CHAPTER 4

Complete Homogeneity and Reversibility in L-Topology

4.1 Introduction

In the paper ‘On the combination of topologies’ [1], G. Birkhoff proved that the collection of all topologies on a given set $X$ forms a complete lattice. Birkhoff’s ordering was the natural one of ‘set inclusion’, that is, if $\tau$ and $\tau'$ are topologies on a given set $X$, $\tau$ is less than or equal to $\tau'$ if and only if $\tau$ is a subset of $\tau'$. In the above lattice, the greatest lower bound corresponds to intersection and the least upperbound of a collection of topologies is the topology which has as a subbase, the union of topologies

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Some of the results appearing in this chapter have been included in a paper published in the Far East Journal of Mathematical Sciences Vol. 16 (3) (2005)
in the collection. Since 1936, many topologists, Vaidynathaswamy [10], Otto Fröhlich [3], Hartmanis [4], Steiner [9], Van Rooij [11] have investigated further properties of this lattice.

When Birkhoff introduced the lattice of topologies, he posed a question as to which subfamilies of topologies do possess minimal and maximal elements. Also one can describe all minimum $p$ and maximum $p$ topologies without knowing precisely which topological property $p$, one is working with. For example, if we assume the continuum hypothesis for the reals, there are only four topologies on the reals which are maximum or minimum topologies with respect to a topological property.

In [5,6] Johnson studied lattice structure of the set of all L-topologies on a given set $X$ and proved that the lattice of L-topologies is not complemented. A related problem in the lattice of L-topologies is to determine which subfamilies of L-topologies do possess minimum (maximum) and minimal (maximal) elements with respect to an L-topological property. In [7] Larson characterized all spaces which are minimum or maximum with respect to a topological property by introducing completely homogeneous topological spaces. In 1966 [8] Rajagopalan and Wilansky proved that a topological space is minimal or maximal for some topological property if and only if it is reversible. In this chapter we investigate the concept ‘Complete homogeneity and reversibility’ in general L-setup and L-topology and characterizes maximum (minimum) L-topological spaces. Also we proved that the set of all completely homogeneous L-topological spaces forms a complete sublattice of the lattice of L-topologies. It is also observed that this lattice possess atoms but it is not atomic and complemented.

4.2 Preliminaries

Here we give some preliminary definitions required in this chapter.

Definition 4.2.1 Let $\theta$ be a function from a set $X$ to a set $Y$ and $f$ be an L-subset in $Y$. Then the ‘inverse image’ of $f$, written as $\theta^{-1}(f)$, is an L-subset in $X$ whose
membership function is given by \( \theta^{-1}(f)(x) = f(\theta(x)) \) for all \( x \) in \( X \). Conversely, let \( g \) be an L-subset in \( X \). Then the ‘image of \( g \)’, written as \( \theta(g) \), is an L-subset in \( Y \), whose membership function is given by

\[
\theta(g)(y) = \begin{cases} 
\sup\{g(z); z \in \theta^{-1}(y)\} & \text{if } \theta^{-1}(y) \neq \phi \\
0, & \text{otherwise.}
\end{cases}
\]

**Definition 4.2.2** A function \( \theta \) from an L-topological space \((X, F)\) to an L-topological space \((Y, G)\) is ‘L-continuous’ if and only if the inverse image of each \( G \) open L-subset in \( Y \) is \( F \) open L-subset in \( X \). An ‘L-homeomorphism’ is an L-continuous one to one map of an L-topological space \((X, F)\) onto an L-topological space \((Y, G)\) such that the inverse of the map is also L-continuous.

Here we take the definition of \( L \)-topology in the sense of Chang [2].

**4.3 Complete homogeneity and Reversibility in L-topology**

The notion of complete homogeneity is well known in topology. Here we introduce in an analogous way the concept of complete homogeneity in L-topological space.

**Definition 4.3.1** An L-topological space \((X, F)\) is called ‘completely homogeneous’ if every one to one mapping of \((X, F)\) onto itself is an L-homeomorphism.

Analogous definition in Topology is available in Larson [7]

**Theorem 4.3.1** If \((X, F)\) is discrete, indiscrete or generated by L-points with the same membership value, then \((X, F)\) is completely homogeneous.

**Proof.** Trivial

**Definition 4.3.2** An ‘L-topological property’ is a class of L-topological spaces which is closed under L-homeomorphism.

**Theorem 4.3.2** Given an L-topological space \((X, F)\) the following conditions are
equivalent.

(a) \((X, \mathcal{F})\) is completely homogeneous

(b) \((X, \mathcal{F})\) is minimum \(p\) for some \(L\)-topological property \(p\).

(c) \((X, \mathcal{F})\) is maximum \(p\) for some \(L\)-topological property \(p\).

**Proof.** Suppose \((X, \mathcal{F})\) is completely homogeneous. Define \(p\) by the following: an \(L\)-topological space \((Y, \mathcal{G})\) has property \(p\) if there exists one to one, onto \(L\)-continuous mapping \(\theta : (Y, \mathcal{G}) \to (X, \mathcal{F})\). Then \((X, \mathcal{F})\) has property \(p\). Now assume \(\mathcal{F}'\) is an \(L\)-topology on \(X\) which possess property \(p\). Then there exist a one to one, onto \(L\)-continuous mapping \(\theta : (X, \mathcal{F'}) \to (X, \mathcal{F})\); but then \(\theta^{-1} : (X, \mathcal{F}) \to (X, \mathcal{F})\) is \(L\)-continuous since \((X, \mathcal{F})\) is completely homogeneous. Hence the identity mapping \(\theta^{-1}.\theta \circ i : (X, \mathcal{F'}) \to (X, \mathcal{F})\) is \(L\)-continuous and \(\mathcal{F} \subseteq \mathcal{F}'\). Thus \((X, \mathcal{F})\) is minimum for \(p\). This proves \((a) \Rightarrow (b)\).

Now to prove \((b) \Rightarrow (a)\) assume \((X, \mathcal{F})\) is minimum for some \(L\)-topological property \(p\). Let \(\theta\) be a one to one mapping of \(X\) onto \(X\). Define \(\mathcal{F}(\theta) = \{\theta(g) : g \in \mathcal{F}\}\). Then \(\mathcal{F}(\theta)\) is an \(L\)-topology on \(X\) and \(\theta : (X, \mathcal{F}) \to (X, \mathcal{F}(\theta))\) is an \(L\)-homeomorphism. Then \(\mathcal{F}(\theta)\) is also minimum for \(p\); since \(p\) is a \(L\)-topological property which implies that \(\mathcal{F}(\theta) \subseteq \mathcal{F}\) and \(\mathcal{F} \subseteq \mathcal{F}(\theta)\). Hence \(\mathcal{F}(\theta) = \mathcal{F}\) and \(\theta : (X, \mathcal{F}) \to (X, \mathcal{F})\) is an \(L\)-homeomorphism. Thus \((X, \mathcal{F})\) is completely homogeneous.

In a similar way we can show that \((a)\) and \((c)\) are equivalent.

**Theorem 4.3.3** If \((X, \tau)\) is a topological space, then \((X, \tau)\) is completely homogeneous if and only if \((X, \omega_L(\tau))\) is completely homogeneous.

**Proof.** The proof follows from the observation that \(\theta\) is a homeomorphism of \((X, \tau)\) onto itself if and only if \(\theta\) is an \(L\)-homeomorphism of \((X, \omega_L(\tau))\) onto itself.

**Definition 4.3.3** An \(L\)-topology \((X, \mathcal{F})\) is homogeneous if for any \(x, y \in X\) there exists an \(L\)-homeomorphism \(\theta\) such that \(\theta(x) = y\).

Then we have
Theorem 4.3.4 Every completely homogeneous $L$-topology is hereditarily homogeneous.

Proof. Trivial.

Definition 4.3.4 An $L$-topological space $(X, \mathcal{F})$ is called reversible if it has no strictly stronger (weaker) $L$-topology $\mathcal{F}'$ such that $(X, \mathcal{F})$ and $(X, \mathcal{F}')$ are $L$-homeomorphic.

Lemma 4.3.1 An $L$-topological space $(X, \mathcal{F})$ is reversible if and only if each $L$-continuous one to one map of the space onto itself is an $L$-homeomorphism.

Proof. Let $(X, \mathcal{F})$ be a reversible $L$-topological space and $\theta : (X, \mathcal{F}) \to (X, \mathcal{F})$ be $L$-continuous, one to one and onto. Let $\mathcal{F}' = \{ g \in L^X : \theta(g) \in \mathcal{F} \}$. Then $\mathcal{F}'$ is an $L$-topology weaker than $\mathcal{F}$ since $g \in \mathcal{F}'$, $g = \theta^{-1}(\theta(g)) \in \mathcal{F}$ and $\theta : (X, \mathcal{F}') \to (X, \mathcal{F})$ is an $L$-homeomorphism. Since $(X, \mathcal{F})$ is reversible, it follows that $\mathcal{F}' = \mathcal{F}$ and hence $\theta$ is an $L$-homeomorphism.

Conversely assume that $(X, \mathcal{F})$ is an $L$-topological space such that each one to one $L$-continuous map of $(X, \mathcal{F})$ onto itself is an $L$-homeomorphism. We claim that $(X, \mathcal{F})$ is reversible. Suppose that $(X, \mathcal{F}')$ is a larger $L$-topology such that $(X, \mathcal{F})$ and $(X, \mathcal{F}')$ are $L$-homeomorphic. Let $\theta : (X, \mathcal{F}) \to (X, \mathcal{F}')$ be an $L$-homeomorphism. Then $\theta$ is $L$-continuous map of $(X, \mathcal{F})$ onto itself. By our assumption $\theta$ is an $L$-homeomorphism of $(X, \mathcal{F})$ onto itself. Let $g \in \mathcal{F}'$. Since $\theta$ is an $L$-homeomorphism of $(X, \mathcal{F}) \to (X, \mathcal{F}'), \theta^{-1}(g) \in \mathcal{F}$. Again since $\theta$ is an $L$-homeomorphism of $(X, \mathcal{F})$ onto itself, $\theta(\theta^{-1}(g)) \in \mathcal{F}$. Hence $g \in \mathcal{F}$ and $\mathcal{F} = \mathcal{F}'$. Thus there is no strictly larger $L$-topology $(X, \mathcal{F}')$ such that $(X, \mathcal{F})$ and $(X, \mathcal{F}')$ are $L$-homeomorphic.

Remark 4.3.1 Analogous lemma and proof are available in [8]

Theorem 4.3.5 If $X$ is any finite set and $\mathcal{F}$ is any $L$-topology on $X$, then $(X, \mathcal{F})$ is reversible.
**Proof.** Let $\theta$ be any one to one L-continuous map of the space $(X, \mathcal{F})$ onto itself. Since $\theta$ is L-continuous for any $g \in \mathcal{F}$, we have $\theta^{-1}(g) \in \mathcal{F}$. That is $g \circ \theta \in \mathcal{F}$. Again continuity of $\theta$ implies that $\theta^{-1}(g \circ \theta) \in \mathcal{F}$. That is $g \circ \theta \circ \theta^{-1} \in \mathcal{F}$. Proceeding like this we see that $g \circ \theta^n \in \mathcal{F}$ for every $n \in \mathbb{N}$. But $\theta^m = \theta^{-1}$ for some $m$. Then $\theta(g) = g \circ \theta \circ \theta^{-1} \in \mathcal{F}$. Hence $\theta$ is an L-homeomorphism of $(X, \mathcal{F})$ onto itself. Thus $(X, \mathcal{F})$ is reversible by lemma 4.3.1.

**Theorem 4.3.6** If $(X, \tau)$ is a topological space then $(X, \tau)$ is reversible if and only if $(X, \omega_L(\tau))$ is reversible.

**Proof.** Trivial

**Theorem 4.3.7** If $(X, \tau)$ is a topological space such that $(X, \tau)$ is not reversible, $g$ is a Scott continuous function from $(X, \tau) \to L$ such that $g$ is one-one and $\mathcal{F}$ is the L-topology generated by $S = \{\lambda_A : A \in \tau\} \cup \{g\}$, where $\lambda_A$ is the characteristic function of $A$. Then $\mathcal{F}$ is reversible and $i(\mathcal{F}) = \tau$

**Proof.** Suppose $\theta$ is a bijection on $X$ such that $\theta$ is not an identity map. Then $\theta^{-1}(g) \not\in \mathcal{F}$. That is $\theta$ is not L-continuous. Thus every one to one L-continuous map onto itself is an L-homeomorphism. Thus $(X, \mathcal{F})$ is reversible.

**Remark 4.3.2** There are non reversible L-topological spaces such that its associated topology is reversible.

**Theorem 4.3.8** Every completely homogeneous L-topology is hereditarily reversible.

**Proof.** Trivial

**Remark 4.3.3** Complete homogeneity implies reversibility but reversibility need not imply complete homogeneity.
4.4 Lattice of Completely homogeneous L-topological spaces

Let $L(X)$ denote the lattice of L-topologies on any set $X$ and $CH(X)$ denote the collection of completely homogeneous L-topology on $X$. $CH(X)$ is a lattice with smallest element, indiscrete L-topology and largest element, discrete L-topology, with inclusion order.

Theorem 4.4.1 $CH(X)$ is a complete sublattice of $L(X)$.

Proof. Trivially follows from the observation that L-topology generated by union of completely homogeneous L-topological spaces and intersection of completely homogeneous L-topological spaces is completely homogenous.

Theorem 4.4.2 $CH(X)$ is not atomic.

Proof. The L-topologies of the form $\{0, 1, \alpha\}$ where $\alpha \in L - \{0, 1\}$ are atoms in $CH(X)$.

Let $X$ be any infinite set and $F_\alpha$ be the L-topology defined by $F_\alpha = \{0\} \cup \{f \in L^X: \text{card (} X - \text{supp}(f)) < \alpha\}$ for some infinite cardinal number $\alpha, \alpha \leq |X|$. Then $F_\alpha$ is completely homogeneous and $F_\alpha$ cannot be expressed as the union of atoms.

Theorem 4.4.3 $CH(X)$ is not complemented.

Proof. Consider the completely homogeneous L-topology $F = \{0, 1, \alpha\}, \alpha \in L - \{0, 1\}$ and $L$ is infinite. We claim that $F$ has no complement. Clearly discrete L-topology is not a complement of $F$. Let $G$ be any completely homogeneous L-topology. If $\alpha \in G$, then $G$ is not a complement of $F$ since $G \cap F \neq 0 = \text{discrete L-topology}$. Suppose $\alpha \notin G$.

Since $L$ is infinite, there exist $\beta \in L$ such that $\beta \in G$ and the set $K = \{\gamma/\beta < \gamma < \alpha\}$ of constant L-subsets is not contained in $G$. Now $B = \{f \wedge g : f \in F, g \in G\}$ is a base for $H = F \lor G$. Then $K$ is not contained in $H$. Thus $G$ is not complement of $F$ and hence the theorem.

Theorem 4.4.4 $CH(X)$ has no dual atom in general.
Proof. Take $L = [0, 1]$. Then for any completely homogeneous L-topology $(X, \mathcal{F})$, we can find completely homogeneous L-topology $(X, \mathcal{F}')$ such that $\mathcal{F} \subset \mathcal{F}' \neq$ discrete L-topology.

Remark 4.4.1 The lattice of completely homogeneous topologies forms a linearly ordered sublattice of lattices of topologies. But $CH(X)$ is not linear and the problem of finding whether $CH(X)$ is distributive remain yet to be solved.

References
