3.1 Introduction

Many authors introduced several non-Gaussian stationary autoregressive processes and continuous time Levy processes connected with the Laplace distribution, and pointed out the general theories leading to such models, which shows bright prospects in stochastic modelling. Generalized Type I asymmetric Laplace (AL) distributions are also called Bessel function distribution (McKay, 1932) or variance-gamma distribution (Madan and Seneta, 1990). Due to its flexibility, simplicity and excellent fit to empirical data, variance-gamma model is nowadays widely used in financial modelling. Rowland and Sichel (1960) applied generalized Laplace model to logarithms of the ratios of duplicate check-sampling-values in South African gold mines and Sichel (1973) used the model for modelling size of diamonds. Time series models with marginals as Laplace and $\alpha$–Laplace distributions can

Some results included in this chapter form part of the paper Jose et al. (2009).
be seen in Jayakumar et al. (1995), Seetha Lakshmi et al. (2003). Elizabeth and Susan (2005) derived an error distribution for gene expression data with the help of skew Laplace (SL) distributions. Although the theory and application of SL distributions are well established and there are considerable literature in recent years, their application in time series modelling is not well developed.

In this chapter, autoregressive processes with different forms of AL distribution as marginals are developed. An overview of the inter-relations between AL distributions are given. Autoregressive models with Type I AL marginals and Type III AL marginals are developed and their generalizations are studied.

3.2 Inter-relations between Asymmetric Laplace Distributions

A study on different types of SL distributions and their properties is reported in Kozubowski and Podgorski (1999, 2001) and Kotz et al. (2001). Kozubowski and Podgorski (2008a) classified all those versions of SL distributions into four types: Type I (based on invariance under geometric summation), Type II (based on Azzalini’s method), Type III (based on convolutions of symmetric Laplace and exponential distributions) and Type IV (based on quantile regression) and established the interrelationships between the four types. Kozubowski and Podgorski (2008b) shows that all AL laws are self-decomposable, so that autoregressive models can be constructed.

A symmetric Laplace distribution with mean $\theta$ and variance $\sigma^2$ has probability density function (pdf),

$$f(x) = \frac{1}{\sqrt{2\sigma}} e^{-\sqrt{2}(x-\theta)/\sigma}, x \in \mathbb{R}.$$ This is known as Laplace first law of errors and derives importance under $L_1$ norm minimization. Type I AL distribution ($AL_1$) arises as the limits of sums of independently and identically distributed ($iid$) random variables with finite second moment, where the number of terms summed is geometrically distributed, independent of the terms themselves (See Kozubowski and Podgorski, 2000). When $p \to 0$, the geometric sum such that
$X_1 + \cdots + X_{\nu p}$ where $P(\nu_p = k) = p(1 - p)^{k-1}, k = 1, 2, \ldots$, converges in distribution to a random variable with characteristic function (cf) of the form,

$$
\phi(t) = \frac{e^{i\theta t}}{1 + \frac{1}{2} \sigma^2 t^2 - i\mu t}.
$$

The Type II AL ($AL_2$) distribution is generated from the symmetric Laplace density via the method of generating skewed distributions described in Azzalini (1985). The function $g(x) = 2f(x)F(\lambda x), x \in \mathbb{R}, \lambda \in \mathbb{R}$ where $X$ has a symmetric Laplace distribution with distribution function ($df$) $F$ and pdf $f$, yields the $AL_2$ density whose cf is given by,

$$
\phi(t) = \frac{e^{i\theta t}[\sigma t + (1 + \lambda)^2 i]}{[\sigma t + i][\sigma^2 t^2 + (1 + \lambda)^2]}.
$$

Analogous to the skew double exponential distribution of Jagannathan (2005), derived from the standard form,

$$
X_\lambda = \frac{1}{\sqrt{1 + \lambda^2}} X + \frac{\lambda}{\sqrt{1 + \lambda^2}} |Y|, \lambda \in \mathbb{R}
$$

where $X$ and $Y$ are iid standard Laplace variables, the distribution obtained as the convolution of symmetric Laplace and exponential distributions is named as Type III AL ($AL_3$) distribution whose cf is,

$$
\phi_X(t) = \frac{e^{i\theta t}}{[1 + (1 - c^2)\sigma^2 t^2][1 - ic\sigma t]}.
$$

In connection with quantile regression, a skew normal distribution is derived in Mudholkar and Hutson (2000). The AL distribution generated in such a manner gives Type IV AL ($AL_4$) distribution.

In fact, all these types are related and the interrelations between them are established in Kozubowski and Podgorski (2008a). Even though, Type I and Type IV are equivalent,
the standard Type I and Type IV distributions are different. Links among Type I and Type II, Type I and Type III, and Type II and Type III are explained in Kozubowski and Podgorski (2008a) in detail.

### 3.3 AR Model with Type I Skew Laplace Marginals

Consider the first order autoregressive (AR(1)) process,

\[ X_n = aX_{n-1} + \epsilon_n; \quad a \in (0, 1), \forall n > 0. \] (3.3.1)

In terms of cfs, we have,

\[ \phi_{X_n}(t) = \phi_{X_{n-1}}(at)\phi_{\epsilon_n}(t). \]

This under the assumption of stationarity gives

\[ \phi_{\epsilon}(t) = \frac{\phi_{X}(t)}{\phi_{X}(at)}. \]

Suppose that \( X \) is a convolution of exponential random variables. That is,

\[ X \overset{d}{=} X_1 - X_2 \] (3.3.2)

where \( X_1 \) is exponentially distributed with parameter \( \alpha (> 0) \), centered at \( \theta_1 (\in \mathbb{R}) \) and \( X_2 \) is exponentially distributed with parameter \( \beta (> 0) \), centered at \( \theta_2 (\in \mathbb{R}) \). The corresponding cf is given by,

\[ \phi_X(t) = \frac{e^{i\theta\alpha\beta}}{(\alpha - it)(\beta + it)} \] (3.3.3)

where \( \theta = \theta_1 - \theta_2 \).

Then for the structure in (3.3.1), using (3.3.3) we have the cf of \( \epsilon \) as,

\[ \phi_{\epsilon}(t) = e^{i\theta(1-a)t}\frac{(\alpha - it)(\beta + iat)}{(\alpha - it)(\beta + it)}. \] (3.3.4)
Hence,  
\[ \epsilon_n \equiv U_n + Y_{1n} - Y_{2n}. \]  
(3.3.5)  

where \( U_n \) is a degenerate random variable, degenerated at \( \theta(1 - a) \), \( Y_{1n} \sim ET(a, \alpha) \) and \( Y_{2n} \sim ET(a, \beta) \) (by Definition 3.3.1) and are independent.

**Definition 3.3.1.** A non-negative random variable, which has an atom of mass \( a \) at 0 and which is exponentially distributed with parameter \( \alpha \) if positive is called an Exponentially Tailed random variable with parameters \( a \) and \( \alpha \), denoted by \( ET(a, \alpha) \).

It is worth mentioning that Gaver and Lewis (1980) obtained the innovation distribution of the EAR(1) process as the \( ET(a, \alpha) \) distribution. Littlejohn (1994) introduced the concept of tailed distributions while discussing non-Gaussian time series models.

**Remark 3.3.1.** The cf in (3.3.3) can be considered as a factorization of the cf of \( AL_1 \) distribution defined in Kozubowski and Podgorski (2008a), given by,  
\[ \phi(t) = \frac{e^{i\theta t}}{1 + \frac{1}{2} \sigma^2 t^2 - i\mu t} \]  
with  
\[ \alpha = \frac{2}{\mu + \sqrt{\mu^2 + 2\sigma^2}} \text{ and } \beta = \frac{\mu + \sqrt{\mu^2 + 2\sigma^2}}{\sigma^2}. \]

Now we shall state the following theorem.

**Theorem 3.3.1.** A necessary and sufficient condition for a stationary autoregressive process \( \{X_n\} \) with the structure in (3.3.1) to be strictly Markovian with \( AL_1 \) marginal is that \( \{\epsilon_n\} \) is distributed as defined in (3.3.5), provided \( X_0 \equiv AL_1(\theta, \sigma, \mu) \).

**Proof:** If the initial distribution of \( X_0 \) is \( AL_1(\theta, \sigma, \mu) \), assuming stationarity, it suffices to verify that \( X_n \equiv AL_1(\theta, \sigma, \mu) \) for every \( n \). This can be proved using an inductive argument. Suppose \( X_{n-1} \equiv AL_1(\theta, \sigma, \mu) \). In terms of cf the model in (3.3.1) can be rewritten as,  
\[ \phi_{X_n}(t) = \phi_{X_{n-1}}(at) \phi_{\epsilon_n}(t). \]
Then, from (3.3.3) and (3.3.4), the proof follows under the assumption of stationarity.

Remark 3.3.2. Even if \( X_0 \) is arbitrarily distributed, it can be proved that \( X_n \) converges in distribution to \( AL_1(\theta, \sigma^2, \mu) \), provided \( \epsilon_n \overset{d}{=} U_n + Y_1 - Y_2 \).

Remark 3.3.3. If the structure of the autoregressive process under consideration is

\[
X_n = \begin{cases} 
\epsilon_n, & \text{with probability } p \\
X_{n-1} + \epsilon_n, & \text{with probability } 1 - p
\end{cases}, \quad 0 \leq p \leq 1 \tag{3.3.6}
\]

having marginals \( X_n \overset{d}{=} AL_1(0, \sigma, \mu) \), then the innovation \( \epsilon_n \overset{d}{=} AL_1(0, \sqrt{\sigma}, \mu) \).

Definition 3.3.2. (Marshall and Olkin, 1997) The cf of one parameter Marshall-Olkin family of distributions is defined by

\[
\phi(t) = \frac{b \phi(t)}{1 - (1 - b) \phi(t)}, \quad b > 0
\]

where \( \phi(t) \) is the cf of a random variable \( X \).

Remark 3.3.4. When \( \phi(t) \) is as given in (3.3.3) with \( \theta = 0 \), then the cf of Marshall-Olkin asymmetric Laplace random variable \( Y \) is,

\[
\phi(t) = \left[ 1 + \frac{\alpha - \beta}{b^2 \alpha \beta} it + \frac{t^2}{b^2 \alpha \beta} \right]^{-1}
\]

and we write \( Y \sim MOAL(b, \alpha, \beta) \).

Remark 3.3.5. For the model defined in (3.3.6) with \( p = \frac{1}{b}, \quad b > 1 \), if \( X \overset{d}{=} MOAL(b, \alpha, \beta) \) then the cf of the innovation sequence will be,

\[
\left[ 1 + \frac{\alpha - \beta}{b^2 \alpha \beta} it + \frac{t^2}{b^2 \alpha \beta} \right]^{-1}.
\]

Hence the innovation sequence will also be Marshall-Olkin asymmetric Laplace.
3.3.1 Generalized Type-I Skew Laplace Models

A generalization of the random variable defined in (3.3.2) can be given by the cf,

$$\phi_X(t) = \left( \frac{e^{i\theta t} \alpha \beta}{(\alpha - it)(\beta + it)} \right)^p. \quad (3.3.7)$$

This can be considered as a factorization of the cf of a generalized $AL_1$ distribution (Mathai, 1993a). The corresponding random variable is called generalized Type-I AL ($GAL_1$) random variable. From the structure of the cf (3.3.7), it may be noted that the $GAL_1$ random variable $X$ is a convolution of some degenerate random variable and gamma random variables. Hence,

$$X = Z + X_1 - X_2$$

where $Z$ is a degenerate random variable, degenerates at $\theta$, $X_1 \sim G(\alpha, p)$ and $X_2 \sim G(\beta, p)$. Then for the structure in (3.3.1) with marginals having the cf in (3.3.7), the cf of the innovations will be,

$$\phi_\epsilon(t) = e^{i\theta p(1-a)t} \left( \frac{(\alpha - iat)}{(\alpha - it)} \right)^p \left( \frac{(\beta + iat)}{(\beta + it)} \right)^p. \quad (3.3.8)$$

Hence,

$$\epsilon_n \overset{d}{=} U_n + Y_{1n} - Y_{2n}$$

where $U$ is a degenerate random variable, degenerates at $\theta p(1-a)$, $Y_1$ is a generalization of exponential tailed random variable with parameter $\alpha$ and $Y_2$ is a generalization of exponential tailed random variable with parameter $\beta$, where $\alpha$ and $\beta$ are as defined in Remark 3.3.1. For integer values of $p$, the random variable $\epsilon$ can be expressed as

$$\epsilon = U + (U_1 + \cdots + U_p) - (V_1 + \cdots + V_p) \quad (3.3.9)$$
where $U_i$’s and $V_i$’s are exponential tailed with parameters $\alpha$ and $\beta$ respectively. The density corresponding to (3.3.8) and (3.3.9) can be obtained in terms of the modified Bessel function of the third kind as in Kozubowski and Podgorski (2008b). For different combinations of parameters $GAL_1$ distribution gives exponential, gamma, symmetric Laplace and $AL_1$ distribution (Refer Kotz et al., 2001), and hence the corresponding autoregressive process includes autoregressive processes with exponential, gamma, symmetric Laplace and $AL_1$ marginals.

### 3.4 AR Model with Type III Skew Laplace marginals

For the model defined in (3.3.1) we consider marginals distributed as $X_c \sim AL_3(\theta, \sigma, c)$. Then,

$$
\phi_\epsilon(t) = \exp(i\theta(1-a)t) \left[ \frac{1 + (1 - c^2)\sigma^2 a^2 t^2}{1 + (1 - c^2)\sigma^2 t^2} \right] \frac{1 - i c a \sigma t}{1 - i c \sigma t}.
$$

(3.4.1)

This implies that the innovation sequence is a convolution of Laplace tailed and exponential tailed independent random variables. That is, $\epsilon_n$ can be written as

$$
\epsilon_n \overset{d}{=} Y + Y_1 + Y_2
$$

where $Y$ is a degenerate random variable that degenerates at $\theta(1-a)$, $Y_1$ is a Laplace tailed random variable and $Y_2$ is an exponential tailed random variable, i.e., $Y_1 \sim LT(a^2, \theta, c, \sigma)$ and $Y_2 \sim ET(a, c\sigma)$ and are independent.

In order to find the joint distribution function of $(X_n, X_{n-1})$, consider

$$
\phi_{X_n,X_{n-1}}(t_1, t_2) = E[\exp(it_1X_n + it_2X_{n-1})] = \phi_\epsilon(t_2)\phi_X(t_1 + at_2).
$$

(3.4.2)
In the case where \( X_n \sim \text{AL}_3(\theta, c, \sigma) \), (3.4.2) becomes

\[
\phi_{X_n, X_{n-1}}(t_1, t_2) = \exp(i\theta(t_1 + t_2)) \frac{[1 + (1 - c^2)\sigma^2 a^2 t_2^2]}{[1 + (1 - c^2)\sigma^2 a^2 t_2^2]} \frac{[1 - ic\sigma t_2][1 - ic\sigma t_2]}{[1 + (1 - c^2)\sigma^2 a^2 t_1 + at_2^2][1 - ic\sigma(t_1 + at_2)]}.
\]

But the above equation is not symmetric in the arguments \( t_1 \) and \( t_2 \). Hence the process is not time reversible.

Using the AR(1) structure \( X_n = a X_{n-1} + \epsilon_n \), we can write,

\[
\phi_{X_n}(t) = \phi_{X_0}(a^n t) \prod_{k=0}^{n-1} \phi_{\epsilon_k}(a^k t).
\]

Suppose \( X_n \sim \text{AL}_3(\theta, c, \sigma) \). It can be seen that,

\[
\prod_{k=0}^{n-1} \phi_{\epsilon_k}(a^k t) = \exp \left( i\theta (1 - a)(1 + a + \cdots + a^{k-1}) t \right) \frac{[1 + (1 - c^2)\sigma^2 a^n t^2]}{[1 + (1 - c^2)\sigma^2 t^2]} \frac{[1 - ic a^n \sigma t]}{[1 - ic\sigma t]}.
\]

When \( n \to \infty \),

\[
\phi_{X_n}(t) \to \frac{\exp(i\theta t)}{[1 + (1 - c^2)\sigma^2 t^2][1 - ic\sigma t]}.
\]

Hence \( X_n \) is asymptotically distributed as \( \text{AL}_3(\theta, \sigma, c) \).

Then we have the first derivative with respect to \( t \),

\[
\phi'_{X_n}(t) = \frac{\exp(i\theta t) \left[ 1 + (1 - c^2)\sigma^2 t^2 \right] \left( [1 - ic\sigma t][i\theta + ic\sigma] + [1 - ic\sigma t]2\sigma^2 t(1 - c^2) \right)}{\left( [1 + (1 - c^2)\sigma^2 t^2][1 - ic\sigma t]^2 \right)^2}.
\]

When \( t = 0 \), we get \( E(X) = \theta + c\sigma \). Therefore \( E(\epsilon_n) = (1 - a)(\theta + c\sigma) \). Hence,

\[
E(X_n | X_{n-1} = x) = ax + (1 - a)(\theta + c\sigma).
\]
Thus the conditional expectation of the process is linear in $x$.

Now, for convenience choose $\theta = 0$ in (3.4.1), the same can be rewritten as,

$$
\phi_\epsilon(t) = \frac{\pi_1}{1 - i\sqrt{1 - c^2}\sigma t} + \frac{\pi_2}{1 + i\sqrt{1 - c^2}\sigma t} + \frac{\pi_3}{1 - ic\sigma t}
$$

where $\pi_1$, $\pi_2$, and $\pi_3$ are given as

$$
\pi_1 = ac \left\{ \frac{\sqrt{1 - c^2} - c}{\sqrt{1 - c^2} + c} \right\} - (1 - a^2)\sqrt{1 - c^2}
$$

$$
\pi_2 = ac \left\{ \frac{\sqrt{1 - c^2} - \sqrt{1 - c^2} + c}{\sqrt{1 - c^2} + c} \right\} \pi_1.
$$

$$
\pi_3 = 1 - \pi_1 - \pi_2.
$$

Therefore, we can represent the error variable $\epsilon_n$ as

$$
\epsilon_n = \begin{cases} 
E_1 , \text{ with probability } q\pi_1 \\
-E_2 , \text{ with probability } q\pi_2 \\
E_3 , \text{ with probability } q\pi_3 
\end{cases}
\hspace{1cm} (3.4.6)
$$

where $E_i \sim$ Exponential distribution with parameter $(\sqrt{1 - c^2})\sigma$ for $i=1, 2$ and $E_3 \sim$ Exponential distribution with parameter $c\sigma$.

A simple quantification of sample path behaviour is provided by the probabilities $P(X_n > X_{n-1})$. By using (3.4.6), this probability can be calculated as

$$
P(X_n > X_{n-1}) = \pi_1 P(E_1 > (1 - a)X_{n-1}) + \pi_2 P(-E_2 > (1 - a)X_{n-1})
$$

$$
+ \pi_3 P(E_3 > (1 - a)X_{n-1}).
$$

where $\pi_i$’s are as defined in (3.4.3), (3.4.4) and (3.4.5). Using simple algebraic calcula-
tions, it can be shown that

\[
\begin{align*}
P(E_1 > (1 - a)X_{n-1}) &= \sqrt{1 - c^2}(1 - a) \\
&\times \left\{ \frac{1}{2(2 - a)[\sqrt{1 - c^2} - (1 - 2c^2)(1 - a)]} - \frac{c^2}{(1 - 2c^2)(1 - a + c)} \right\}. 
\end{align*}
\]

\[
P(-E_2 > (1 - a)X_{n-1}) = \frac{\sqrt{1 - c^2}}{2(2 - a)[\sqrt{1 - c^2} + c]}. 
\]

\[
P(E_3 > (1 - a)X_{n-1}) = \frac{\sqrt{1 - c^2}}{2[\sqrt{1 - c^2} - c]} \left\{ 1 - \frac{c}{(1 - a)\sqrt{1 - c^2} + c} - \frac{c^2}{1 - 2c^2} \right\}. 
\]

On substituting (3.4.8), (3.4.9) and (3.4.10) in (3.4.7) we get the required probability.

### 3.5 Estimation of Parameters

Let \(x_0, x_1, \ldots, x_n\) be a given set of observations of a Type III skew Laplace AR(1) process (3.3.1) with unknown parameters \(\vartheta = (a, \theta, \sigma, c)\). Then the conditional least squares estimates of \(\vartheta\) is obtained by minimizing the sum of squares

\[
C_n(\vartheta) = \sum_{i=1}^{n} (x_i - E(X_i|X_{i-1}))^2. 
\]

But when the observations are from AL\(_3(\theta, \sigma, c)\), (3.5.1) becomes,

\[
C_n(\vartheta) = \sum_{i=1}^{n} ((x_i - ax_{i-1}) - (1 - a)(\theta + c\sigma))^2. 
\]

On differentiating the above with respect to the unknown parameters,

\[
\hat{a} = \frac{\sum_{i=1}^{n} x_i x_{i-1} - \frac{1}{n} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_{i-1}}{\sum_{i=1}^{n} x_{i-1} - 2\frac{n}{n} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_{i-1} + \frac{1}{n} (\sum_{i=1}^{n} x_i)^2}. 
\]
\[ \vartheta + c\sigma = \frac{1}{1 - a} \frac{1}{n} \sum_{i=1}^{n} x_i - \frac{a}{1 - a} \sum_{i=1}^{n} x_{i-1}. \]  

(3.5.2)

Direct estimation of the parameters from (3.5.2) is not possible. By taking 
\[ T_n = \frac{1}{n} \sum_{i=1}^{n} U_j, \]
where 
\[ U_j = \begin{cases} 
1 & \text{if } X_j > X_{j-1} \\
0 & \text{otherwise}
\end{cases} \]
as the estimate for \( P(X_j > X_{j-1}) \) in the (3.4.7), we can estimate the value for \( c \) and then using (3.5.2) the estimate for \( \sigma \) can be obtained for fixed values of \( \theta \).

### 3.6 Generalized Type III Skew Laplace Models

Mathai (1993a) introduced and studied a class of distribution namely, generalized Laplace distribution with \( \phi \),
\[ \phi(t) = \left( \frac{1}{1 + \sigma^2 t^2} \right)^\tau, \sigma \geq 0, \tau \geq 0. \]

The application of generalized Laplace distributions in different contexts such as the production of a chemical called melatonin in human body, solar neutrino fluxes in cosmos, growth decay mechanism like formation of sand dunes in nature etc. are discussed in Mathai (1993a,b,c, 1994, 2000). The application of generalized Laplace distribution in the field of time series modelling is discussed in Seetha Lekshmi et al. (2003).

In this section, we introduce the generalized skew Laplace distribution of Type III with \( \phi \),
\[ \phi(t) = \left( \frac{1}{[1 + (1 - c^2)\sigma^2 t^2][1 - i\sigma t]} \right)^\tau, \sigma \geq 0, \tau \geq 0. \]  

(3.6.1)

From the structure of the \( \phi \) (3.6.1), it can be seen that it is the \( \phi \) of a random variable having distribution as the convolution of a generalized Laplace and a gamma distributed random variables.

When \( \tau \) is a positive integer, then the generalized skew Laplace distribution of Type III is self-decomposable, being \( \tau \)-fold convolution of skew Laplace distributions of Type III. We can construct an AR(1) process (3.3.1), where marginal distribution is the generalized skew Laplace distribution of Type III with parameters \( c, \sigma \) and \( \tau \). For such a case, the \( \phi \) of
the error term can be obtained as,

\[ \phi_\epsilon (t) = \left( \frac{\pi_1}{(1 - i\sqrt{1 - c^2}\sigma t)} + \frac{\pi_2}{(1 + i\sqrt{1 - c^2}\sigma t)} + \frac{\pi_3}{(1 - i\gamma \sigma t)} \right) \tau \]

where \( \pi_1, \pi_2, \) and \( \pi_3 \) are given as (3.4.3), (3.4.4) and (3.4.5). The corresponding distribution is a finite mixture of various convolutions of exponential distributions.

### 3.7 Tailed Type III Skew Laplace Models

Suppose \( X \) is a random variable with \( \text{cf} \phi_X (t) \), then a tailed random variable \( Y \) associated with \( X \) assuming zero with probability \( \theta \) and tail probability \( (1 - \theta) \) has the \( \text{cf} \)

\[ \phi_Y (t) = \theta + (1 - \theta)\phi_X (t). \]

If \( X \sim AL(0, c, \sigma) \) then,

\[ \phi_Y (t) = \frac{1 + \theta(1 - c^2)\sigma^2 t^2 - i\theta c\sigma t(1 + (1 - c^2)\sigma^2 t^2)}{[1 + (1 - c^2)\sigma^2 t^2][1 - i\gamma \sigma t]} . \quad (3.7.1) \]

Now we develop an autoregressive process model by using the \( \text{cf} \) (3.7.1). Define

\[ Y_n = \begin{cases} \epsilon_n, \text{ with probability } p \\ Y_{n-1} + \epsilon_n, \text{ with probability } 1 - p. \end{cases} \quad (3.7.2) \]

Let the random variable \( Y \) has the \( \text{cf} \) (3.7.1), then by substituting (3.7.1) in (3.7.2) we get the \( \text{cf} \) for the error term \( \epsilon_n \) as

\[ \phi_{\epsilon_n} (t) = \frac{1 + \theta(1 - c^2)\sigma^2 t^2(1 - i\gamma \sigma t) - i\theta c\sigma t}{1 + \gamma(1 - c^2)\sigma^2 t^2(1 - i\gamma \sigma t) - i\gamma \sigma t}. \]

### 3.8 Conclusion

In this chapter, we have developed autoregressive processes with AL marginals. There are four different types of AL distributions as explained in Kozubowski and Podgorski (2008a).
Here we have considered the time series models with Type I and Type III AL marginals. The generalized cases are also studied. We also dealt with the problem of estimation of parameters by the method of conditional least squares. Tailed Type III Skew Laplace Models are also developed. Tailed distributions are used as a model in several situations where the variable can assume either zero value with a certain probability or a continuous value with the remaining probability. Thus, tailed distribution can be used as a model in different areas like dose response in the medical field, flow of water in rivers that are dry for some period in a year, economies that show dull behaviour in money circulation for a specific period of time etc.

References


Kotz, s., Kozubowski, T.J., Podgorski, K. (2001). *The Laplace distribution and generalizations: A Revisit with Applications to Communications, Economics, Engineering and*


