CHAPTER III
FUZZY MEASURE SPACES
AND COMPLEX VALUED B-FUNCTIONS

3.0. Introduction

This chapter contains some very important definitions and results which are essential for developing the theory.

3.1. Definitions

The mathematical formality of a fuzzy measure is defined as follows:

Let $X$ be a set, $\mathcal{A}$ be a $\sigma$–algebra of subsets of $X$. By a fuzzy measure we mean a positive, extended real-valued set function $\delta : \mathcal{A} \rightarrow [0, +\infty]$ with properties:

1. $\delta(\emptyset) = 0$ when $\emptyset \in \mathcal{A}$ (vanishing at $\emptyset$);
2. $A \subseteq B \Rightarrow \delta(A) \leq \delta(B)$ if $A, B \in \mathcal{A}$ (monotonicity);
3. $A_1 \subseteq A_2 \subseteq \ldots, A_n \in \mathcal{A}$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ imply $\lim_{n \to \infty} \delta(A_n) = \delta(\bigcup_{n=1}^{\infty} A_n)$ (continuity from below);
4. $A_1 \supset A_2 \supset \ldots, A_n \in \mathcal{A}, \delta(A_1) < \epsilon$ and $\bigcap_{n=1}^{\infty} E_n \in \mathcal{A}$ imply $\lim_{n \to \infty} \delta(E_n) = \delta(\bigcap_{n=1}^{\infty} E_n)$ (continuity from above).
Observe that no additivity assumption is made on $\nu$.

**Definition 3.1.1**

$\delta$ is called a lower semi continuous fuzzy measure on $(X, A)$ iff it satisfies the above conditions 1, 2 and 3.

**Definition 3.1.2**

$\delta$ is called an upper semi continuous fuzzy measure on $(X, A)$ iff it satisfies the above conditions 1, 2 and 4.

**Definition 3.1.3**

A fuzzy measure $\delta$ is said to be regular iff $X \in A$ and $\delta(X) = 1$.

**Definition 3.1.4**

When $A$ is a $\delta$-algebra of subsets of $X$, we call $(X, A)$ a measurable space. If $\delta$ is a fuzzy measure on $(X, A)$ then $(X, A, \delta)$ is called a fuzzy measure space.

**Definition 3.1.5**

$\delta$ is subadditive iff $\delta(E) \leq \delta(E_1) + \delta(E_2)$ whenever $E_1 \in A$, $E_2 \in A$, $E \in A$ and $E = E_1 \cup E_2$.

**Definition 3.1.6**

$\delta$ is superadditive iff $\delta(E) \geq \delta(E_1) + \delta(E_2)$ whenever $E_1 \in A$, $E_2 \in A$, $E \in A$, $E_1 \cap E_2 = \emptyset$ and $E = E_1 \cup E_2$. 

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**Definition 3.1.7**

\( \delta \) is fuzzy additive iff \( \delta(E) = \delta(E_1) \lor \delta(E_2) \) whenever \( E_1, E_2 \in A \), \( E \in A \) and \( E = E_1 \cup E_2 \).

Thus, the fuzzy measure, according to Sugeno is a monotonic set function which is an extension of the probability measure. From the mathematical point of view, the monotonicity is inessential. Aumann and Shapely have investigated fuzzy measures without monotonicity. A non-monotonic fuzzy measure on \( (X, A) \) is real-valued set function \( \delta : A \rightarrow \mathbb{R} \) satisfying \( \delta(\emptyset) = 0 \).

We shall give another definition of a fuzzy measure due to D. Butnariu [4] and state some of its properties. Butnariu [4] introduced the concept of an additive fuzzy measure and discussed a series of properties of additive fuzzy measures which are very similar to those in the classical definition of measure. In order to make our work more meaningful and fruitful, we consider fuzzy measure due to Butnariu.

**Definition 3.1.8**

Let \( X \) be a non-empty set, \( C \) be a \( \delta \)-additive class of fuzzy sets in \( X \). Let \( R^* \) denote the extended real numbers and let \( m : C \rightarrow R^* \) be a function. Then, \( m \) is called an additive function over \( C \) iff

1. \( m(\emptyset) = 0 \).
2. If \( A, B \in C \) and \( A \oplus B = \emptyset \), then \( m(A \oplus B) = m(A) + m(B) \).

Let \( C^* \) denote the set of all additive functions over \( C \) and \( C^*_+ \) denote the set of all non-negative additive functions over \( C \).
Results 3.1.9

1. If \( m \in \mathcal{C}_+^* \) then \( m \) is monotonic, that is, \( m(B) \leq m(A) \) if \( B \subseteq A \).
2. If \( m \) is monotonic, then \( m \) is non-negative.
3. If \( m \in \mathcal{C}_+^* \) then \( m \) is subadditive, that is \( m(A \oplus B) \leq m(A) + m(B) \) if \( A, B \in \mathcal{C} \).

Definition 3.1.10

\( m \) is said to be a \( \sigma \)-additive function over \( \mathcal{C} \) if

1. \( m(\emptyset) = 0 \).
2. If \( (A_n)_{n \in \mathbb{N}} \) is a sequence in \( \mathcal{C} \), then \( m(\bigoplus_{n \in \mathbb{N}} A_n) = \sum_{i=1}^{\infty} m(A_n) \).

A nonnegative \( \sigma \)-additive function over \( \mathcal{C} \) is called a measure or fuzzy measure over \( \mathcal{C} \).

Definition 3.1.11

Let \( X \) be any set and \( \mathcal{C} \) be a \( \sigma \)-algebra of fuzzy subsets. By a fuzzy measure we mean a non-negative function \( m \) defined over \( \mathcal{C} \) with properties:

1. \( m(\emptyset) = 0 \).
2. If \( (A_n)_{n \in \mathbb{N}} \) is a sequence in \( \mathcal{C} \), then \( m(\bigoplus_{n \in \mathbb{N}} A_n) = \sum_{i=1}^{\infty} m(A_n) \).

Let \( \mathcal{C}^- \) denote the class of the \( \sigma \)-additive functions over \( \mathcal{C} \) and \( \mathcal{C}_r^\times \) the class of the fuzzy measure over \( \mathcal{C} \).
Remark 3.1.12

If \( m \) is a fuzzy measure on \( \mathcal{C} \), \( A_n \supseteq A_{n+1} \) (\( n \in \mathbb{N} \)) and these exists \( p \in \mathbb{N} \) with \( m(A_p) < \infty \) then \( m(\lim A_n) = \lim m(A_n) \).

Any \( \sigma \)-additive function is also additive.

Definition 3.1.13

A Borel Space is a pair \( \mathcal{B} = (X, \mathcal{C}) \) where \( \mathcal{C} \) is a \( \sigma \)-algebra of fuzzy sets of \( X \).

Definition 3.1.14

A measure space is a triple \( \mathcal{M} = (X, \mathcal{C}, m) \) where \( (X, \mathcal{C}) \) is a Borel space and \( m \) is a fuzzy measure on \( \mathcal{C} \).

Definition 3.1.15

Let \( A \) be a fuzzy set. A finite fuzzy partition of \( A \) is a finite family of disjoint fuzzy sets whose sum is equal to \( A \).

Definition 3.1.16

\( s = (\Lambda, a) \) is called a simple B-function where \( \Lambda = \{A_1, A_2, \ldots A_n\} \), a finite fuzzy partition of \( X \) and \( a = (a_1, a_2, \ldots a_n) \in \mathbb{R}^n \) such that \( s(x) = \sum_{i=1}^{n} a_i A_i(x) \).

Definition 3.1.17

Let \( f: X \rightarrow \mathbb{R}^+ = (0, +\infty] \). We call \( f \) a B-function over \( A \), if there exists a sequence of simple B-functions
\[ \{ s_n = \sum_{i=1}^{k(n)} a_{n,i} A_{n,i} \} \] such that

i. \[ s_{n+1} > A_s(A \in C) \]

ii. \[ \lim_{n \to \infty} s_n(x) A(x) = f(x) A(x) \quad (x \in X) \]

iii. \[ (f(x) - a_{n,i}) A_{n,i}(x) A(x) \geq 0 \quad (x \in X, i = 1,2,\ldots,k(n)) \]

**Proposition 3.1.18**

If \( f, g : X \to \mathbb{R}^* \) are B- functions and \( f + g \) exists then \( f + g \) is a B- function.

**Lemma 3.1.19**

The difference of two non-negative B- functions, if it exists, is also a B- function.

**Proposition 3.1.20**

If \( f : X \to \mathbb{R} \) is a B- function and \( c \in \mathbb{R} \), then \( cf \) is also a B function.

**Proposition 3.1.21**

If \( f,g : X \to \mathbb{R}^* \) are B- functions then \( f \cdot g \) is also a B- function.

**Lemma 3.1.22**

If \( u \) and \( v \) are non-negative B- functions then \( w(x) = \max \{ u(x), v(x) \} \) is also a B- function.
Remark 3.1.23

Any finite fuzzy measure is equivalent to a fuzzy measure in Sugeno’s sense, but the converse is not true.

Definition 3.1.24

The minimal $\sigma$-algebra $B$ which contains all the closed fuzzy subsets of the real line is a reduced $\sigma$-algebra whose intersection with $P(R)$ is exactly the family of Borel subsets of $R$.

Proposition 3.1.25

If $C$ is a reduced $\sigma$-algebra, then the sum and the product of two B-fns $f, g : X \to R$ are also B-functions.

3.2. Complex Valued B-functions

In this section we introduce and study the concept of B-functions for complex valued functions.

Definition 3.2.1

If $(X, C)$ is a Borel space and $f$ be a complex-valued function defined on $X$ so that $f = \text{Re } f + i \text{ Im } f$. Then $f$ is said to be a complex-valued B-function on the Borel space $(X, C)$ if both Re $f$ and Im $f$ are real-valued B-functions.
Proposition 3.2.2

If \( f \) is a complex-valued B-function then \( \bar{f} \) is also a B-function.

Proof:

Since \( f = \text{Re} f - i \text{Im} f \), the result follows.

Proposition 3.2.3

If \( f \) and \( g \) are complex-valued B-functions, then \( f + g \) is also a complex B-function.

Proof:

Since the sum of two real-valued B-functions is a B-function, the result follows.

Corollary 3.2.4

If \( f \) is a complex valued B-function then \( f + \bar{f} \) is a real B-function.

Proposition 3.2.5

Let \( f \) be a complex-valued B-function. If \( c \in \mathbb{C} \), then \( cf \) is a complex valued B-function.

The result follows from proposition 3.1.20.
Proposition 3.2.6

If f and g are complex B-functions, then fg is also a complex B-function.

Proof:

Since the product of two real-valued B-functions is a B-function, the result follows.

Corollary 3.2.7

If f is a complex valued B-function then $\overline{f} = |f|^2$ is a real B-function.