CHAPTER I

* FUZZY TOPOLOGICAL SPACES AND
* FUZZY COMPACTNESS

1.0. Introduction

The chapter begins with some preliminaries taken from [4] which are being used in the following discussions.

1.1. Preliminaries

Definition 1.1.1

Let $X$ be a non empty set. A mapping $\mu : X \rightarrow [0,1]$ is called a fuzzy set in $X$ where $[0,1]$ is the closed unit interval, while an ordinary set $A \subseteq X$ is identified with its characteristic function $\chi_A : X \rightarrow \{0,1\}$.

Let $I(X)$ be the family of all the fuzzy sets in $X$ called fuzzy space and $P(X)$ be the class of fuzzy sets whose membership functions have all their values in $\{0,1\}$. $X$ is called the carrier domain of each fuzzy subset in it, and $I$ is called the value domain of each fuzzy subset of $X$. 
Note 1.1.2.

The fuzzy sets 0 and 1 are given by $0(x) = 0$ and $1(x) = 1$ for all $x \in X$.

Definition 1.1.3

Let $\mu$ and $\sigma$ be any two fuzzy sets in a set $X$. Then $\mu$ is said to be contained in $\sigma$, denoted by $\mu \leq \sigma$, if $\mu(x) \leq \sigma(x)$ for all $x \in X$.

If $\mu(x) = \sigma(x)$ for all $x \in X$, $\mu$ and $\sigma$ are said to be equal and we write $\mu = \sigma$.

Definition 1.1.4

The support of a fuzzy set $\mu$, denoted by $\text{supp}(\mu)$, is given by $\text{Supp}(\mu) = \{x \in X : \mu(x) > 0\}$.

Definition 1.1.5

A fuzzy set $\mu$ is normal if $\sup_x \mu(x) = 1$. If the supremum is less than 1, then $\mu$ is called subnormal.

Definition 1.1.6

A fuzzy set in $X$ is called a fuzzy point iff it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at $x$ is $\lambda$ for $\lambda \in (0,1]$, we denote this point by $x_\lambda$ where the point $x$ is called its support.
Definition 1.1.7

The fuzzy point $x_\lambda$ is contained in a fuzzy set $\mu$ or $\mu$ is said to contain $x_\lambda$, denoted by $x_\lambda \in \mu$ iff $\lambda \leq \mu(x)$.

Definition 1.1.8

Let $X, Y$ be non-empty sets.

Let $f : X \rightarrow Y$ be a mapping on $X$ to $Y$. If $\lambda$ is a fuzzy set of $X$, $f(\lambda)$ is defined as follows:

$$f(\lambda)(y) = \sup_{x \in f^{-1}(y)} \lambda(x) \quad \text{if } f^{-1}(y) \neq \emptyset$$

$$= 0 \quad \text{otherwise}$$

For each $y \in Y$, and if $\mu$ is a fuzzy set of $Y$, $f^{-1}(\mu)$ is defined as follows.

$$f^{-1}(\mu)(y) = \mu(f(x)) \quad \text{for each } x \in X.$$

1.2. Additive operations with fuzzy sets

In $I(X)$ the following additive operations can be introduced:

Definition 1.2.1

Let $\mu$ and $\sigma$ be two fuzzy sets. Then,

i. The sum $\mu \oplus \sigma$ is the fuzzy set whose membership function is $(\mu \oplus \sigma)(x) = \min(1, \mu(x) + \sigma(x))$ for all $x \in X$
ii. The difference $\mu \ominus \sigma$ is the fuzzy set whose membership function is $(\mu \ominus \sigma)(x) = \max (0, \mu(x) - \sigma(x))$ for all $x \in X$

iii. The conjunction $\mu \& \sigma$ is the fuzzy set whose membership function is $(\mu \& \sigma)(x) = \max(0, \mu(x) + \sigma(x) - 1)$ for all $x \in X$

iv. The product $\mu \cdot \sigma$ is the fuzzy set whose membership function is $(\mu \cdot \sigma)(x) = \mu(x) \cdot \sigma(x)$ for all $x \in X$.

In his classical paper, Zadeh have introduced the union, the intersection, the complementary and the inclusion as follows:

i. The union $\mu \cup \sigma$ is the fuzzy set whose membership function is $(\mu \cup \sigma)(x) = \max(\mu(x), \sigma(x))$ for all $x \in X$

ii. The intersection $\mu \cap \sigma$ is the fuzzy set whose membership function is $(\mu \cap \sigma)(x) = \min(\mu(x), \sigma(x))$ for all $x \in X$

iii. The complement $\mu'$ or $\mu^c$ is the fuzzy set whose membership function is $\mu'(x) = 1 - \mu(x)$ for all $x \in X$.

iv. The inclusion of fuzzy sets is given by $\mu \subseteq \sigma$ iff $\mu(x) \leq \sigma(x)$ for all $x \in X$.

Properties 1.2.2

Let $\mu$ and $\sigma$ be two fuzzy sets. Then

1. $\mu' = X \ominus \mu$
2. $\mu \cup \sigma = (\mu' \oplus \sigma)' \oplus \sigma$
3. $\mu \cap \sigma = ((\mu \oplus \sigma') \oplus \sigma')'$
4. $\mu \& \sigma = (\mu' \oplus \sigma')'$
5. $\mu \oplus 0 = \mu$
6. $\mu \oplus \mu' = 1$
7. $\mu \oplus 1 = 1$
8. $\mu \& \mu' = 0$
9. $\mu \& 0 = 0$
10. $\mu \& 1 = \mu$
11. $\mu \ominus \mu = 0$
12. $X \ominus \mu' = \mu$
13. $\mu \ominus \sigma = (\mu' \oplus \sigma)'$
14. $\mu \& \sigma \subseteq \mu \& \sigma \subseteq \mu \oplus \sigma$
15. $\mu \oplus \sigma = \sigma \oplus \mu$
16. $\mu \& \sigma = \sigma \& \mu$
17. $\mu, \sigma = \sigma, \mu$
18. $(\mu \oplus \sigma) \oplus \nu = \mu \oplus (\sigma \oplus \nu)$
19. $(\mu \& \sigma) \& \nu = \mu \& (\sigma \& \nu)$
20. $(\mu, \sigma) \cdot \nu = \mu, (\sigma, \nu)$

The sum of the sequence $(\mu_n)_{n \in \mathbb{N}}$ is the fuzzy set $\bigoplus_{n \in \mathbb{N}} \mu_n$ whose membership function is defined by

$$(\bigoplus_{n \in \mathbb{N}} \mu_n) (x) = \lim_{n \rightarrow \infty} \bigoplus_{i=1}^{n} \mu_i (x) \quad (x \in X)$$

The conjunction of the sequence $(\mu_n)_{n \in \mathbb{N}}$ is the fuzzy set $\bigwedge_{n \in \mathbb{N}} \mu_n$ whose membership function is defined by

$$(\bigwedge_{n \in \mathbb{N}} \mu_n) (x) = \lim_{n \rightarrow \infty} \bigwedge_{i=1}^{n} \mu_i (x) \quad (x \in X)$$

**Definition 1.2.3**

The fuzzy sets $\mu_1, \mu_2, \ldots, \mu_n$ are said to be *disjoint* iff

$$(\bigoplus_{i=1}^{k} \mu_i) \& \mu_{k+1} = 0 \quad \text{for } k = 1, 2, \ldots, (n-1).$$
Definition 1.2.4

Let $J$ be an infinite index set and $J_i \subset J$ be finite / countable set.

Similar to operations on ordinary sets, we can generalize the sum $\bigoplus_{i \in J} \mu_i$ of any family $\{\mu_i / i \in J\}$ of fuzzy sets of a set $X$ as $(\bigoplus_{i \in J} \mu_i) (x) = \sup_{J_i \subset J} (\bigoplus_{i \in J_i} \mu_i) (x) \ x \in X.$

In a similar way, we define the conjunction of any family $\{\mu_i / i \in J\}$ of fuzzy sets of a set $X$ as $(\&_{i \in J} \mu_i) (x) = \inf_{J_i \subset J} (\&_{i \in J_i} \mu_i) (x), \ x \in X.$

Definition 1.2.5

The sequence $(\mu_n)_{n \in \mathbb{N}}$ is said to be a disjoint sequence of fuzzy sets if $\mu_1, \mu_2, \ldots, \mu_n$ are disjoint fuzzy sets for each $n$ in $\mathbb{N}, \ n \geq 2.$

Definition 1.2.6

Let $\mu$ be a fuzzy set. A finite (infinite) fuzzy partition of $\mu$ is a finite (countable) family of disjoint fuzzy sets whose sum is equal to $\mu.$

1.3. Additive class of fuzzy sets

In this section $C$ will denote a family of fuzzy sets.

Definition 1.3.1

We say that $C$ is an additive class of fuzzy sets iff

1. $1 \in C$
2. If $\mu, \sigma \in C$ then $\mu \oplus \sigma$ and $\mu \ominus \sigma \in C$
Remark 1.3.2

Let \( C \) be an additive class of fuzzy sets and \( \mu, \sigma \) are in \( C \). Then we have \( 0 \in C, \mu' \in C, \mu \cup \sigma \in C, \mu \cap \sigma \in C \) and \( \mu \& \sigma \in C \).

Definition 1.3.3

We say that \( C \) is a \( \sigma \)-additive class of fuzzy sets iff

1. \( 1 \in C \)
2. If \( \mu, \sigma \in C \) then \( \mu \ominus \sigma \in C \)
3. If \( (\mu_n)_{n \in \mathbb{N}} \) is a sequence of fuzzy sets contained in \( C \), then \( \bigoplus_{n \in \mathbb{N}} \mu_n \in C \).

Remark 1.3.4

If \( C \) is a \( \sigma \)-additive class of fuzzy sets then it is an additive and monotonic.

Definition 1.3.5

\( C \) is called an algebra of fuzzy sets iff it is an additive class of fuzzy sets and it is closed under products of its elements.

Definition 1.3.6

\( C \) is called a \( \sigma \)-algebra of fuzzy sets iff it is a \( \sigma \)-additive class of fuzzy sets and is closed under products of its elements.
Definition 1.3.7

A class $C$ of fuzzy sets on $X$ is said to be a $\sigma$-algebra if $C$ satisfies the following properties:

1. $1 \in C$
2. If $\{\mu_n\}_{n \in N} \subseteq C$ then $\bigoplus_{n \in N} \mu_n \in C$
3. If $\mu \in C$ and $\delta \in C$ then $\mu \ominus \delta \in C$ and $\mu \cdot \delta \in C$

Note 1.3.8

If $C$ is a $\sigma$-additive class of fuzzy sets and $C \subseteq P(X)$ then $C$ is a $\sigma$-algebra of sub sets.

1.4. *Fuzzy Topological Spaces

Now we are going to introduce the concept of *fuzzy topological spaces. In this connection we quote below the definitions given by Chang and Lowen before defining *fuzzy topological space as given in [14] and [37].

C.L Chang [14] defined fuzzy topological spaces (1968) as a generalization of topological spaces. He introduced the structure of a fuzzy topology on a set as follows:

Definition 1.4.1

(Chang) Let $X$ be a non empty set. Let $\delta$ be a family of fuzzy subsets of $X$. Then $\delta$ is called a fuzzy topology on $X$ if $\delta$ satisfies the following axioms:
1. $0, 1 \in \delta$
2. $\mu \land \nu \in \delta$ whenever $\mu, \nu \in \delta$
3. $\{ \forall_{\mu} \}_{\alpha} \subset \delta$ whenever $\{ \mu_{\alpha} : \alpha \in J \} \in \delta$

The ordered pair $(X, \delta)$ is called a fuzzy topological space (fts, for short). The members of $\delta$ are called open fuzzy sets and their complements are called closed fuzzy sets.

**Remark 1.4.2**

Every topological space is a fuzzy topological space but not conversely.

R. Lowen [36] also defined fuzzy topological spaces in a different way, as follows:

**Definition 1.4.3**

(Lowen) Let $X$ be a non empty set and $\delta$ be a family of fuzzy subsets of $X$. $\delta$ is called a fuzzy topology on $X$ if the following axioms are satisfied.

1. Each constant function $\alpha : X \rightarrow [0,1]$ is a member of $\delta$
2. $\mu \land \nu \in \delta$ whenever $\mu, \nu \in \delta$
3. $\lor_{\alpha \in J} \mu_{\alpha} \in \delta$ whenever $\{ \mu_{\alpha} : \alpha \in J \} \subset \delta$

The ordered pair $(X, \delta)$ is called a fuzzy topological space (fts) in the sense of Lowen. Members of $\delta$ are called open fuzzy sets and their complements are called closed fuzzy sets.
Remark 1.4.4

1. If $(X, \delta)$ is a fts in the sense of Lowen then $(X, \delta)$ is a fts in the sense of Chang.
2. A fuzzy topology in the sense of Chang need not be a fuzzy topology in the sense of Lowen.
3. A topological space need not be a fts in sense of Lowen.

Aiming to our study, we introduce an alternate definition of fuzzy topology called *fuzzy topology and the corresponding fuzzy topological space called *fuzzy topological space as follows:

**Definition 1.4.5**

Let $X$ be a non empty set and $\delta$ be a family of fuzzy subsets of $X$. $\delta$ is called a *fuzzy topology on $X$ if the following axioms are satisfied:

1. $0, 1 \in \delta$
2. If $\mu, \nu \in \delta$ then $\mu \& \nu \in \delta$
3. For any subfamily $\{\mu_\alpha\}_{\alpha \in J} \subset \delta$, $(\oplus_\alpha \mu_\alpha) \in \delta$

The ordered pair $(X, \delta)$ is called a *fuzzy topological space or *fts, for short. The members of $\delta$ are called *fuzzy open sets and their complements are called *closed fuzzy sets.

**Definition 1.4.6**

A fuzzy set $\mu$ in $(X, \delta)$ is called a neighborhood of fuzzy point $x_\lambda$ if there exists a $\sigma \in \delta$ such that $x_\lambda \in \sigma \subset \mu$; A neighborhood $\mu$ is *fuzzy open.
Remark 1.4.7

If \( \mu, \sigma \in P(X) \), then \( \mu \oplus \sigma = \mu \cup \sigma \), \( \mu \& \sigma = \mu \cap \sigma \) and \( \mu \ominus \sigma = \mu \setminus \sigma \). Thus the ordinary topology and ordinary topological spaces become special cases of *fuzzy topology and *fuzzy topological spaces.

Examples 1.4.8

a. Take \( G = \{0, 1\} \subset I(X) \), then \( G \) is a *fuzzy topology on \( X \). We call this *fuzzy topology the trivial *fuzzy topology and call the corresponding *fuzzy topological space \((X, G)\) a trivial *fuzzy topological space.

b. Take \( \delta = I(X) \), then \( \delta \) is a *fuzzy topology on \( X \) and we call the corresponding *fuzzy topological space \((X, \delta)\) a discrete *fuzzy topological space.

c. Let \( X = \{a, b, c\} \) and \( \mu = \{(a,0), (b,0.4), (c,1)\} \) be a fuzzy subset of \( X \). Then, \( \delta = \{0, 1, \mu\} \) is a *fuzzy topology on \( X \) and corresponding topological space \((X, \delta)\) is a *fuzzy topological space.

Proof:

This is straightforward.

Example 1.4.9

Let \( X = \{a, b, c, d\} \). Define \( \acute{a}, \hat{a}, \gamma : X \to [0, 1] \) by

\[
\acute{a}(x) = 1 \text{ if } x = a \text{ and } 0 \text{ elsewhere}
\]

\[
\hat{a}(x) = 1 \text{ if } x = b \text{ and } 0 \text{ elsewhere}
\]
\( \tilde{\alpha}(x) = 1 \) if \( x = a, b \) and 0 elsewhere.

Let \( \delta = \{ 0, 1, \tilde{a}, \tilde{\beta}, \gamma \} \). Then, \((X, \delta)\) is a *fts.

For,
\[
\tilde{\alpha} \oplus \tilde{\alpha} = \gamma, \quad \tilde{\alpha} \oplus \gamma = \gamma, \quad \tilde{\beta} \oplus \tilde{\alpha} = \gamma
\]
\[
\tilde{\alpha} \oplus 1 = 1, \quad \tilde{\beta} \oplus 1 = 1, \quad \tilde{\alpha} \oplus 1 = 1
\]
\[
\tilde{\alpha} \oplus 0 = \tilde{\alpha}, \quad \tilde{\beta} \oplus 0 = \tilde{\beta}, \quad \tilde{\alpha} \oplus 0 = \gamma, \quad 0 \oplus 1 = 1
\]
\[
\tilde{\alpha} \& \tilde{\alpha} = 0, \quad \tilde{\alpha} \& \tilde{\beta} = \tilde{\alpha}, \quad \tilde{\beta} \& \tilde{\beta} = \tilde{\beta}
\]
\[
\tilde{\alpha} \& 1 = \tilde{\alpha}, \quad \tilde{\beta} \& 1 = \tilde{\beta}, \quad \tilde{\alpha} \& 1 = \tilde{\alpha}
\]
\[
\tilde{\alpha} \& 0 = \tilde{\beta} \& 0 = \tilde{\alpha} \& 0 = 1 \& 0 = 0
\]

Here *open fuzzy sets are 0, 1, \( \tilde{a}, \tilde{\alpha}, \gamma \) and *closed fuzzy sets are 0, 1, 1 - \( \tilde{a}, \tilde{\alpha}, 1 - \gamma \).

**Definition 1.4.10**

Let \( \mu \) be a fuzzy set in a *fts \((X, \delta)\). Then the *closure of \( \mu \) denoted by *\( \text{cl} \mu \) or \( \text{cl} \mu \) or \( \mu^- \) is defined as the conjunction of all *closed fuzzy sets containing \( \mu \).

**Definition 1.4.11**

The *interior of \( \mu \) denoted by *\( \text{int} \mu \) or \( \text{int} \mu \) or \( \mu^\circ \) is defined as the sum of all the *open fuzzy sets contained in \( \mu \).

Thus, \( \mu^\circ \) is the largest *open fuzzy set contained in \( \mu \) and \( \mu^- \) is the smallest *closed fuzzy set containing \( \mu \).

**Definition 1.4.12**

A collection \( \{ \mu_\alpha \}_{\alpha \in J} \) of *fuzzy open sets in \( X \) is called a *fuzzy open cover of a fuzzy set \( \mu \) in \( X \) if \( \mu \subset \bigoplus_{\alpha \in J} \mu_\alpha \).
Definition 1.4.13

A fuzzy set $\kappa$ of a *fuzzy* topological space $X$ is called *fuzzy compact* if every *fuzzy* open cover of $\kappa$ has a finite sub cover.

Definition 1.4.14

A *fuzzy* topological space $X$ is called *fuzzy compact* if every *fuzzy* open cover of $X$ has a finite subcover.

Definition 1.4.15

A *fuzzy* topological space $(X, \delta)$ is called locally *compact* if every point $x$ in $X$ there exists a member $\mu \in \delta$ such that $x \in \mu$ and $\mu$ is *fuzzy compact*.

Definition 1.4.16

Let $(X, \delta)$ and $(Y, \eta)$ be *fuzzy* topological spaces and $f$ be a function from $X$ to $Y$. Then, $f$ is said to be *fuzzy continuous* iff $f^{-1}(\mu) \in \delta$ for all $\mu \in \eta$.

Definition 1.4.17

A *fts* $(X, \delta)$ is said to have the Hausdorff property or to be Hausdorff if $x, y \in X$ with $x \neq y$, implies that there exist *fuzzy* open sets $\mu$ and $\nu$ with $\mu(x) = 1 = \nu(y)$ and $\mu \& \nu = 0$.

Normality in fts was introduced by B.W. Hutton [26]

Definition 1.4.18

A *fts* $(X, \delta)$ is said to be normal if for every *closed* fuzzy set $\kappa$ and every open fuzzy set $\beta$ such that $\kappa \subseteq \beta$ there exists a fuzzy set $\mu$ such that $\kappa \subseteq \mu^* \subseteq \cl_\mu \subseteq \beta$.
Regularity in fts was defined and studied by S. R Malghan and S.S. Benchalli [3]

Definition 1.4.19

A *fts \((X, \delta)\) is said to be regular if for each \(x \in X\) and a *closed fuzzy set \(\kappa\) with \(\kappa(x) = 0\), there exist open fuzzy sets \(\beta\) and \(\gamma\) such that \(\beta(x) = 1\), \(\kappa \subseteq \gamma\) and \(\beta \subseteq 1 - \gamma\).

1.5. Theorems on *fuzzy Compactness

In the point set topology, the compactness is one of the most important notions. Since the fuzzy topology was introduced by Prof. C.L. Chang, the fuzzy compactness were investigated by many authors like T.E Canter, Hu Cheng-ming, R.A Lowen, Guojun, Wang et al. Chang proved that compactness is hereditary for the closed subset and is invariant under fuzzy continuous surjections. Now we prove some similar properties and results of *fuzzy compactness.

Theorem 1.5.1

If \(\kappa\) is *compact and \(\sigma\) is *closed in a *fuzzy topological space \(X\) and \(\sigma \subseteq \kappa\), then \(\sigma\) is *compact.

Proof:

Let \(\{\mu_{\alpha}\}_{\alpha \in I}\) be a family of *fuzzy open sets in \(X\) such that \(\emptyset \subseteq \bigoplus_{\alpha \in I} \mu_{\alpha}\). Then \(\sigma \cap (\bigoplus_{\alpha \in I} \mu_{\alpha})\) covers \(X\) and hence there is a finite collection \(\{\mu_{\alpha_i}\}\) such that...
\[ \kappa \subseteq \sigma^c \oplus \mu_{\alpha_1} \oplus \ldots \oplus \mu_{\alpha_n}. \]

Then, \( \sigma \subseteq \mu_{\alpha_1} \oplus \ldots \oplus \mu_{\alpha_n}. \)

Hence \( \sigma \) is *fuzzy compact.

**Theorem 1.5.2**

A *closed fuzzy subset of a *fuzzy compact space is *fuzzy compact.

**Proof:**

Let \( \sigma \) be a *-closed fuzzy subset of a *fuzzy topological space \( X \) and let \( \{\mu_{\alpha}\}_{\alpha \in J} \) be a *fuzzy open cover of \( \sigma \). Then \( X \) has *fuzzy open sets \( \nu_\alpha \) such that \( \nu_\alpha = \mu_\alpha \& \sigma \) and \( \{\nu_\alpha : \alpha \in J, \sigma^c \} \) is a *fuzzy open cover of \( X \). Hence we get a finite *fuzzy open subcover \( \{\nu_{\alpha_1}, \ldots, \nu_{\alpha_n}, \sigma^c\} \) of \( X \) and hence a finite *fuzzy open subcover \( \{\mu_{\alpha_1}, \ldots, \mu_{\alpha_n}\} \). Hence the theorem.

**Theorem 1.5.3**

Suppose \( X \) is a Hausdorff *fuzzy topological space. \( \kappa \subseteq X, \kappa \) is *fuzzy compact and \( x \in \kappa^c \). Then, there are *fuzzy open sets \( \mu \) and \( \omega \) such that \( x \in \mu, \kappa \subseteq \omega \) and \( \mu \& \omega = 0. \)

**Proof:**

If \( y \in \kappa \), the *Hausdorff separation axiom implies that there exists *fuzzy open sets \( \mu_y \) and \( \nu_y \) with \( x \in \mu_y \) and \( y \in \nu_y \) and \( \mu_y \& \nu_y = 0. \) Since \( \kappa \) is *fuzzy compact there are points \( y_1, y_2, \ldots, y_n \in \kappa \) such
that $\kappa \subset \nu_{y_1} \oplus \nu_{y_2} \oplus \ldots \oplus \nu_{y_n}$. Take $\mu = \mu_{y_1} \oplus \mu_{y_2} \oplus \ldots \oplus \mu_{y_n}$ and

$\omega = \nu_{y_1} \oplus \nu_{y_2} \oplus \ldots \oplus \nu_{y_n}$. Then, $\mu \& \omega = 0$, $x \in \mu$ and $\kappa \subseteq \omega$.

**Corollary 1.5.4**

A *fuzzy compact subsets of a Hausdorff *fuzzy topological space is *fuzzy closed.

**Corollary 1.5.5**

If $\sigma$ is *fuzzy closed and $\kappa$ is *fuzzy compact in a Hausdorff *fuzzy topological space, then $\sigma \& \kappa$ is *fuzzy compact.

**Proof:**

As $\kappa$ is a *fuzzy compact subset of $X$, it is *closed, by cor.1 of theorem 1.5.3, $\sigma$ is *fuzzy closed implies $\sigma \& \kappa$ is *fuzzy closed subset of $\kappa$. By theorem 1.5.1, $\sigma \& \kappa$ is *fuzzy compact.

**Theorem 1.5.6**

If $\{\kappa_\alpha : \alpha \in J\}$ is a collection of *compact fuzzy subsets of a Hausdorff *fuzzy topological space $X$ and if $(\&\kappa_\alpha)(x) = 0$, then some finite subcollection of $\{\kappa_\alpha\}$ also has empty intersection.

**Proof:**

Put $\mu_\alpha = \kappa_\alpha^c$. Fix a member $\kappa_1$ of $\{\kappa_\alpha : \alpha \in J\}$. Since no point of $\kappa_1$ belongs to every $\kappa_\alpha$, $\{\mu_\alpha : \alpha \in J\}$ is a *fuzzy open cover of $\kappa_1$. 23
Hence $\kappa_1 \subset \mu_{\alpha_1} \oplus \mu_{\alpha_2} \oplus \ldots \oplus \mu_{\alpha_n}$ for some finite collection \{\mu_{\alpha_i}\}. This implies that $(\kappa_1 \land \kappa_{\alpha_1} \land \kappa_{\alpha_2} \land \ldots \land \kappa_{\alpha_n})(x) = 0$. Hence the theorem.

**Theorem 1.5.7**

Let $X$ be a Hausdorff locally compact fuzzy topological space. Assume that $\mu$ is a *open fuzzy set and $\kappa$ is a *fuzzy compact set satisfying $\kappa \subset \mu$. Then there exists a *open fuzzy set $\nu$ such that

\[ \kappa \subset \nu \subset \ast \text{cl} (\nu) \subset \mu. \]

**Proof:**

Since each point of $\kappa$ has a neighborhood with compact closure, and since $\kappa$ is covered by the union of finitely many of these neighborhoods, $\kappa$ lies in an *fuzzy open set $\varphi$ with compact closure. If $\mu = X$, take $\nu = \varphi$. Otherwise, if $\psi$ is the complement of $\mu$, Theorem 1.5.3. shows that to each $x$ in $\psi$ there exists an *open fuzzy set $\omega_x$ such that $\kappa \subset \omega_x$ and $x \notin \omega_x$.

Hence, $\{\psi \land \overline{\varphi} \land \omega_x \}_{x \in \psi}$ is a collection of *compact fuzzy sets with empty intersection.

Therefore, by theorem 1.5.4 there are points $x_1, x_2, \ldots, x_n \in \psi$ such that $\psi \land \omega_{x_1} \land \ldots \land \omega_{x_n} = 0$

Then the set $\nu = \varphi \land \omega_{x_1} \land \ldots \land \omega_{x_n}$ has the required properties, since $\nu = \omega_{x_1} \land \ldots \land \omega_{x_n}$. 

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**Theorem 1.5.8**

If \( f : (X, \delta) \to (Y, \eta) \) is *fuzzy continuous function and \( \kappa \) is a *fuzzy compact set in \( (X, \delta) \) then \( f(\kappa) \) is *fuzzy compact in \( (Y, \eta) \).

**Proof:**

Let \( \{ \mu^1_{\alpha}, \alpha \in J \} \) be a *fuzzy open cover of \( f(\kappa) \).

Then \( \{ f^{-1}(\mu^1_{\alpha}) \}_{\alpha \in J} \) is a *fuzzy open cover of \( \kappa \).

As \( \kappa \) is *fuzzy compact, \( \kappa \subset f^{-1}(\mu^1_{\alpha_1}) \oplus f^{-1}(\mu^1_{\alpha_2}) \oplus \ldots \oplus f^{-1}(\mu^1_{\alpha_n}) \) for some \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and therefore \( f(\kappa) \subset \mu^1_{\alpha_1} \oplus \mu^1_{\alpha_2} \oplus \ldots \oplus \mu^1_{\alpha_n} \). Hence \( f(\kappa) \) is *fuzzy compact in \( (Y, \eta) \).

**Theorem 1.5.9**

Let \( (X, \delta) \) be a *fuzzy compact topological space and suppose \( f : (X, \delta) \to (Y, \eta) \) is a surjective *fuzzy continuous mapping. Then \( (Y, \eta) \) is *fuzzy compact.

**Proof:**

Let \( \{ \mu^1_{\alpha}, \alpha \in J \} \) be a *fuzzy open cover of \( (Y, \eta) \).

Then \( \{ f^{-1}(\mu^1_{\alpha}) \}_{\alpha \in J} \) is a *fuzzy cover of \( (X, \delta) \) and since \( f \) is *fuzzy continuous, it is a *fuzzy open cover of \( (X, \delta) \). Since \( X \) is *fuzzy compact, \( X \subset f^{-1}(\mu^1_{\alpha_1}) \oplus f^{-1}(\mu^1_{\alpha_2}) \oplus \ldots \oplus f^{-1}(\mu^1_{\alpha_n}) \) for some \( \alpha_1, \alpha_2, \ldots, \alpha_n \). Then \( Y = f(X) \subset \mu^1_{\alpha_1} \oplus \mu^1_{\alpha_2} \oplus \ldots \oplus \mu^1_{\alpha_n} \). Hence \( (Y, \eta) \) is *fuzzy compact.