CHAPTER 7

Stable-Linnik Processes and Generalizations

7.1 Introduction


Laplace distribution and its generalizations are of important consideration in statistical literature, because of the nice properties held by them. A widely used generalization of Laplace distribution is Linnik distribution (Linnik, 1953). A convolution of stable and Linnik

Some results included in this chapter form part of the paper Lishamol and Jose (2009).
(also known as \(\alpha\)-Laplace) distributions are significant in the same background, having the combined properties.


In this chapter, a convolution of stable and Linnik distributions is introduced. Stable-Linnik distribution is studied and its properties are discussed. The corresponding first order autoregressive processes are developed. Generalizations of stable-Linnik distribution namely, semi stable-Linnik, generalized semi stable-Linnik, geometric semi stable-Linnik, geometric generalized semi stable-Linnik, and bivariate semi stable-Linnik distribution are studied. Autoregressive processes corresponding to the generalizations are also discussed. Discrete stable-Linnik distribution and processes are developed.

### 7.2 Stable-Linnik Distribution

Stable-Linnik (StLi) distribution can be obtained as the convolution of stable and Linnik random variables. The characteristic function (cf) of StLi random variable is given by,

\[
\phi(t) = e^{-|t|^\alpha_1} \frac{1}{1 + |t|^\alpha_2}, \quad 0 < \alpha_1, \alpha_2 \leq 2.
\]  

#### 7.2.1 Properties of Stable-Linnik Distribution

1. Closed under linear transformations

Consider the linear transformation, \(Z = aX + b\). The cf of \(Z\) can be obtained as, \(\phi_Z(t) = \phi_{aX+b}(t) = e^{-|t|^\alpha_1 + ibt} \left( \frac{1}{1 + |at|^\alpha_2} \right)\), which is in the form of the cf in (7.2.1), and is a shewed
StLi random variable.

2. Infinite Divisibility

The cf in (7.2.1) can be rewritten as,

$$
\phi_X(t) = \left[ e^{-\frac{|t|^\alpha_1}{\alpha_1}} \left( \frac{1}{1 + |t|^\alpha_2} \right)^\frac{1}{\pi} \right]^n
$$

for any integer $n > 0$. Here the term in square bracket is the cf of a generalized StLi (see Remark 7.2.1) random variable. Hence StLi is infinitely divisible (id). As StLi is a convolution of stable and Linnik distributions which is not geometrically infinitely divisible (gid), it follows that the StLi is not gid.

3. Self-decomposability

Consider the cf of StLi,

$$
\phi_X(t) = e^{-|t|^\alpha_1} \frac{1}{1 + |t|^\alpha_2}
$$

$$
= e^{-|at|^\alpha_1} \frac{1}{1 + |at|^\alpha_2}
$$

$$
\times e^{-(1-a^\alpha_1)|t|^\alpha_1} \left( \frac{1 + a^\alpha_2 |t|^\alpha_2}{1 + |t|^\alpha_2} \right)
$$

$$
= \phi_X(at) \phi_a(t)
$$

where $\phi_a(t)$ is a cf as in (7.3.2). Hence the StLi is self-decomposable.

4. Related Distributions

For various combinations of values of $\alpha_1$ and $\alpha_2$, we get many distributions as special cases. When $\alpha_1 = 2$, $\alpha_2 = 2$ it reduces to normal-Laplace distribution (Reed, 2004). If only $\alpha_2 = 2$ it reduces to stable-Laplace distribution. Some generalizations of StLi distributions can be obtained as follows:
Semi Stable-Linnik Distribution

Semi Stable-Linnik random variable is obtained as the convolution of semi-stable and semi Linnik random variables and is having the cf of the form

\[ \phi_X(t) = e^{-\psi_1(t)} \frac{1}{1 + \psi_2(t)} \]  
(7.2.2)

where \( \psi_i(t) \) satisfies certain conditions.

**Definition 7.2.1.** A distribution function \( F \) of a random variable \( X \) with cf in (7.2.2) is called semi stable-Linnik if \( \psi_j(t) \) of (7.2.2) satisfies the property

\[ \psi_j(t) = c^j \psi_j(c^{1/\alpha_j} t), \quad 0 < \alpha_j \leq 2, \quad 0 < c < 1, \quad j = 1, 2. \]  
(7.2.3)

A solution of the functional equation (7.2.3) can be obtained as, \( \psi_j(t) = |t|^{\alpha_j} h_j(t), \quad j = 1, 2 \) where \( h_j(t) \) is a periodic function in \( \ln t \) with period \( \frac{2\pi}{\ln \alpha_j} \). When \( \psi_j(t) = |t|^{\alpha_j} \), we have the StLi random variable.

**Generalized Semi Stable-Linnik Distribution**

Generalized semi stable-Linnik distribution is defined as the one with cf,

\[ e^{-\psi_1(t)} \left[ \frac{1}{1 + \psi_2(t)} \right]^\beta. \]

**Remark 7.2.1.** As a subclass of generalized semi stable-Linnik distribution, we can define generalized StLi distribution, when \( \psi_j(t) = |t|^{\alpha_j} \).

**Bivariate Semi Stable-Linnik Distribution**

The bivariate semi stable-Linnik distribution is obtained as the convolution of bivariate semi stable and bivariate semi Linnik distributions.

**Definition 7.2.2.** The distribution function of a random variable \( X \) is called bivariate
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A semi stable-Linnik if its cf is given by,

$$\phi(t_1, t_2) = e^{-\psi_1(t_1, t_2)} \frac{1}{1 + \psi_2(t_1, t_2)}$$

if $$\psi_j(t_1, t_2)$$ satisfies the property

$$\psi_j(t_1, t_2) = \frac{1}{a} \psi_j(a^{\frac{1}{\alpha_1}} t_1, a^{\frac{1}{\alpha_2}} t_2), \quad 0 < \alpha_1, \alpha_2 \leq 2, \quad 0 < a < 1.$$

### 7.3 Stable-Linnik Processes

Let $$\{\epsilon_n, \ n \geq 1\}$$ be a sequence of independently and identically distributed (iid) random variables. Define $$\{X_n, n \geq 1\}$$ by a first order autoregressive (AR(1)) model given by,

$$X_n = aX_{n-1} + \epsilon_n; \quad |a| < 1 \text{ and } \forall \ n > 0. \quad (7.3.1)$$

Here $$X_n$$ depends on $$X_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_n$$ and is independent of $$\{\epsilon_i, i > n\}$$. Suppose that $$\{X_n\}$$ has the StLi distribution with cf (7.2.1). Then assuming stationarity, (7.3.1) satisfies the relation,

$$\phi_\epsilon(t) = \frac{\phi_X(t)}{\phi_X(at)}$$

and hence on substitution,

$$\phi_\epsilon(t) = e^{-(1-|a|^{\alpha_1})|t|^{\alpha_1}} \frac{1 + |at|^{\alpha_2}}{1 + |t|^{\alpha_2}}. \quad (7.3.2)$$

**Remark 7.3.1.** A random variable which has an atom of mass ‘a’ at 0 and which is Linnik distributed with probability $$(1 - a)$$ for non-zero values is called Linnik tailed random variable.

Hence $$\epsilon$$ is a convolution of stable and Linnik tailed random variables, where the Linnik tailed random variable has an atom of mass $$|a|^{\alpha_2}$$ at 0 and is Linnik distributed for non-zero values.
7.3.1 Properties of Stable-Linnik Processes

1. The distribution of $\epsilon$ can be obtained as a mixture distribution given by,

$$
\epsilon = \begin{cases} 
Y & \text{with probability } |a|^{\alpha_2} \\
X & \text{with probability } (1 - |a|^{\alpha_2})
\end{cases}
$$

where $Y$ is stable distribution and $X$ is StLi. Hence the probability density function (pdf) of $\epsilon$ can be obtained as,

$$
g_{\epsilon}(x) = |a|^{\alpha_2}f_Y(x) + (1 - |a|^{\alpha_2})f_X(x).
$$

2. Since the distribution of $\epsilon$ has no zero component, the StLi model is free from zero deficiency.

3. The AR(1) process $X_n = aX_{n-1} + \epsilon_n$, $|a| < 1$ is strictly stationary with StLi marginal distribution with cf as in (7.2.1) if and only if $\{\epsilon_n\}$ are iid with cf as defined in (7.3.2) provided $X_0 \sim$ StLi and is independent of $\epsilon_1$.

4. If $X_0$ is distributed arbitrarily, then also the process is asymptotically Markovian with StLi distribution, provided $\epsilon$ is as in (7.3.2).

5. If $T_r = X_n + X_{n+1} + \cdots + X_{n+r-1}$, then the distribution of $T_r$ is uniquely determined by the cf,

$$
\phi_{T_r}(t) = \phi_{X_n} \left( \frac{1 - a^r}{1 - a} t \right) \prod_{j=1}^{r-1} \phi_{\epsilon} \left( \frac{1 - a^{r-j}}{1 - a} t \right)
$$

$$
= e^{-\frac{|1-a^r|t^{\alpha_1}}{1+|1-a^r|t^{\alpha_2}}} \prod_{j=1}^{r-1} e^{-\frac{(1-|a|^\alpha_1)|1-a^{r-j}|t^{\alpha_1}}{1+|1-a^{r-j}|t^{\alpha_2}}} \left( \frac{1 + |a|^{1-a^{r-j}}t^{\alpha_2}}{1 + |1-a^{r-j}|t^{\alpha_2}} \right).
$$

6. The joint distribution of contiguous observations $(X_n, X_{n+1})$ of the process can be
obtained in terms of the bivariate $\phi$,
\[
\phi_{X_n, X_{n+1}}(t) = e^{-(t_1 + at_2)^{\alpha_1} + (1-|a|^{\alpha_1})|t_2|^{\alpha_1}} \left( \frac{1 + |at_2|^{\alpha_2}}{(1 + |t_1 + at_2|^{\alpha_2})(1 + |t_2|^{\alpha_2})} \right).
\]

The obtained expression is not symmetric in $t_1$ and $t_2$. Hence it follows that the process with StLi marginals is not time reversible.

### 7.3.2 Geometric Semi Stable-Linnik Distribution and Processes

We have the StLi distribution and hence the semi stable-Linnik distributions are id. Now,

\[
e^{-\psi_1(t)} \frac{1}{1 + \psi_2(t)} = \exp \left\{ 1 - \frac{1}{\left[ 1 + \psi_1(t) + \log(1 + \psi_2(t)) \right]^{-1}} \right\}.
\]

Hence $\frac{1}{1 + \psi_1(t) + \log(1 + \psi_2(t))}$ is gid (Klebanov et al., 1984).

A distribution with $\phi = \frac{1}{1 + \psi_1(t) + \log(1 + \psi_2(t))}$ is called geometric semi stable-Linnik distribution.

Like semi stable-Linnik distribution mentioned above, the geometric generalized semi stable-Linnik distribution corresponding to generalized semi stable-Linnik distribution is obtained as the one with $\phi$,

\[
\frac{1}{1 + \psi_1(t) + \beta \log (1 + \psi_2(t))}
\]

**Remark 7.3.2.** Geometric StLi distribution can be defined as the distribution with $\phi$ and the $\phi = \frac{1}{1 + |t|^{\alpha_1 + \beta \log (1 + |t|^{\alpha_2})}}$ corresponds to geometric generalized StLi distribution.

**Theorem 7.3.1.** Geometric semi stable-Linnik distribution is the limit distribution of geometric sum of generalized semi stable-Linnik variables.

**Proof:**

\[
\phi_n(t) = \left\{ 1 + n \left[ \left( \frac{1 + \psi_2(t)}{e^{-\psi_1(t)}} \right)^{\frac{1}{n}} - 1 \right] \right\}^{-1}.
\]  

(7.3.3)

By Lemma 3.2 of Pillai (1990), (7.3.3) is the $\phi$ of the geometric sum of iid generalized
semi stable-Linnik variables. Taking limit as $n \to \infty$,

$$
\phi(t) = \lim_{n \to \infty} \phi_n(t)
= \left\{ 1 + \lim_{n \to \infty} n \left[ \left( \frac{1 + \psi_2(t)}{e^{-\psi_1(t)}} \right)^{\frac{1}{n}} - 1 \right] \right\}^{-1}
= \frac{1}{[1 + \psi_1(t) + \log(1 + \psi_2(t))]}.
$$

Then for the geometric generalized semi stable-Linnik distribution we have the following theorem.

**Theorem 7.3.2.** Geometric generalized semi stable-Linnik distribution is the limit distribution of geometric sum of generalized semi stable-Linnik variables.

**Proof:** By Lemma 3.2 of Pillai (1990), following is the $cf$ of the geometric sum of iid generalized semi stable-Linnik variables.

$$
\phi_n(t) = \left\{ 1 + n \left[ \left( \frac{1 + \psi_2(t)}{e^{-\psi_1(t)}} \right)^{\frac{1}{n}} - 1 \right] \right\}^{-1}.
$$

Then by taking limit as $n \to \infty$ of $\phi_n(t)$, the theorem follows.

For the first order autoregressive processes with geometric semi stable-Linnik as marginals, the following structure for $\{X_n\}$ is considered.

$$
X_n = \begin{cases} 
\epsilon_n, & \text{with probability } p \\
X_{n-1} + \epsilon_n, & \text{with probability } 1 - p.
\end{cases} \tag{7.3.4}
$$

In terms of $cf$ the model defined in (7.3.4) can be given as

$$
\phi_{X_n}(t) = p\phi_{\epsilon_n}(t) + (1 - p)\phi_{X_{n-1}}(t) \phi_{\epsilon_n}(t). \tag{7.3.5}
$$
Assuming stationarity, we have
\[ \phi_\epsilon(t) = \frac{\phi_X(t)}{p + (1 - p)\phi_X(t)} = \frac{1}{[1 + p\psi_1(t) + p\log (1 + \psi_2(t))]} . \]

Hence the innovation sequence \( \{\epsilon_n\} \) is distributed \( iid \) as geometric generalized semi stable-Linnik if and only if \( \{X_n\} \) is distributed marginally as geometric semi stable-Linnik.

Now we construct an AR(1) model with geometric generalized semi stable-Linnik marginals. Consider an autoregressive process \( \{X_n\} \) with structure given by (7.3.4). Suppose that \( \{X_n\} \) has the geometric generalized semi stable-Linnik distribution. Then with the assumption of stationarity, (7.3.5) gives
\[ \phi_\epsilon(t) = \frac{1}{[1 + p\psi_1(t) + p\beta \log (1 + \psi_2(t))]} . \]

Hence \( \{\epsilon_n\} \) also is distributed as geometric generalized semi stable-Linnik.

### 7.3.3 Stable-Gamma distribution and Processes

The stable-gamma distribution (SG) results from the convolution of independent stable and gamma components. In terms of \( cf \), the SG can be defined as which, for all real \( x \) is,
\[ \phi_X(t) = e^{i\mu t - \frac{c}{2}t^\alpha (1 + i\gamma \frac{c}{2}t^\alpha \omega(t, \alpha))} \left( \frac{1}{1 - i\lambda_1 t} \right)^{\beta_1} \left( \frac{1}{1 + i\lambda_2 t} \right)^{\beta_2} \]
where \( \mu, \alpha, \gamma \) and \( c \) are constants with \( c \geq 0, \ 0 < \alpha \leq 2, \ |\gamma| \leq 1 \) and
\[ \omega(t, \alpha) = \begin{cases} \tan \frac{\pi \alpha}{2} & \text{if } \alpha \neq 1 \\ \frac{2}{\pi} \ln |t| & \text{if } \alpha = 1 \end{cases} \]
We shall write \( X \sim SG(\mu, \alpha, \gamma, c, \lambda_1, \lambda_2, \beta_1, \beta_2) \) to indicate that a random variable \( X \) has such a distribution. For convenience we are considering the form,

\[
\phi_X(t) = e^{-ct|t|^\alpha} \left( \frac{1}{1 - i\lambda_1 t} \right)^{\beta_1} \left( \frac{1}{1 + i\lambda_2 t} \right)^{\beta_2}.
\] (7.3.6)

That is, we are considering the random variable, \( X \sim SG(0, \alpha, 0, c, \lambda_1, \lambda_2, \beta_1, \beta_2) \). Since it is a convolution, \( X \) can be represented as,

\[ X \overset{d}{=} Y + G_1 - G_2 \]

Where \( Y, G_1 \) and \( G_2 \) are independent random variables with \( Y \) is stable, denoted by \( Y \sim S(0, \alpha, 0, c) \), \( G_1 \) and \( G_2 \) are independent gamma with parameters \((\lambda_1, \beta_1)\) and \((\lambda_2, \beta_2)\) respectively. Hence \( \phi_X(t) \) in (7.3.6) is the product of the cf's of its stable and gamma components.

**Remark 7.3.3.** It can be proved that the SG distribution is

1. Closed under linear transformations
2. Infinitely Divisible
3. Self-decomposable
4. Asymmetric

**Proof:** These properties can be easily proved in a way similar to the one that considered for the properties of StLi distributions.

For \( \lambda_1 = \lambda_2 = \lambda \) and \( \beta_1 = \beta_2 = \beta \), SG is symmetric and in that case the cf is,

\[
\phi_X(t) = e^{-ct|t|^\alpha} \left( \frac{1}{1 + \lambda^2 t^2} \right)^\beta.
\]

As a convolution, it can be expressed as \( X \overset{d}{=} Y + L \) where \( Y \) and \( L \) are independent.
stable and symmetric generalized Laplace (Refer Mathai, 1993a,b) random variables respectively.

**Remark 7.3.4.** For various combinations of values of \( \alpha, \beta_1 \) and \( \beta_2 \), we get many distributions as special cases. When \( \alpha = 2, \beta_1 = \beta_2 = \beta \) it reduces to generalized normal-Laplace distribution (Reed, 2004). If only \( \beta_1 = \beta_2 = \beta \) it reduces to generalized stable-Laplace distribution. For \( \alpha = 2, \beta_1 = \beta_2 = 1 \) it reduces to stable-Laplace distribution. For \( \alpha = 2, \beta_1 = \beta_2 = 1 \) it reduces to normal-Laplace distribution (Reed and Jorgensen, 2004).

The first order stable-gamma autoregressive process (SGAR(1)) is constituted by \( \{X_n, n \geq 1\} \) where \( X_n \) satisfies the AR(1) model \( X_n = aX_{n-1} + \epsilon_n, a \in (0, 1) \) with \( \{\epsilon_n\} \) is a sequence of iid random variables such that \( X_n \) is stationary Markovian with SG marginal distribution. On assuming stationarity, the cf of the innovation sequence, \( \{\epsilon_n\} \) can be obtained as,

\[
\phi_\epsilon(t) = e^{-\lambda(1-a^\alpha)} |t|^{\beta_1} \left( \frac{1 - i\lambda_1 a t}{1 - i\lambda_1 t} \right)^{\beta_1} \left( \frac{1 + i\lambda_2 a t}{1 + i\lambda_2 t} \right)^{\beta_2} \tag{7.3.7}
\]

Hence, \( \{\epsilon_n\} \) can be regarded as the convolution of stable and generalized exponential tailed random variables.

### 7.4 Discrete Stable-Linnik Distribution and Processes

Consider a convolution of a discrete stable random variable \( X \ (DS(\lambda, \gamma)) \) and a discrete generalized Linnik random variable \( Y \ (L(\theta, \alpha, \beta)) \) given by

\[
U = X + Y
\]

where \( X \) and \( Y \) are independent. The probability generating function \( (pgf) \) of \( U \) is given by

\[
P_U(s) = e^{-\lambda(1-s)^{\gamma}(1 + \theta(1 - s)^{\alpha})^{-\beta}} \tag{7.4.1}
\]
where \(|s| \leq 1, \lambda > 0, \gamma \in (0, 1), \alpha \in (0, 1), \beta > 0, \theta > 0\). The random variable having the above pgf is named herein as Discrete Stable-Generalized Linnik (DStLi) random variable. When \(\gamma = 1, \beta = 1\), reduces to the Poisson-discrete Mittag-Leffler distribution.

The first order integer valued autoregressive (INAR(1)) process is defined by the model structure,

\[
U_n = a \ast U_{n-1} + \epsilon_n, n \in \mathbb{Z}, a \in [0, 1] \tag{7.4.2}
\]

where \(U\) is a non-negative integer valued random variable, \(\{\epsilon_n\}\) is sequence of uncorrelated non-negative integer valued random variables and \(\ast\) denotes the binomial thinning operator defined by,

\[
a \ast U = \sum_{i=1}^{U} Y_i
\]

where \(Y_i\) is a sequence of iid random variables independent of \(U\), such that \(P[Y_i = 1] = 1 - P[Y_i = 0] = a\).

**Theorem 7.4.1.** Let \(P(s)\) be the pgf of a DStLi distribution with \(\lambda > 0, \gamma \in (0, 1), \alpha \in (0, 1), \beta > 0, \theta > 0, a \in (0, 1)\). There exists a stationary INAR(1) process \(\{U_n, n \in \mathbb{Z}\}\) with \(P(s)\) as the pgf of its marginal distribution. Also the marginal distribution of the innovation sequence \(\{\epsilon_n, n \in \mathbb{Z}\}\) has pgf \(P_\epsilon(s)\) given by

\[
P_\epsilon(s) = \frac{e^{-\lambda s^\gamma} (1 + \theta s^\alpha)^{-\beta}}{e^{-\lambda a s^\gamma} (1 + \theta a^\alpha s^\alpha)^{-\beta}}.
\]

**Proof:** In terms of alternate probability generating function (apgf) defined as \(G(s) = P(1 - s)\), the INAR(1) model defined in (7.4.2) can be rewritten as,

\[
G_{U_n}(s) = G_{U_{n-1}}(as)G_{\epsilon_n}(s).
\]

Under stationarity it reduces to,

\[
G_U(s) = G_U(as)G_\epsilon(s).
\]
Hence,
\[ G_\epsilon(s) = \frac{G_U(s)}{G_U(as)}. \]

The INAR(1) process with DStLi marginals is defined, if there exists an innovation sequence \( \{\epsilon_n\} \) such that \( G_\epsilon(s) \) is an apgf.

From (7.4.1),
\[ G_U(s) = e^{-\lambda s^\gamma (1 + \theta s^\alpha)^{-\beta}}. \]

Then we have
\[ G_\epsilon(s) = \frac{e^{-\lambda s^\gamma (1 + \theta s^\alpha)^{-\beta}}}{e^{-\lambda a s^\gamma s^\alpha (1 + \theta a s^\alpha)^{-\beta}}} = e^{-\lambda(1-a) s^\gamma} \left( s^\alpha + (1-a)^\alpha \frac{1}{1+\theta s^\alpha} \right)^\beta. \]

Therefore, \( \{\epsilon_n\} \) has the convolution structure,

\[ \epsilon_n = W_n + V_n \]

where \( W_n \) follows discrete stable \( DS(\lambda(1-a)^\gamma), \gamma) \) and \( V_n \) is a \( \beta \)-fold convolution of discrete Mittag-Leffler tailed random variable \( ML_T(a^\alpha, \theta, \alpha, \beta) \).

For checking the time reversibility of the process, since the process is Markovian, it is sufficient to determine the joint distribution of \( (U_{n-1}, U_n) \). The joint pgf of \( (U_{n-1}, U_n) \) is given by,

\[
P_{U_{n-1}, U_n}(s_1, s_2) = E(s_1^{U_{n-1}} s_2^{U_{n-1}+\epsilon_n})
= P_{\epsilon_n}(s_2) P_{U_{n-1}}(s_1(1-a+as_2))
= e^{-\lambda(1-a) s_2^\gamma} \left( 1 + \theta a^\alpha s_2^\alpha \right)^\beta e^{-\lambda(1-s_1(1-a+as_2)) s_2^\gamma} \left( 1 + \theta(1 - s_1(1-a+as_2))^\alpha \right)^\beta.
\]

As the obtained expression is not symmetric in \( s_1 \) and \( s_2 \), the INAR(1) process is not time reversibly.
reversible and hence the backward regression would be of interest.

By differentiating $P_{U_{n-1}, U_n}(s_1, s_2)$ with respect to $s_1$ and setting $s_1 = 0$,

$$E(U_{n-1}s_2^{U_n}) = (1 - a + as_2) \left( \lambda \gamma + \frac{\alpha \beta \theta}{(1 + \theta)} \right) e^{-\lambda((1-a)\gamma)s_2 + 1} \left( \frac{1 + \theta_\alpha s_2^\alpha}{(1 + \theta s_2^\alpha)(1 + \theta)} \right) \beta.$$  \tag{7.4.4}

But

$$E(U_{n-1}s_2^{U_n}) = \sum_{m=0}^{\infty} s_2^m E(U_{n-1}|U_n = m) P(U_n = m).$$

Hence $E(U_{n-1}|U_n = m)$ can be obtained from the coefficient of $s_2^m$ in the expansion of right hand side of (7.4.4).

### 7.4.1 INAR(p) Processes

Here we define a $p^{th}$ order integer valued AR (INAR(p)) with the probability structure,

$$U_n = \begin{cases} 
  a_1 * U_{n-1} + \epsilon_n, & \text{with probability } \delta_1 \\
  a_2 * U_{n-2} + \epsilon_n, & \text{with probability } \delta_2 \\
  \vdots \\
  a_p * U_{n-p} + \epsilon_n, & \text{with probability } \delta_p 
\end{cases} \tag{7.4.5}$$

where $0 < a_i, \delta_i < 1, i = 1, \ldots, p, \sum_{i=1}^{p} \delta_i = 1$.

In terms of $apgf$, (7.4.5) can be rewritten as,

$$G_{U_n}(s) = G_{\epsilon_n}(s) \left[ \sum_{i=1}^{p} \delta_i G_{U_{n-i}}(a_i s) \right].$$

Assuming stationarity,

$$G_{U}(s) = G_{\epsilon}(s) \left[ \sum_{i=1}^{p} \delta_i G_{U}(a_i s) \right].$$
Hence,
\[ G_\epsilon(s) = \frac{G_U(s)}{\sum_{i=1}^{p} \delta_i G_U(a_i s)} . \]

For the DStLi marginals, the innovation sequence of the process has apgf,
\[ G_\epsilon(s) = \frac{e^{-\lambda s^\gamma} (1 + \theta s^\alpha)^{-\beta}}{\sum_{i=1}^{p} \delta_i e^{-\lambda a_i^\gamma s^\gamma} (1 + \theta a_i^\alpha s^\alpha)^{-\beta}}. \]  
\[ (7.4.6) \]

For the particular case of \( a_i = a \), for \( i = 1, \ldots, p \), (7.4.6) yields the apgf defined in (7.4.3).

Hence with an error sequence \( \{\epsilon_n\} \) distributed as the convolution of discrete stable and convolution of Mittag-Leffler tailed random variables, the \( p^{th} \) order discrete stable-Linnik autoregressive processes are properly defined.

### 7.5 Conclusion

In this chapter, generalizations of the distributions and processes developed in the previous chapters are done. Thus, it gives a generalization of the Gaussian non-Gaussian autoregressive processes. Hence, those Gaussian non-Gaussian models are extended to the more general class stable non-Gaussian processes viz., semi stable-Linnik autoregressive models. This model contains a wide variety of time series models as special cases, including the normal-Laplace model. We have also developed integer valued stable-Linnik processes in this final chapter of the present thesis. The problems which are unsolved and remained unexplored in the present work are the areas for further research in future.

### References


