2.1 Introduction

The concept of induced fuzzy topological spaces was introduced by Weiss [14]. Subsequently, Lowen [9] established relationship between fuzzy topological spaces and topological spaces and proved some deeper results on induced fuzzy topological spaces which he called as topologically generated spaces. Martin [10] introduced a generalized concept weakly induced space, which was called semi induced space by Mashhour, Ghanim, Wakeil and Morsi [11]. The notion of lower semi continuous functions plays an

Some of the results appearing in this chapter have been included in a paper published in the Journal of Fuzzy Mathematics Vol. 16, No.2, 2008.
important tool in defining the above concepts. In [2,3] R.N. Bhaumik and A. Mukherjee introduced two classes of I-topological spaces- completely induced I-topological space and completely semi induced I-topological space using the tool completely lower semi continuous functions [4]. These are defined with the generalized concept of completely continuous functions introduced by Arya and Gupta [1]. In [8] Aygün, Warner and Kudri introduced a new class of functions from a topological space $(X, T)$ to a fuzzy lattice $L$ with its Scott topology called (completely) Scott continuous functions as a generalization of (completely) lower semi continuous functions from $(X, T)$ to the closed unit interval $I$. Using this tool, the above authors generalized Lowen's induced spaces to general L-topological setup. Here in this chapter we generalize the weakly induced space introduced by Martin [10] using the tool (completely) Scott continuous functions and study the lattice structure of the set $W(X)$ of all weakly induced L-topologies on a given set $X$. Here we take the definition of L-topology in the sense of Chang [6]. For a given topology $\tau$ on $X$, we study properties of the lattice $W_\tau$ of weakly induced L-topologies defined by families of (completely) Scott continuous functions with reference to $\tau$ on $X$. From the lattice $W_\tau$, we deduce that lattice $W(X)$ of weakly induced L-topologies on $X$ is not complemented but join complemented. Also it is proved that induced L-topologies and crisp topologies have complements.

### 2.2 Lattice of Weakly Induced L-topologies

For a given topology $\tau$ on $X$, the family $W_\tau$ of all weakly induced L-topologies defined by families of Scott continuous functions from $(X, \tau) \rightarrow L$ forms a lattice under the natural order of set inclusion. The least upperbound of a collection of weakly induced L-topologies belonging to $W_\tau$ is the weakly induced L-topology which is generated by their union and the greatest lowerbound is their intersection. The smallest and largest element in $W_\tau$ are denoted by $O_\tau$ and $1_\tau$ respectively.

Also for a given topology $\tau$ on $X$, the family $CW_\tau$ of all weakly induced L-topologies
defined by families of completely Scott continuous functions from \((X, \tau) \to L\) forms a lattice under the natural order of set inclusion. Since every completely Scott continuous function is Scott continuous, it follows that \(CW_\tau\) is a sublattice of \(W_\tau\). Also note that \(W_\tau\) and \(CW_\tau\) coincide when each open set in \(\tau\) is regular open.

When \(\tau = D\), the discrete topology on \(X\), these lattices coincide with lattice of weakly induced L-topologies.

**Definition 2.2.1** [7] An element \(p\) of \(L\) is called prime if \(p \neq 1\) and whenever \(a, b \in L\) with \(a \land b \leq p\), then \(a \leq p\) or \(b \leq p\). The set of all prime elements of \(L\) will be denoted by \(\text{pr}(L)\).

**Definition 2.2.2** [13] The Scott topology on \(L\) is the topology generated by the sets of the form \(\{t \in L : t \not\leq p\}\), where \(p \in \text{pr}(L)\). Let \((X, \tau)\) be a topological space and \(f : (X, \tau) \to L\) be a function, where \(L\) has its Scott topology. We say that \(f\) is Scott continuous if for every \(p \in \text{pr}(L)\), \(f^{-1}(\{t \in L : t \not\leq p\}) \in \tau\).

**Remark 2.2.1** When \(L = [0, 1]\), the Scott topology coincides with the topology of topologically generated spaces of Lowen [9]. The set \(\omega_L(\tau) = \{f \in L^X ; f : (X, \tau) \to L\}\) is Scott continuous \(\in W_\tau\).

**Definition 2.2.3** [8] Let \((X, \tau)\) be a topological space and \(a \in X\). A function \(f : (X, \tau) \to L\), where \(L\) has its Scott topology, is said to be completely Scott continuous at \(a \in X\) if for every \(p \in \text{pr}(L)\) with \(f(a) \not\leq p\), there is a regular open neighbourhood \(U\) of \(a\) in \((X, \tau)\) such that \(f(x) \not\leq p\) for every \(x \in U\). That is, \(U \subset f^{-1}(\{t \in L : t \not\leq p\})\).

\(f\) is called completely Scott continuous on \(X\) if \(f\) is completely Scott continuous at every point of \(X\).

**Note 2.2.1** Let \(F\) be a L-topology on the set \(X\), let \(F_c\) denote the 0-1 valued members of \(F\), that is, \(F_c\) is the set of all characteristic mappings in \(F\). Then \(F_c\) is a L-topology.
on $X$. Define $F^*_c = \{ A \subset X : \mu_A \in F_c \}$. The L-topological space $(X, F_c)$ is same as the topological space $(X, F^*_c)$.

**Definition 2.2.4** An L-topological space $(X, F)$ is said to be a weakly induced L-topological space if for each $f \in F$, $f$ is a Scott continuous function from $(X, F^*_c) \to L$.

**Definition 2.2.5** If $F$ is the collection of all Scott continuous functions from $(X, F^*_c) \to L$ then $F$ is an induced space and $F = \omega_L(F^*_c)$. This is equivalent to the definition given by Warner [12].

**Theorem 2.2.1** $W_\tau$ is complete.

**Proof.** Let $S$ be a subset of $W_\tau$ and let $G = \bigcap_{F \in S} F$. Clearly $G$ is an L-topology. Let $g \in G$. Since each $F \in S$ is weakly induced, $g$ is a Scott continuous map from $(X, F^*_c) \to L$. That is, $g^{-1}\{ t \in L : t \not\leq p \text{ where } p \in pr(L) \}$ belongs to $F^*_c$ for each $F \in S$. Therefore $g^{-1}\{ t \in L : t \not\leq p \text{ where } p \in pr(L) \} \subseteq \bigcap_{F \in S} F^*_c$. In other words, $g^{-1}\{ t \in L : t \not\leq p \text{ where } p \in pr(L) \} \subseteq \bigcap_{F \in S} F^*_c$. Hence $g$ is a Scott continuous function from $(X, G^*_c) \to L$. Then $(X, G^*_c) = (X, \bigcap_{F \in S} F^*_c)$. That is $G \in W_\tau$ and $G$ is the greatest lowerbound of $S$.

Let $K$ be the set of upperbounds of $S$. Then $K$ is nonempty since $1_\tau = \omega_L(\tau) \in K$. Using the above argument $K$ has a greatest lowerbound, say $H$. Then this $H$ is a least upperbound of $S$. Thus every subset $S$ of $W_\tau$ has greatest lowerbound and least upperbound. Hence $W_\tau$ is complete.

**Theorem 2.2.2** $W_\tau$ is not atomic.

**Proof.** Atoms in $W_\tau$ are either of the form $\{ 0, 1, \alpha \}$ or $\{ 0, 1, \mu_A \}$ where $\mu_A$ is the characteristic function of open subset $A$ of $(X, \tau)$. Let $X = \{ a, b, c \}$, $\tau = \{ \phi, X, \{ a \}, \{ a, b \} \}$ and
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\[ F = \begin{cases} 
  a \rightarrow .8 & a \rightarrow .8 & a \rightarrow .8 & a \rightarrow 1 & a \rightarrow 1 \\
  0, 1, \mu_{\{a\}}, \mu_{\{a,b\}} & f : b \rightarrow .7 & g : b \rightarrow 0 & h : b \rightarrow .7 & k : b \rightarrow .7 & l : b \rightarrow 1 \\
  c \rightarrow .6 & c \rightarrow 0 & c \rightarrow 0 & c \rightarrow .6 & c \rightarrow .6 
\end{cases} \]

Here \( F^*_c = \tau \) and \( F \in W_\tau \). But this \( F \) cannot be expressed as join of atoms. Hence \( W_\tau \) is not atomic.

**Theorem 2.2.3** \( W_\tau \) is not distributive.

**Proof.** Since every distributive lattice is necessarily modular we prove that \( W_\tau \) is not modular. For example let \( X = \{a, b, c\} \) and \( \tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} \). Suppose \( F_1 = \{0, 1\} \), \( F_2 = \{0, 1, \mu_{\{a\}}\} \), \( F_3 = \{0, 1, \mu_{\{b\}}\} \), \( F_4 = \{0, 1, \mu_{\{a\}}, \mu_{\{a,b\}}\} \), \( F_5 = \{0, 1, \mu_{\{a\}}, \mu_{\{b\}}, \mu_{\{a,b\}}\} \). Then each element in the collection \( S = \{F_1, F_2, F_3, F_4, F_5\} \) belongs to \( W_\tau \) and \( S \) is a sublattice of \( W_\tau \) isomorphic to \( N_5 \). Therefore \( W_\tau \) is not modular and hence not distributive.

**Proposition 2.2.1** If \( \tau_1 \) and \( \tau_2 \) are adjacent topologies on \( X \), that is there is no topology \( \tau \) between \( \tau_1 \) and \( \tau_2 \) then \( \omega_L(\tau_1) \) is a dual atom in \( W_{\tau_2} \).

Proof is easy.

2.3 Complementation in the lattice of weakly induced L-Topologies

In this section, from the lattice \( W_\tau \) we deduce that lattice \( W(X) \) of weakly induced L-topologies on \( X \) is not complemented but join complemented. Also it is proved that induced L-topologies and crisp topologies have complements with smallest element 0 and largest element 1.

**Definition 2.3.1** A bounded lattice \( S \) is said to be join complemented if for every \( x \) in \( S \) there exist \( y \) in \( S \) such that \( x \lor y = 1 \)
Definition 2.3.2 A bounded lattice $S$ is said to be meet complemented if for every $x$ in $S$ there exist $y$ in $S$ such that $x \land y = 0$.

Definition 2.3.3 A bounded lattice $S$ is said to be complemented if for every $x$ in $S$ there exist $y$ in $S$ such that $x \land y = 0$ and $x \lor y = 1$.

Definition 2.3.4 A bounded lattice $S$ is said to be semi complemented if it is either join complemented or meet complemented.

Example 2.3.1 Join complemented lattice but not complemented

![Diagram of a lattice](image-url)
Example 2.3.2 Meet complemented lattice but not complemented

Proposition 2.3.1 If $L$ has no dual atom, then atoms in $W_\tau$ of the form $\{0, 1, \alpha\}$ have no complements in $W_\tau$.

Proof. Let $\mathcal{F} = \{0, 1, \alpha\}$ be an atom in $W_\tau$. We claim that $\mathcal{F}$ has no complement. $1_\tau$ is not a complement of $\mathcal{F}$ since $1_\tau \wedge \mathcal{F} \neq O_\tau$. Let $\mathcal{P}$ be a weakly induced L-topology in $W_\tau$ other than $1_\tau$. If $\alpha \in \mathcal{P}$, then $\mathcal{P}$ is not a complement of $\mathcal{F}$ since $\mathcal{F} \wedge \mathcal{P} \neq O_\tau$.

Suppose that $\alpha \notin \mathcal{P}$. Then $\mathcal{P}$ cannot contain simultaneously all characteristic functions of open sets in $\tau$ and all constant L-subsets since they together generate $1_\tau$. Then the set $K = \{h : h \text{ is a Scott continuous function from } (X, \tau) \to L \text{ and } h \notin \mathcal{P}\}$ is non empty and two cases arise:

(i) $K$ contain constant L-subsets

(ii) $K$ contain at least one characteristic function corresponding to an openset in $\tau$.

In either case $B = \{f \wedge g : f \in \mathcal{F}, \ g \in \mathcal{P}\}$ is a base for $\mathcal{R} = \mathcal{F} \vee \mathcal{P}$. Then at least one subset of $K$ is not contained in $\mathcal{R}$. Hence $\mathcal{R} \neq 1_\tau$. Thus $\mathcal{P}$ is not a complement of $\mathcal{F}$. 

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Remark 2.3.1 The above proposition is not true for an arbitrary lattice $L$. For example, take $L = \{0, \alpha, 1\}$ ordered by $0 < \alpha < 1$. If $(X, \tau)$ is a topological space and $\Delta_1 = \{0, 1, \alpha\}$, then clearly $\Delta_1$ is an atom in $W_\tau$, when $\alpha$ is not a characteristic function. Let $\Delta_2 = \{0, 1\} \cup \{\mu_A : A \in \tau\}$. Clearly $\Delta_2$ is an element of $W_\tau$ and $\Delta_1 \land \Delta_2 = \emptyset_\tau$ and $\Delta_1 \lor \Delta_2 = 1_\tau$. Hence $\Delta_2$ is a complement of $\Delta_1$.

**Theorem 2.3.1** $W_\tau$ is not complemented.

**Proof.** This follows from the above proposition.

Remark 2.3.2 When $\tau = D$, the discrete topology on $X$, $W_D = W(X)$, the collection of all weakly induced L-topologies on $X$. Let $\sum$ denote the family of all weakly induced L-topologies defined by Scott continuous functions where each Scott continuous function is a characteristic function. Then $\sum$ is a sublattice of $W(X)$ and is lattice isomorphic to the lattice of all topologies on $X$. The elements of $\sum$ are called crisp topologies.

**Theorem 2.3.2** The lattice of weakly induced L-topologies $W(X)$ is not complemented.

**Proof.** The proof follows from remark 2.3.2

**Theorem 2.3.3** Every induced L-topology has complement.

**Proof.** Let $\mathcal{F}$ be an induced L-topology. Since $\mathcal{F}$ is induced, $\mathcal{F}$ is the collection of all Scott continuous functions from $(X, F^*_\tau) \rightarrow L$. Let $F^*_\tau = \tau \in \sum(X)$. Since $\sum(X)$ is complemented, there exists $\tau'$ in $\sum(X)$ such that $\tau \land \tau'$ equal to the indiscrete topology on $X$ and $\tau \lor \tau'$ equal to the discrete topology on $X$. Then $\mathcal{F} \land \tau'$ is indiscrete L-topology and $\mathcal{F} \lor \tau'$ is the discrete L-topology.

**Theorem 2.3.4** Every atom in $W(X)$ of the form $\{0, 1, \mu_A\}$ has complement.
Proof. Let $\mathcal{F} = \{0, 1, \mu_A\}$. Then $\mathcal{F}$ is an element of $\sum(X)$. Since $\sum(X)$ is complemented there exists $\tau$ in $\sum(X)$ such that $\tau \lor \mathcal{F}$ equal to the discrete topology and $\tau \land \mathcal{F}$ equal to the indiscrete topology on $X$. Then $\mathcal{F} \lor \omega_L(\tau) = 1$ (the discrete L-topology on $X$) and $\mathcal{F} \land \omega_L(\tau) = 0$ (the indiscrete L-topology on $X$).

Theorem 2.3.5 The Lattice $W(X)$ of all weakly induced L-topologies on any set $X$ is semi complemented.

Proof. Let $\mathcal{F} \in W(X)$. Since $\mathcal{F}$ is weakly induced there is a topology $\tau$ on $X$ such that each element $f \in \mathcal{F}$ is a Scott continuous function from $(X, \tau) \to L$. Since the lattice of topology $\sum(X)$ is complemented, we can find a topology $\tau'$ in $\sum(X)$ such that $\mathcal{F} \lor \omega_L(\tau') = 1$ (discrete L-topology) and $\mathcal{F} \land \omega_L(\tau')$ need not be equal to $O$, the indiscrete L-topology on $X$. Thus every $\mathcal{F}$ in $W(X)$ has a join complement. Hence $W(X)$ is semi complemented.

Remark 2.3.3 Dual atoms in $W(X)$ are of the form $\omega_L(\tau)$ where $\tau$ is a dual atom (ultratopology) in $\sum(X)$. Each induced L-topology other than the discrete L-topology can be expressed as meet of dual atoms. But an arbitrary weakly induced L-topology, for example the weakly induced L-topology $F = \{0, 1, \alpha\}$ cannot be expressed as meet of dual atoms.

Theorem 2.3.6 The lattice $W(X)$ of all weakly induced L-topologies on any set $X$ is not dually atomic.

Proof. The proof follows from remark 2.3.3.
References


