X)X''(1) + \lambda^3 E((P_b)^3)(E(X))^3$$

$$F'_1 = k - \lambda E(A_b)E(X) \qquad F''_1 = k(k-1) - E(A_b)\lambda X''(1) + \lambda^2 E((A_b)^2)\left(E(X)\right)^2$$

$$F'''_1 = k(k-1)(k-2) - A_3 \qquad \Phi(\mathrm{X},\delta) = A_1 h_1 + \sum_{j=0}^{k-1} j(G_j + S_j) + F'_2$$

$$\psi(\mathrm{X},\lambda) = 2\mathrm{P}_1(h_1 F'_1) + 2A_1(h'_1 + P_1 h_1) + A_2 h_1 - \sum_{j=0j}^{k-1} j(j-1)\left(G_j + S_j\right) + F''_2 + 2\chi_1 F'_1 \lambda E(X) + g''_1$$

$$\Omega(\mathrm{X},\lambda) = \begin{pmatrix} 3\mathrm{P}_1(h_1 F''_1 + 2h'_1 F'_1) + \mathrm{h}\mathrm{P}_2 F'_1 + A_1(2h''_1 + 2\mathrm{P}_1 h'_1 + 3\mathrm{P}_2 h_1) \\ +A_2(2h'_1 + \mathrm{P}_1 h_1) + A_1 h''_1 + 2\mathrm{P}_1 A_1 h'_1 + 2\mathrm{P}_1 A_2 h_1 \\ +5F'_1(\lambda E(X) + A_3 h_1 + F'''_2 \\ +3\chi_1(F'_1\left(\lambda X''(1) + F'_1 \lambda E(X)\right) + 5\chi'_1 F'_1 \lambda E(X) + g'''_1)(I'(1)) \\ -\sum_{j=0}^{k-1} j(j-1)(j-2)\left(G_j + S_j\right) + \left(3F'_1 \lambda X''(1) + 3F''_1 E(X)\right)\left(I(1)\right) \end{pmatrix}$$

$$\chi_1 = \frac{W_0(\lambda)}{\delta-(1-\lambda g_a)-(1-\lambda g_k)} \quad x''_1 = 2F'_1\big(B_1(1-\lambda g_a) + C_1(1-\lambda g_k)\big)$$

$$h_1 = \sum_{n=a}^{k-1}(\mathrm{S_n} + \mathrm{G_n} + W_n + \sum_{m=1}^{a-1} I_m \lambda g_{n-m}) - \sum_{n=a}^{k-1}\sum_{m=0}^{a-1}(V^1{}_m \lambda \mathrm{g_{n-m}} + V^2{}_m \lambda \mathrm{g_{n-m}})$$

$$g''_1 = x''_1 - 2(a+n+k) \quad g_1''' = 2(a+n+k)(x''_1) + x'''_1 + x''_1(a+n+k)$$

$$x'''_1 = 3F'_1 \begin{Bmatrix} B_1(k-\lambda g_k) + (C_2 + C_1 + B_2 + 1)(1-\lambda g_k)(1+3F''_1) \\ +C_1(k-\lambda g_k)(1+3F''_1) + C_1(n-\lambda g_k) \end{Bmatrix}$$

$$+3F''_1\big((B_1 + C_1)(1-\lambda g_k)\big)$$

$$n_1 = a+n+k; n_2 = n_1(a+n+k-1) \quad n_3 = n_2(a+n+k-2)$$

$$m''_1 = 2F'_1(B_1 + C_1) \quad m'''_1 = 3F'_1(2kB_1 + B_2 + 2nC_1 + C_2)$$

$$F_2 = I(1) - \delta \sum_{m=0}^{a-1} I_m \sum_{j=0}^{k-m-1} g_j - \lambda\big(\sum_{m=0}^{a-1} V_m{}^1 - \sum_{j=0}^{k-m-1} g_j\big)$$

$$-\lambda\big(\sum_{m=0}^{k-1} V_m{}^2 - \sum_{j=0}^{k-m-1} g_j\big)$$

$$F'_2 = I(1)E(X) + I'(1) - \lambda \sum_{m=0}^{a-1} mI_m \sum_{j=0}^{k-m-1} jg_j - \lambda\begin{pmatrix} \sum_{m=0}^{a-1} mV_m{}^1 + E(X)\sum_{m=0}^{k-1} V_m{}^1 \\ -\sum_{j=0}^{k-m-1} jg_j \end{pmatrix}$$

$$-\lambda\big(\sum_{m=0}^{k-1} mV_m{}^2 + E(X)\sum_{m=0}^{k-1} V_m{}^1 - \sum_{j=0}^{k-m-1} jg_j\big)$$

$$F''_2 = \lambda\big(I(1)\big)X''(1) - 3I'(1)E(X) + I''(1))$$

$$-\lambda \sum_{m=0}^{a-1} m(m-1)I_m \sum_{j=0}^{k-m-1} j(j-1)g_j$$

$$-\lambda\begin{pmatrix} \sum_{m=0}^{a-1} m(m-1)V_m{}^1 + 2E(X)\sum_{m=0}^{k-1} mV_m{}^1 + X''(1)\sum_{m=0}^{k-1} V_m{}^1 \\ -\sum_{j=0}^{k-m-1} j(j-1)g_j - \sum_{j=0}^{k-m-1} j(j-1)g_j \end{pmatrix}$$

$$-\lambda\big(\sum_{m=0}^{k-1} m(m-1)V_m{}^2 + 2E(X)\sum_{m=0}^{k-1} mV_m{}^2 + X''(1)\sum_{m=0}^{k-1} V_m{}^2\big)$$

$$F'''_3 = \lambda\big(I(1)X'''(1) - 3\delta X''(1) + 3I''(1)\lambda E(X) + I'''(1)\big) - \alpha_1 - \alpha_2$$

$$+3X''(1)\sum_{m=0}^{k-1} mV_m{}^2 - X'''(1) - 3\lambda X''(1) + 3I''(1)\lambda E(X)$$

$$+I'''(1)\sum_{m=0}^{a-1} m(m-1)(m-2)I_m \sum_{j=0}^{k-m-1} j(j-1)(j-2)g_j$$

$$\alpha_1 = \lambda\left(\begin{matrix}\sum_{m=0}^{k-1} m(m-1)(m-2){V_m}^1 + 3\lambda E(X)\sum_{m=0}^{k-1}\big(m(m-1) + X'''(1)\big){V_m}^1 \\ -\sum_{j=0}^{k-m-1} j(j-1)(j-2)g_j + 3\lambda E(X)\sum_{m=0}^{k-1} m(m-1){V_m}^2 +\end{matrix}\right)$$

$$\alpha_2 = \lambda\left(\begin{matrix}\sum_{m=0}^{k-1} m(m-1)(m-2){V_m}^2 + X'''(1)\big(3\sum_{m=0}^{a-1} m{V_m}^1 + \sum_{m=0}^{a-1}{V_m}^2\big) \\ -\sum_{j=0}^{k-m-1} j(j-1)(j-2)g_j\end{matrix}\right)$$

4.7.2 Expected Waiting Time in the Queue

The mean waiting time of the customers in the queue E (W) can be easily obtained using Little's formula.

$$\text{E (W)} = \frac{E(Q)}{\lambda E(X)}$$

4.7.3 Probability that the server is not busy during working vacation

$$\text{P(NBWV)} = \frac{W_0(\lambda)}{\lambda - (1-\lambda g_a) - (1-\lambda g_k)}$$

4.7.4 Probability that the server is not busy and the server is not on working vacation

$$P(I) = \frac{W_0(\lambda) + ({V^1}_{1n}(0) + {V^2}_{kn}(0))(\lambda - (1-\lambda g_a) - (1-\lambda g_k))}{\lambda - (1-\lambda g_a) - (1-\lambda g_k)}$$

4.7.5 Probability that the server is busy during working vacation

$$\text{P(BWV)} = \frac{(B_1 + C_1)(W(1) - W_0) - (\lambda g_a + \lambda g_k)W'(1)}{\lambda E(X)}$$

4.7.6 Probability that the server is busy without working vacation

$$P(B) = \frac{2(P_1)\big(\mathrm{k} - \lambda \mathrm{E}(\mathrm{A_b})\mathrm{E}(\mathrm{X})\big)h_1 + 2A_1(g_2) + A_2(g_1)}{2(\mathrm{k} - \lambda \mathrm{E}(\mathrm{A_b})\mathrm{E}(\mathrm{X}))(\lambda \mathrm{E}(\mathrm{X}))}$$

where

$$h_1 = \sum_{n=a}^{k-1}(\mathrm{S_n} + \mathrm{G_n} + W_n + \sum_{m=1}^{a-1} I_m \lambda g_{n-m})$$

$$-\sum_{n=a}^{k-1}\sum_{m=0}^{a-1}({V^1}_m \lambda \mathrm{g_{n-m}} + {V^2}_m \lambda \mathrm{g_{n-m}})$$

$$g_1 = h_1 - \sum_{n=0}^{k-1} \mathrm{S_n} + \mathrm{G_n} + W_n + I(1) - \delta \sum_{m=0}^{a-1} I_m \sum_{j=0}^{k-m-1} g_j$$

$$-\lambda\left(\sum_{m=0}^{a-1} V_m{}^1 - \sum_{j=0}^{k-m-1} g_j\right) - \lambda\left(\sum_{m=0}^{k-1} V_m{}^2 - \sum_{j=0}^{k-m-1} g_j\right)$$

$$g_2 = y_1(\lambda) + y_2(\lambda) + y_3(\lambda)$$

$$y_1(\lambda) = \sum_{n=a}^{k-1}(n-1)(S_n + G_n + W_n + \sum_{m=1}^{a-1} I_m \lambda g_{n-m}) - \sum_{n=0}^{k-1} n(S_n + G_n + W_n)$$

$$y_2(\lambda) = -\sum_{n=a}^{k-1}\sum_{m=0}^{a-1}(n-1)(V^1{}_m \lambda g_{n-m} + V^2{}_m \lambda g_{n-m}) - I(1)E(X)$$

$$y_3(\lambda) = I'(1) - \lambda \sum_{m=0}^{a-1} m I_m \sum_{j=0}^{k-m-1} j g_j - x_1 - x_1$$

$$x_1 = -\lambda\left(\sum_{m=0}^{a-1} m V_m{}^1 + E(X)\sum_{m=0}^{k-1} V_m{}^1 - \sum_{j=0}^{k-m-1} j g_j\right)$$

$$x_2 = -\lambda\left(\sum_{m=0}^{k-1} m V_m{}^2 + E(X)\sum_{m=0}^{k-1} V_m{}^1 - \sum_{j=0}^{k-m-1} j g_j\right)$$

4.8 Special Cases

The model so developed is general in nature as the service time is arbitrary. But for practical purposes, service time with a particular distribution is required. In this section, some special cases of the proposed model by specifying service time random variables as exponential, Erlang and hyper exponential distribution are discussed.

Case 1: $M^X/M(a,b)/1$ queue with two patterns of working vacation (Exponential service time both for the service during working vacation and for the service when the server is not on working vacation)

The probability density function of exponential service time is given as follows:

$s_i(x) = \mu_i e^{-\mu_i x}, i = 1, 2$ where μ is the parameter. Then

the PGF of the queue size distribution of this model can be obtained by

$$\widetilde{A_b}(\lambda - \lambda X(z)) = \widetilde{P_b}(\lambda - \lambda X(z)) = \left(\frac{\mu_1}{\mu_1 + \lambda(1 - X(z))}\right)$$

$$\widetilde{A_v}(\lambda - \lambda X(z)) = \widetilde{P_v}(\lambda - \lambda X(z)) = \left(\frac{\mu_2}{\mu_2 + \lambda(1 - X(z))}\right)$$

Substituting in Eqn. 4.40, the PGF of the queue size distribution for single server queue with two patterns of working vacation policy is given as

$$P(z) = \frac{\left(\left(\frac{\mu_1}{\mu_1+\lambda(1-X(z))}\right)-1\right)h(P_b,\ \lambda)F_1+\left(\left(\frac{\mu_1}{\mu_1+\lambda(1-X(z))}\right)-1\right)\left(\left(\frac{\mu_1}{\mu_1+\lambda(1-X(z))}\right)h(P_b,\lambda)-\sum_{j=0}^{k-1}c_jz^j+F_2\right) \\ +\Psi(P_b,\lambda)F_1\{(-\lambda+\lambda X(z))+g(P_b,\lambda)\}+F_1(-\lambda+\lambda X(z))\{M(P_b,\lambda)W_0+I(z)\}+F_3}{(-\lambda+\lambda X(z))F_1}$$

where

$$h(P_b,\lambda) = \sum_{n=a}^{k-1}(C_n + W_n + \sum_{m=1}^{a-1} I_m\lambda g_{n-m})z^{n-1}$$

$$+\sum_{m=0}^{a-1}\tilde{C}_{1m}(\lambda-\lambda X(z))\lambda g_{a+1-m} - \sum_{m=0}^{a-1}\tilde{C}_{1m}(\theta)\lambda g_{a+1-m}$$

$$+\sum_{m=0}^{a-1}\widetilde{D}_{km}(\lambda-\lambda X(z))\lambda g_{a+1-m} - \sum_{m=0}^{a-1}\widetilde{D}_{km}(\theta)\lambda g_{a+1-m}$$

$$g(P_b,\lambda) = \frac{F_1\left(\widetilde{P_v}\left(\left(\frac{\mu_2}{\mu_2+\lambda(1-X(z))}\right)-1\right)(z^k-\lambda g_a)+\widetilde{A_v}\left(\left(\frac{\mu_2}{\mu_2+\lambda(1-X(z))}\right)-1\right)(z^n-\lambda g_k)\right)}{z^{a+n+k}}$$

$$\Psi(P_b,\lambda) = \frac{W_0\left(\lambda z^{n+a+k}-\left(\frac{\mu_2}{\mu_2+\lambda(1-X(z))}\right)z^{n+k}-\left(\frac{\mu_2}{\mu_2+\lambda(1-X(z))}\right)z^{n+a}\right)}{\lambda z^{n+a+k}-\left(\frac{\mu_2}{\mu_2+\lambda(1-X(z))}\right)(z^{n+k}-\lambda z^{n+a+k})-\left(\frac{\mu_2}{\mu_2+\lambda(1-X(z))}\right)(z^{n+a}-\lambda g_k z^{n+a+k})}$$

$$M(P_b,\lambda) = \frac{F_1\left(\left(\left(\frac{\mu_2}{\mu_2+\lambda(1-X(z))}\right)-1\right)z^k+\left(\left(\frac{\mu_2}{\mu_2+\lambda(1-X(z))}\right)-1\right)z^n\right)}{z^{a+n+k}}$$

$$F_1 = \left(z^k - \left(\frac{\mu_1}{\mu_1+\lambda(1-X(z))}\right)\right)$$

$$F_2 = \begin{pmatrix} I(z)X(z) - \lambda\sum_{m=0}^{a-1} I_m z^m \sum_{j=0}^{k-m-1} g_j z^j \\ +\lambda\left(\sum_{i=0}^{a-1} z^i\left(\tilde{E}_{1m}(\lambda-\lambda X(z)) - \tilde{E}_{1m}(\theta)\right)\left(X(z)-\sum_{j=0}^{k-m-1} g_j z^j\right)\right) \\ -\lambda\left(\sum_{i=0}^{a-1} z^{i-b}\left(\tilde{C}_{1m}(\lambda-\lambda X(z))\widetilde{P_b}(\theta) - \tilde{C}_{1m}(\theta)\widetilde{P_b}(\lambda-\lambda X(z))\right)\right)\end{pmatrix} -$$

$$F_3 = \begin{pmatrix} \lambda\left(\sum_{i=0}^{a-1} z^i \left(\tilde{F}_{km}(\lambda-\lambda X(z)) - \tilde{F}_{km}(\theta)\right)\left(X(z) - \sum_{j=0}^{k-m-1} g_j z^j\right)\right) \\ -\lambda\left(\sum_{i=0}^{a-1} z^{i-b}\left(\widetilde{D}_{km}(\lambda-\lambda X(z))\widetilde{P_b}(\theta) - \widetilde{D}_{km}(\theta)\left(\frac{\mu_1}{\mu_1+\lambda(1-X(z))}\right)\right)\right) \end{pmatrix}$$

Case. 2: M^X / hyper exponential / 1 queue with two patterns of working vacation (Hyper exponential service time both for the service during working vacation and for the service when the server is not on working vacation)

When the service time follows hyper exponential distribution with probability density function, then $a(x) = cde^{-dx} + (1-c)\, fe^{-fx}$, where d and f are parameters, then,

$$\widetilde{A_b}(\lambda-\lambda X(z)) = \widetilde{P_b}\,(\lambda-\lambda X(z)) = \left(\frac{d_1 c}{d_1+(\lambda-\lambda x(z))}\right) + \left(\frac{f_1(1-c)}{f_1+(\lambda-\lambda x(z))}\right)$$

$$\widetilde{A_v}(\lambda-\lambda X(z)) = \widetilde{P_v}(\lambda-\lambda X(z)) = \left(\frac{d_2 c}{d_2+(\lambda+\eta-\lambda x(z))}\right) + \left(\frac{f_2(1-c)}{f_2+(\lambda+\eta-\lambda x(z))}\right)$$

Substituting in Eqn. 4.40, the PGF of the queue size distribution for single server queue with two patterns of working vacation policy is given as

$$P(z) = \frac{\begin{array}{c}\left(\left(\frac{d_1 c}{d_1+(\lambda-\lambda x(z))}+\frac{f_1(1-c)}{f_1+(\lambda-\lambda x(z))}\right)-1\right)h(P_b,\lambda)F_1 \\ +\left(\left(\frac{d_1 c}{d_1+(\lambda-\lambda x(z))}+\frac{f_1(1-c)}{f_1+(\lambda-\lambda x(z))}\right)-1\right)\left(\left(\frac{d_1 c}{d_1+(\lambda-\lambda x(z))}+\frac{f_1(1-c)}{f_1+(\lambda-\lambda x(z))}\right)h(P_b,\lambda)-\sum_{j=0}^{k-1} c_j z^j+F_2\right) \\ +\Psi(P_b,\lambda)F_1\{(-\lambda+\lambda X(z))+g(P_b,\lambda)\}+F_1(-\lambda+\lambda X(z))\{M(P_b,\lambda)W_0+I(z)\}+F_3\end{array}}{(-\lambda+\lambda X(z))F_1}$$

where

$$F_1 = \left(z^k - \left(\frac{d_1 c}{d_1+(\lambda-\lambda x(z))}\right) + \left(\frac{f_1(1-c)}{f_1+(\lambda-\lambda x(z))}\right)\right)$$

$$\begin{aligned} h(P_b,\lambda) = {} & \sum_{n=a}^{k-1}(C_n + W_n + \sum_{m=1}^{a-1} I_m \lambda g_{n-m}) z^{n-1} \\ & + \sum_{m=0}^{a-1} \tilde{C}_{1m}(\lambda-\lambda X(z))\lambda g_{a+1-m} - \sum_{m=0}^{a-1} \tilde{C}_{1m}(\theta)\lambda g_{a+1-m} \\ & + \sum_{m=0}^{a-1} \widetilde{D}_{km}(\lambda-\lambda X(z))\lambda g_{a+1-m} - \sum_{m=0}^{a-1} \widetilde{D}_{km}(\theta)\lambda g_{a+1-m} \end{aligned}$$

$$g(P_b,\lambda)=\frac{F_1\begin{pmatrix}\left(\left(\frac{d_2c}{d_2+(\lambda-\lambda x(z))}+\frac{f_2(1-c)}{f_2+(\lambda-\lambda x(z))}\right)-1\right)(z^k-\lambda g_a)\\+\left(\left(\frac{d_2c}{d_2+(\lambda-\lambda x(z))}+\frac{f_2(1-c)}{f_2+(\lambda-\lambda x(z))}\right)-1\right)(z^n-\lambda g_k)\end{pmatrix}}{z^{a+n+k}}$$

$$\Psi(P_b,\lambda)=\frac{W_0\left(\lambda z^{n+a+k}-\widetilde{P_v}(\lambda-\lambda X(z))z^{n+k}-\widetilde{A_v}(\lambda-\lambda X(z))z^{n+a}\right)}{\begin{pmatrix}\lambda z^{n+a+k}-\left(\frac{d_2c}{d_2+(\lambda-\lambda x(z))}+\frac{f_2(1-c)}{f_2+(\lambda-\lambda x(z))}\right)(z^{n+k}-\lambda z^{n+a+k})\\-\left(\frac{d_2c}{d_2+(\lambda-\lambda x(z))}+\frac{f_2(1-c)}{f_2+(\lambda-\lambda x(z))}\right)(z^{n+a}-\lambda g_k z^{n+a+k})\end{pmatrix}}$$

$$M(P_b,\lambda)=\frac{F_1\left(\left(\left(\frac{d_2c}{d_2+(\lambda-\lambda x(z))}+\frac{f_2(1-c)}{f_2+(\lambda-\lambda x(z))}\right)-1\right)z^k+\left(\left(\frac{d_2c}{d_2+(\lambda-\lambda x(z))}+\frac{f_2(1-c)}{f_2+(\lambda-\lambda x(z))}\right)-1\right)z^n\right)}{z^{a+n+k}}$$

$$F_2=I(z)X(z)-\lambda\sum_{m=0}^{a-1}I_m z^m\sum_{j=0}^{k-m-1}g_j z^j-\lambda\left(X(z)\sum_{m=0}^{a-1}{V_m}^1 z^m-\sum_{j=0}^{k-m-1}g_j z^j\right)$$

$$-\lambda\left(X(z)\sum_{m=0}^{k-1}{V_m}^2 z^m-\sum_{j=0}^{k-m-1}g_j z^j\right)$$

$$F_2=\begin{pmatrix}I(z)X(z)-\lambda\sum_{m=0}^{a-1}I_m z^m\sum_{j=0}^{k-m-1}g_j z^j\\+\lambda\left(\sum_{i=0}^{a-1}z^i\left(\tilde{E}_{1m}(\lambda-\lambda X(z))-\tilde{E}_{1m}(\theta)\right)\left(X(z)-\sum_{j=0}^{k-m-1}g_j z^j\right)\right)\\-\lambda\left(\sum_{i=0}^{a-1}z^{i-b}\left(\tilde{C}_{1m}(\lambda-\lambda X(z))\widetilde{P_b}(\theta)-\tilde{C}_{1m}(\theta)\widetilde{P_b}(\lambda-\lambda X(z))\right)\right)\end{pmatrix}-$$

$$F_3=\begin{pmatrix}\lambda\left(\sum_{i=0}^{a-1}z^i\left(\tilde{F}_{km}(\lambda-\lambda X(z))-\tilde{F}_{km}(\theta)\right)\left(X(z)-\sum_{j=0}^{k-m-1}g_j z^j\right)\right)\\-\lambda\left(\sum_{i=0}^{a-1}z^{i-b}\left(\widetilde{D}_{km}(\lambda-\lambda X(z))\widetilde{P_b}(\theta)-\widetilde{D}_{km}(\theta)\left(\frac{d_1c}{d_1+(\lambda-\lambda x(z))}+\frac{f_1(1-c)}{f_1+(\lambda-\lambda x(z))}\right)\right)\right)\end{pmatrix}$$

Case. 3: Single server bulk arrival queue with Erlangian service time and two patterns of working vacation (Erlang service time both for the service during working vacation and for the service when the server is not on working vacation)

When the service time follows k - Erlang distribution with probability density function, then $a(x) = \frac{(k_i\mu_i)^{k_i} x^{k_i-1} e^{-(k_i\mu_i x)}}{(k_i-1)!}$, $i = 1,2$, $\mu_i > 0$, $x > 0$, k_i is a positive integer and μ_i is the parameter, then

$$\widetilde{A_b}\big(\lambda - \lambda X(z)\big) = \widetilde{P_b}\,(\lambda - \lambda X(z)) = \frac{(k_1\mu_1)^{k_1} x^{k_1-1} e^{-(k_1\mu_1 x)}}{(k_1-1)!},$$

$$\widetilde{A_v}\big(\lambda - \lambda X(z)\big) = \widetilde{P_v}\big(\lambda - \lambda X(z)\big) = \frac{(k_2\mu_2)^{k_2} x^{k_2-1} e^{-(k_2\mu_2 x)}}{(k_2-1)!}$$

Substituting in Eqn. 4.40, the PGF of the queue size distribution for single server queue with two patterns of working vacation policy is obtained.

4.9 Numerical Illustrations

In this section theoretical results are justified with numerical examples. To study the effect of arrival rate on performance measures the following notations are used and some assumptions are made.

Batch size distribution of the arrival is geometric with mean	2
Regular service time with single service is 2-Erlang with parameter	$\mu^1{}_b$
Regular service time with batch service is 4- Erlang with parameter	$\mu^2{}_b$
Service time (working vacation) with single service is exponential with parameter	$\mu^1{}_v$
Service time (working vacation) with batch service is exponential with parameter	$\mu^2{}_v$
Vacation duration is exponential with parameter	η

An influence of arrival rate on expected queue length and expected waiting time is given in Table 4.2, Fig. 4.2 and Fig. 4.3 shows the way in which the expected queue length and the expected waiting time in the queue change for increasing the values of arrival rate. The service times are considered as exponential, Erlang-2 and hyper exponential with the subsequent assumptions $\mu^1{}_b = 4$; $\mu^2{}_b = 3$; $\mu^1{}_v = 3$; $\mu^2{}_v = 2; \eta = 2$.

It is observed that

- When arrival rate increases expected queue length increases.
- Expected waiting time in the queue increases when arrival rate increases.

Table 4.3 presents the effect of arrival rate on the probability that the server is busy during working vacation with ${\mu^1}_b = 4$; ${\mu^2}_b = 3$; ${\mu^1}_v = 3$; ${\mu^2}_v = 2$; $\eta = 2$ when the service time distribution follows Exponential, Erlang-2 and Hyper exponential respectively.

- Probability that the server is busy during working vacation increases when arrival rate increases.

Table 4.4 presents the effect of arrival rate on the Probability that the server is busy but not in working vacation with ${\mu^1}_b = 4$; ${\mu^2}_b = 3$; ${\mu^1}_v = 3$; ${\mu^2}_v = 2$; $\eta = 2$ when the service time distribution follows exponential, Erlang-2 and hyper exponential respectively.

- Probability that the server is busy but not in working vacation increases when arrival rate increases.

Table 4.5 presents the effect of arrival rate on the probability that the server is not busy during working vacation with ${\mu^1}_b = 4$; ${\mu^2}_b = 3$; ${\mu^1}_v = 3$; ${\mu^2}_v = 2$; $\eta = 2$ when the service time distribution follows exponential, Erlang-2 and hyper exponential respectively.

- Probability that the server is not busy during working vacation decreases when arrival rate increases.

Table 4.6 presents the effect of arrival rate on the probability that the server is not busy and the server is not on working vacation with ${\mu^1}_b = 4$; ${\mu^2}_b = 3$; ${\mu^1}_v = 3$; ${\mu^2}_v = 2$; $\eta = 2$ when the service time distribution follows exponential, Erlang-2 and hyper exponential respectively.

- Probability that the server is not busy and the server is not on working vacation decreases when arrival rate increases.

Table 4.2 Arrival Rate versus Performance Measures

λ	Exponential		Erlang-2		Hyper-Exponential	
	E(Q)	**E(W)**	**E(Q)**	**E(W)**	**E(Q)**	**E(W)**
2.0	0.9532	0.5284	0.9274	0.3972	0.9418	0.4931
2.5	1.6429	1.1763	1.1842	0.8470	1.3917	0.9527
3.0	3.0659	1.9527	1.9273	1.3859	2.5292	1.7762
3.5	4.8427	2.6439	3.2984	1.8321	3.5792	2.1863
4.0	6.1842	3.8215	4.6428	2.3752	5.8251	2.9528
4.5	7.6214	4.6328	5.9273	3.5528	6.5295	4.1736

Table 4.3 Arrival Rate versus Probability that the Server is Busy with Working Vacation

λ	**P(BWV)**		
	Exponential	Erlang-2	Hyper-Exponential
0.3	0.0453	0.0459	0.0457
0.5	0.0464	0.0468	0.0466
0.7	0.0472	0.0476	0.0474
0.9	0.0473	0.0480	0.0476
1.1	0.0474	0.0485	0.0481

Table 4.4 Arrival Rate versus Probability that the Server is Busy but not in Working Vacation

λ	P(BNWV)		
	Exponential	Erlang-2	Hyper-Exponential
0.3	0.0872	0.0883	0.0875
0.5	0.0891	0.0901	0.0897
0.7	0.1032	0.1041	0.1035
0.9	0.1421	0.1432	0.1427
1.1	0.1735	0.1745	0.1739

Table. 4.5 Arrival Rate versus Probability that the Server is not Busy During Working Vacation

λ	P(NBWV)		
	Exponential	Erlang-2	Hyper-Exponential
0.3	0.0027	0.0035	0.0032
0.5	0.0025	0.0032	0.0030
0.7	0.0023	0.0029	0.0027
0.9	0.0020	0.0028	0.0025
1.1	0.0018	0.0025	0.0023

Table 4.6 Arrival Rate versus Probability that the Server is not Busy and the Server is not on Working Vacation

λ	$P(I)$		
	Exponential	Erlang-2	Hyper-Exponential
0.3	0.0722	0.0745	0.0734
0.5	0.0637	0.0651	0.0640
0.7	0.0569	0.0590	0.0575
0.9	0.0343	0.0372	0.0359
1.1	0.0076	0.0098	0.0087

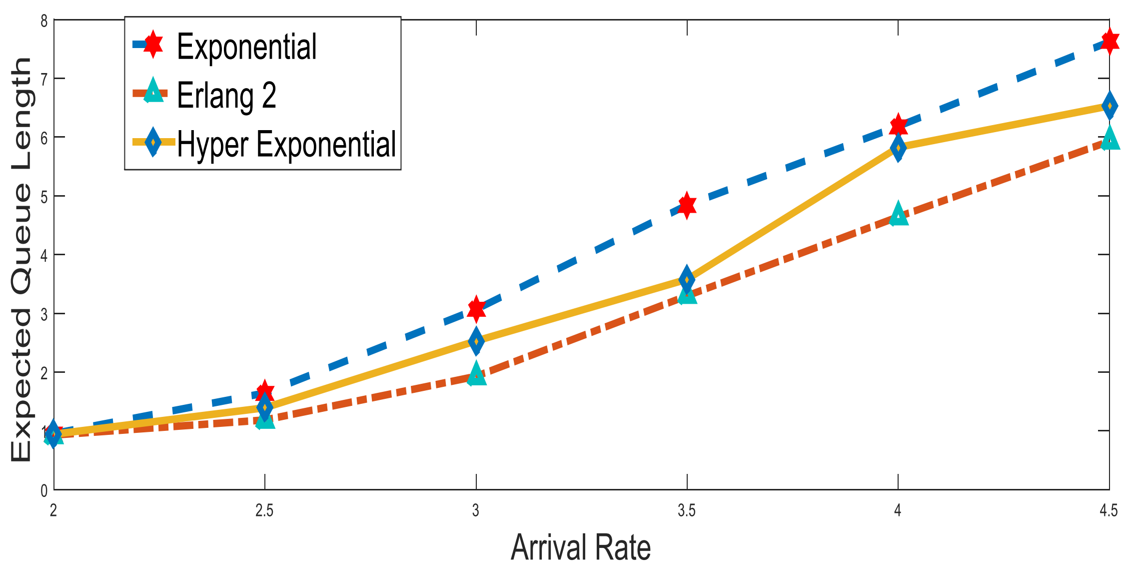

Fig. 4.2 Arrival Rate versus Expected Queue Length

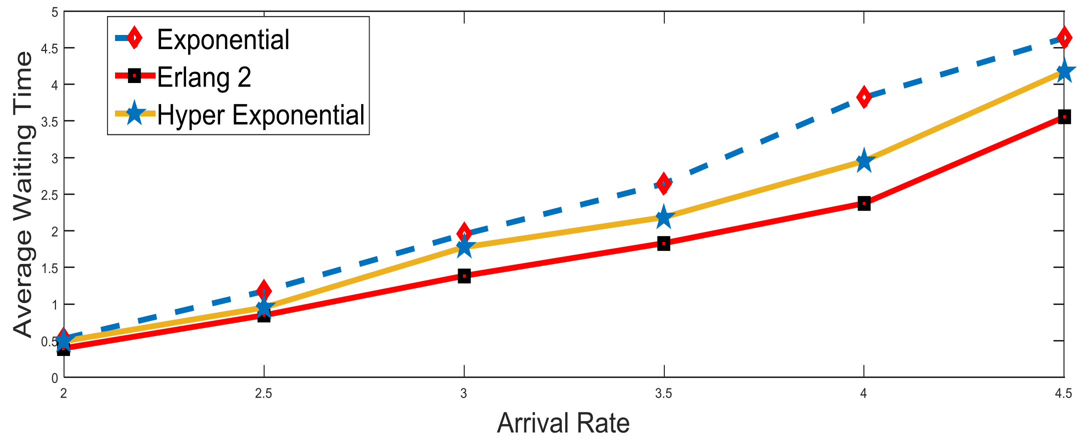

Fig. 4.3 Arrival Rate versus Average Waiting Time in the Queue

4.10 Conclusion

This chapter has analyzed single server and bulk arrival queueing system with batch size dependent service and two patterns of working vacation. The server provides service in two service modes as single service and fixed batch service depending upon the queue length. The server affords service in two service modes during working vacation also. For the designed queueing system, probability generating function of the queue size at an arbitrary time is derived. Some system performance measures are computed in steady state. The theoretical development of the model has been justified with numerical illustrations.