4.1 INTRODUCTION

Queuing Theory Basics

We have seen that as a system gets congested, the service delay in the system increases. A good understanding of the relationship between congestion and delay is essential for designing effective congestion control algorithms. Queuing Theory provides all the tools needed for this analysis.

Communication Delays:

Let us understand the different components of delay in a messaging system. The total delay experienced by messages can be classified into the following categories:

Processing Delay:

* This is the delay between the times of receipt of a packet for transmission to the point of putting it into the transmission queue.

* On the receive end, it is the delay between the time of reception of a packet in the receive queue to the point of actual processing of the message.
This delay depends on the CPU speed and CPU load in the system.

**Queuing Delay**

* This is the delay between the point of entry of a packet in the transmit queue to the actual point of transmission of the message.

* This delay depends on the load on the communication link.

**Transmission Delay**

* This is the delay between the transmission of first bit of the packet to the transmission of the last bit.

* This delay depends on the speed of the communication link.

**Propagation Delay**

* This is the delay between the point of transmission of the last bit of the packet to the point of reception of last bit of the packet at the other end.

* This delay depends on the physical characteristics of the communication link.
Retransmission Delay

* This is the delay that results when a packet is lost and has to be retransmitted.

* This delay depends on the error rate on the link and the protocol used for retransmissions.

Little's Theorem

The average number of customers \(N\) can be determined from the following equation:

\[
N = \lambda T
\]

Here \(\lambda\) is the average customer arrival rate and \(T\) is the average service time for a customer.

Proof of this theorem can be obtained from any standard textbook on queueing theory. Consider the example of a restaurant where the customer arrival rate (\(\lambda\)) doubles but the customers still spend the same amount of time in the restaurant (\(T\)). This will double the number of customers in the restaurant (\(N\)). By the same logic if the customer arrival rate remains the same but the customers service...
time doubles, this will also double the total number of customers in the restaurant.

Waiting in line for service is one of the most unpleasant experiences of life on this world. Barrers (1957) says, in certain queuing processes a potential customer is considered “lost” if the system is busy at the time service is demanded. If not served during this time, the customer leaves the system and is considered lost. In queuing system the customer satisfaction can be increased by constructing control charts for $N$ and providing control limits for $N$ so as to make effective utilization of time. If control limits are displayed so that customer can have prior idea about control limits. For any queuing system/model, average queue length and average waiting time are the main observable characteristics. Customer wants to have the waiting time in the system as minimum as possible that is queue length should be small. To monitor the average queue length of the system through control charts, In this paper Control limits are established. Construction of Control chart for random queue length $N$ is done using method suggested by Haim Shore and also by using traditional
Shewhart method. The performance of these two control charts is compared using performance measure ARL. It is observed that using these control limits, the performance of the system may be improved. With this factor, an attempt is made to find control limits for random queue length \( N \) for Power Supply Model \((M/M/\infty) : (\infty/FIFO)\) queuing model. The pioneering work in this direction was made by Haim Shore in 2000. Two control chart \( C_1 \) and \( C_2 \) are constructed for random queue length \( N \) for \((M/M/1):(\infty/FCFS)\) queuing model by Khaparde and Dhabe (2010) and also \( C_3 \) and \( C_4 \) are constructed for random queue length \( N \) for \((M/M/1):(\infty/FCFS)\) queuing model by Khaparde and Dhabe (2011) where

Control Chart \( C_1 \): is the Shewhart control chart
Control Chart \( C_2 \): is the control chart using method of weighted variance
Control Chart \( C_3 \): is control chart for random queue length \( N \) based on skewness and
Control Chart \( C_4 \): is control chart using Nelson’s Power transformation

In this paper construction of control charts \( C_3 \) for random queue length \( N \) for “power supply model” queuing model is done using method based on skewness. Performance of control chart \( C_3 \) is compared with Control chart \( C_1 \)
4.2. **Generalized model: Birth and Death Process**

This model deals with a queuing system having single service channel, Poisson input with no limit on system capacity. Arrivals can be considered as birth to the system, whereas a departure can be looked upon as a death.

\[ n = \text{number of customers in the system} \]
\[ \lambda_n = \text{arrival rate of customers given to } n \text{ customers in the system}, \]
\[ \mu_n = \text{departure rate of customers given } n \text{ customers in the system}, \]
\[ P_n = \text{steady − state probability of } n \text{ customers in the system} \]

The model determines the values of \( P_n \) in terms of \( \lambda_n \) and \( \mu_n \). Now we use the following Axioms of Poisson Process:

**Axiom 1:**

The number of arrivals is non-overlapping intervals are statistically independent, that is, the process has independent increments.

**Axiom 2:**

The probability of more than one arrival occurs between time \( t \) and time \( t + \Delta t \) is \( o(t) \); that is the probability of two or more arrivals during the small time interval \( \Delta t \) is negligible.

Thus

\[ P_0(t) + P_1(\Delta t) + o(\Delta t) = 1 \]

**Axiom 3:**

The probability that an arrival occurs between time \( t \) and time \( t + \Delta t \) is equal to \( \lambda \Delta t + o(\Delta t) \)

Thus

\[ P_1(\Delta t) = \lambda \Delta t + o(\Delta t) \]

Where \( \lambda \) is constant and is independent of the total number of arrivals up to time \( t \), \( \Delta t \) is an increment element, and \( o(\Delta t) \) represents the terms such that
\[
\lim_{\Delta t \to \infty} \left( \frac{o(\Delta t)}{\Delta t} \right) = 0.
\]

From the axioms we observed that an arrival during the small time interval \( \Delta t \) is negligible. This implies that for \( \Delta t > 0 \), state \( n \) can change only two possible states: states \( n - 1 \) when a departure occurs at the rate \( \mu_n \) and states \( n + 1 \) when an arrival occurs at rate \( \lambda_n \). State 0 can only change to state 1 when an arrival occurs at the rate \( \lambda_0 \). Since no departure is possible when the system is empty, \( \mu_0 \) is undefined.

Under steady-state conditions, for \( \Delta t > 0 \), the rate of flow into and out of state \( n \) must be equal. This is illustrated in the transition-rate diagram given below:

The balance equation is:

Expected rate of flow into state \( n \) = Expected rate of flow out of state \( n \)
i.e. 
\[
\lambda_{n-1}P_{n-1} + \mu_{n+1}P_{n+1} = \lambda_nP_n + \mu_nP_n
\]

and 
\[
\mu_1P_1 = \lambda_0P_0
\]

using the iterative procedure, we have

\[
P_1 = \frac{\lambda_0}{\mu_1}P_0
\]

\[
P_2 = \frac{\lambda_0 + \mu_1}{\mu_2}P_1 - \frac{\lambda_0}{\mu_2}P_0
\]
In general, we can write the following formula

\[ P_n = \frac{\lambda_{n-1} \lambda_{n-2} \lambda_{n-3} \ldots \lambda_0}{\mu_n \mu_{n-1} \mu_{n-2} \ldots \mu_1} P_0 \]  

or

\[ P_n = P_0 + \prod_{i=1}^{n} \frac{\lambda_i}{\mu_{i+1}} \]

Now

\[ P_{n+1} = P_0 \prod_{i=1}^{n} \frac{\lambda_i}{\mu_{i+1}} \]

Thus, by mathematical induction the general value of \( P_n \) holds for all .

To obtain the value of \( P_0 \), we use the boundary condition

\[ \sum_{n=0}^{\infty} P_n = 1 \text{ and } P_0 + \sum_{n=1}^{\infty} P_n = 1 \]

\[ P_0 = \left( 1 + \sum_{i=1}^{n-1} \prod_{i=1}^{\infty} \frac{\lambda_i}{\mu_{i+1}} \right)^{-1} \]
4.3. **Power Supply Model:**

This model describes a situation where an electric circuit supplies power to \( a \) customers. The requirements of customers are assumed to follow Poisson distribution with parameter \( \lambda \), and the supply schedule also follows Poisson distribution with parameter \( \mu \).

Using the balance equation for the generalized model (Birth and Death Process) discussed above with

\[
\lambda_n = (a - n)\lambda \quad \text{and} \quad \mu_n = n\mu
\]

We have,

\[
P_n = \frac{1}{n!} a(a-1) \ldots (a-n+1) \left( \frac{\lambda}{\mu} \right)^n P_0
\]

And \( P_a = \frac{a!}{a!} \left( \frac{\lambda}{\mu} \right)^a P_0 \)

Using the boundary condition

\[
\sum_{n=0}^{\infty} P_n = 1
\]

We get,

\[
P_0 \left[ 1 + \frac{\lambda}{\mu} + a(a - 1)/2! \left( \frac{\lambda}{\mu} \right)^2 + \ldots + \left( \frac{\lambda}{\mu} \right)^a \right] = 1
\]

i.e.,

\[
P_0 \left[ 1 + \frac{\lambda}{\mu} \right]^a = 1 \quad \text{or} \quad P_0 = \left[ \frac{\mu}{\mu + \lambda} \right]^a
\]

Therefore
\[ P_n = \frac{1}{n!} a(a-1) \ldots \ldots (a-n+1) \left( \frac{\lambda}{\mu} \right)^n \left( \frac{\mu}{\mu+\lambda} \right)^a \]

\[ = \frac{a!}{(a-n)!n!} \left( \frac{\lambda}{\mu} \right)^n \left( \frac{\mu}{\mu+\lambda} \right)^a \]

\[ P_n = \binom{a}{n} \left( \frac{\lambda}{\mu+\lambda} \right)^a \left( \frac{\mu}{\mu+\lambda} \right)^{a-n} \]  \hspace{1cm} (1)

which is a binomial distribution.

Let random variable \( W_S \) denote the waiting time spent in the system by the customer. This includes both the waiting time and service time. The p.d.f. of random variable \( W_S \) is given by:

\[ f(W_S) = (\mu - \lambda) e^{-(\mu-\lambda)W_S} \quad , W_S > 0 \]  \hspace{1cm} (2)

Thus \( W_S \) follows an exponential distribution with mean

\[ E(W_S) = \int_0^\infty W_S f(W_S) dW_S \]

\[ = \int_0^\infty W_S (\mu - \lambda) e^{-(\mu-\lambda)W_S} dW_S \]

\[ E(W_S) = \frac{1}{\mu-\lambda} \]  \hspace{1cm} (3)

\[ E(W_S^2) = \frac{2}{(\mu-\lambda)^2} \]  \hspace{1cm} (4)

Using (3) and (4), the variance is given by

\[ V(W_S) = E(W_S^2) - [E(W_S)]^2 \]

\[ V(W_S) = \frac{1}{(\mu-\lambda)^2} \]  \hspace{1cm} (5)

The distribution function of \( W_S \) is

\[ F(x) = P(W_S \leq x) \]
\[ F(x) = 1 - e^{-(\mu - \lambda)x}, x > 0 \]

Hence

The probability distribution of \( N \) is a geometric distribution with parameter \( (1 - \rho) \). The steady state distribution of the random variable \( N \) depends on two parameters \( \mu \) and \( \lambda \) only through their ratio.

Using equation (1), the average number of customers and variance of queue length in the system is given by

\[ E(N) = \mu_1' \]

\[ = \sum_{a=1}^{\infty} n \cdot P_n \]

\[ = \sum_{a=1}^{\infty} n \cdot \left( \frac{a}{n(a-1)!} \right) \left( \frac{\lambda}{\mu + \lambda} \right)^{a-1} \left( \frac{\mu}{\mu + \lambda} \right)^{a-n} \]

\[ = \left( \frac{\lambda}{\mu + \lambda} \right)^a \sum_{a=1}^{\infty} \left( \frac{a-1}{n(n-1)!} \right) \left( \frac{\lambda}{\mu + \lambda} \right)^{a-1} \left( \frac{\mu}{\mu + \lambda} \right)^{a-n} \]
Using the boundary condition

We get

\[ \mu_1' = \left( \frac{\lambda}{\mu + \lambda} \right) a \]  

\[ E(N^2) = \mu_2' \]

\[ E(N^2) = E[N(N - 1) + N] \]

\[ = \sum_{a=1}^{n} \left[ n(n - 1) \right] \binom{a}{n} \left( \frac{\lambda}{\mu + \lambda} \right)^a \left( \frac{\mu}{\mu + \lambda} \right)^{a-n} + \sum_{a=1}^{n} n \binom{a}{n} \left( \frac{\lambda}{\mu + \lambda} \right)^a \left( \frac{\mu}{\mu + \lambda} \right)^{a-n} \]

\[ = \sum_{a=1}^{n} n(n - 1) \left( \frac{a(a - 1)(a - 2)!}{n(n - 1)(n - 2)! (a - n)!} \right) \left( \frac{\lambda}{\mu + \lambda} \right)^{a-2} \left( \frac{\lambda}{\mu + \lambda} \right)^2 \left( \frac{\mu}{\mu + \lambda} \right)^{a-n} \]

\[ + \sum_{a=1}^{n} \frac{n(a - 1)!}{n(n - 1)!} \left( \frac{\lambda}{\mu + \lambda} \right)^{a-1} \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)^{a-n} \]

\[ = \left( \frac{\lambda}{\mu + \lambda} \right)^2 a(a - 1) \sum_{a=1}^{n} \binom{a - 2}{n - 2} \left( \frac{\lambda}{\mu + \lambda} \right)^{a-2} \left( \frac{\mu}{\mu + \lambda} \right)^{a-n} \]

\[ + \left( \frac{\lambda}{\mu + \lambda} \right) a \sum_{a=1}^{n} \binom{a - 1}{n - 1} \left( \frac{\lambda}{\mu + \lambda} \right)^{a-1} \left( \frac{\mu}{\mu + \lambda} \right)^{a-n} \]
By using the boundary condition, we get

\[ \mu_2' = \left( \frac{\lambda}{\mu + \lambda} \right)^2 a(a - 1) + \left( \frac{\lambda}{\mu + \lambda} \right) a \]

\[ = a \left( \frac{\lambda}{\mu + \lambda} \right) \left[ 1 + (a - 1) \left( \frac{\lambda}{\mu + \lambda} \right) \right] \]

\[ \text{---------------------(7)} \]

Using (6) and (7), the variance of queue length in the system is

\[ V(N) = E(N^2) - [E(N)]^2 \]

\[ V(N) = \]

\[ \left[ \sum_{a=1}^{n} [n(n - 1)] \binom{a}{n} \left( \frac{\lambda}{\mu + \lambda} \right)^a \left( \frac{\mu}{\mu + \lambda} \right)^{a-n} + \sum_{a=1}^{n} n \binom{a}{n} \left( \frac{\lambda}{\mu + \lambda} \right)^a \left( \frac{\mu}{\mu + \lambda} \right)^{a-n} \right] \]

\[ - \left[ \sum_{a=1}^{n} n \binom{a}{n} \left( \frac{\lambda}{\mu + \lambda} \right)^a \left( \frac{\mu}{\mu + \lambda} \right)^{a-n} \right]^2 \]

Hence

\[ V(N) = \left[ a \left( \frac{\lambda}{\mu + \lambda} \right) \left[ 1 + (a - 1) \left( \frac{\lambda}{\mu + \lambda} \right) \right] \right] - \left[ \left( \frac{\lambda}{\mu + \lambda} \right) a \right]^2 \]
\[ V(N) = a \left( \frac{\lambda}{\mu + \lambda} \right) + a(a - 1) \left( \frac{\lambda}{\mu + \lambda} \right)^2 - a^2 \left( \frac{\lambda}{\mu + \lambda} \right)^2 \]

\[ V(n) = a \left( \frac{\lambda}{\mu + \lambda} \right) \left( 1 - \left( \frac{\lambda}{\mu + \lambda} \right) \right) \]

\[ V(N) = a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right) \]

\[ \text{(8)} \]

Using (8) the standard deviation is calculated as

\[ \sigma = \sqrt{V(N)} \]

So,

\[ \sigma = \sqrt{\left[ a \left( \frac{\lambda}{\mu + \lambda} \right) \left( 1 + (a - 1) \left( \frac{\lambda}{\mu + \lambda} \right) \right) \right] - \left[ \left( \frac{\lambda}{\mu + \lambda} \right) a \right]^2} \]

\[ \sigma = \sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)} \]

\[ \text{(9)} \]

Now

\[ E(N^3) = \mu_3' \]

For this we use

\[ N^3 = N(N - 1)(N - 2) + 3N^2 - 2N \]

We have

\[ E(N^3) = E[N(N - 1)(N - 2)] + 3E(N^2) - 2E(N) \]
Now we have to first calculate $E[N(N - 1)(N - 2)]$. By using the (1) we have

$$E[N(N - 1)(N - 2)] = \sum_{a=1}^{n} n(n-1)(n-2) \binom{n}{a} \left( \frac{\lambda}{\mu + \lambda} \right)^a \left( \frac{\mu}{\mu + \lambda} \right)^{a-n}$$

$$= \sum_{a=1}^{n} n(n - 1)(n - 2) \left( \frac{a(a - 1)(a - 2)(a - 3)!}{n(n - 1)(n - 2)(n - 3)! (a - n)!} \right) \left( \frac{\lambda}{\mu + \lambda} \right)^{a-3} \left( \frac{\mu}{\mu + \lambda} \right)^3 \left( \frac{\mu}{\mu + \lambda} \right)^{a-n}$$

$$= a(a - 1)(a - 2) \left( \frac{\lambda}{\mu + \lambda} \right)^3 \sum_{a=1}^{n} \frac{(a - 3)}{(n - 3)} \left( \frac{\lambda}{\mu + \lambda} \right)^{a-3} \left( \frac{\mu}{\mu + \lambda} \right)^{a-n}$$

By using the boundary condition, we have

$$E[N(N - 1)(N - 2)] = a(a - 1)(a - 2) \left( \frac{\lambda}{\mu + \lambda} \right)^3 \mbox{------------------(10)}$$

Hence by using (6), (7) and 10, we have

$$E(N^3) = E[N(N - 1)(N - 2)] + 3E(N^2) - 2E(N)$$

$$= a(a - 1)(a - 2) \left( \frac{\lambda}{\mu + \lambda} \right)^3$$

$$+ 3 \left[ a \left( \frac{\lambda}{\mu + \lambda} \right) \left[ 1 + (a - 1) \left( \frac{\lambda}{\mu + \lambda} \right) \right] - 2 \left[ \left( \frac{\lambda}{\mu + \lambda} \right) a \right] \right]$$
\[ E(N^3) = a(a-1)(a-2) \left( \frac{\lambda}{\mu+\lambda} \right)^3 + 3a(a-1) \left( \frac{\lambda}{\mu+\lambda} \right)^2 + a \left( \frac{\lambda}{\mu+\lambda} \right) \]

Hence we get
\[ E(N^3) = \mu_3' = a(a-1)(a-2) \left( \frac{\lambda}{\mu+\lambda} \right)^3 + 3a(a-1) \left( \frac{\lambda}{\mu+\lambda} \right)^2 + a \left( \frac{\lambda}{\mu+\lambda} \right) \]

\[ \text{------------------------------------(11)} \]

Using the (6), (7) and (11) the third central moment is given by
\[ \mu_3 = \mu_3' - 3\mu_2\mu_1 + 2[\mu_1']^3 \]
\[ \mu_3 = \left[ \left( a(a-1)(a-2) \left( \frac{\lambda}{\mu+\lambda} \right)^3 + 3a(a-1) \left( \frac{\lambda}{\mu+\lambda} \right)^2 + a \left( \frac{\lambda}{\mu+\lambda} \right) \right) \right. \]
\[ - 3 \left( a \left( \frac{\lambda}{\mu+\lambda} \right) \left[ 1 + (a-1) \left( \frac{\lambda}{\mu+\lambda} \right) \right] \right) \left( \frac{\lambda}{\mu+\lambda} \right)^3 \]
\[ + 2 \left( \left( \frac{\lambda}{\mu+\lambda} \right)^3 \right) \left. \right] \]
\[ \mu_3 = a \left( \frac{\lambda}{\mu+\lambda} \right) \left[ -3a \left( \frac{\lambda}{\mu+\lambda} \right)^2 + 3a \left( \frac{\lambda}{\mu+\lambda} \right) + 2 \left( \frac{\lambda}{\mu+\lambda} \right)^2 - 3 \left( \frac{\lambda}{\mu+\lambda} \right) + 1 \right. \]
\[ - 3a \left( \frac{\lambda}{\mu+\lambda} \right) \left( \frac{\mu}{\mu+\lambda} \right) \left. \right] \]
\[ \mu_3 = a \left( \frac{\lambda}{\mu+\lambda} \right) \left[ 3a \left( \frac{\lambda}{\mu+\lambda} \right) \left( 1 - \left( \frac{\lambda}{\mu+\lambda} \right) + 2 \left( \frac{\lambda}{\mu+\lambda} \right)^2 - 3 \left( \frac{\lambda}{\mu+\lambda} \right) + 1 - 3a \left( \frac{\lambda}{\mu+\lambda} \right) \left( \frac{\mu}{\mu+\lambda} \right) \right] \right] \]
\[ \mu_3 = a \left( \frac{\lambda}{\mu + \lambda} \right) \left[ 2 \left( \frac{\lambda}{\mu + \lambda} \right)^2 - 3 \left( \frac{\lambda}{\mu + \lambda} \right) + 1 \right] \]

\[ \mu_3 = a \left( \frac{\lambda}{\mu + \lambda} \right) \left[ 2 \left( \frac{\lambda}{\mu + \lambda} \right)^2 - 2 \left( \frac{\lambda}{\mu + \lambda} \right) + \left( \frac{\mu}{\mu + \lambda} \right) \right] \]

\[ \mu_3 = a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right) \left[ 1 - 2 \left( \frac{\lambda}{\mu + \lambda} \right) \right] \]

\[ \mu_3 = a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right) \left[ \frac{\mu}{\mu + \lambda} - \frac{\lambda}{\mu + \lambda} \right] \]

\[ \frac{\lambda}{\mu + \lambda} \left[ \frac{\mu}{\mu + \lambda} - \frac{\lambda}{\mu + \lambda} \right] \]

\[ \left( \sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)} \right)^3 \]

\[ S_k(N) = \frac{\mu_3}{\sigma^3} \]

where \( \mu_3 \) is the third central moment.

Using (9) and (12)

\[ S_k(N) = \frac{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right) \left[ \frac{\mu}{\mu + \lambda} - \frac{\lambda}{\mu + \lambda} \right]}{\left( \sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)} \right)^3} \]
Chapter IV

\[ S_k(N) = \sqrt{\frac{\left[ \frac{\mu}{\mu + \lambda} - \frac{\lambda}{\mu + \lambda} \right]^{2}}{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)}} \]

-----------------------------------(13)

4.4 **SHEWHART'S CONTROL CHART C1**

The basic principle underlying all the shewhart type control charts is that the distribution of the plotted statistic may be approximated by a normal distribution, with parameters that preserve the true mean and standard deviation. The control charts have very widely spread applications which means that they are very useful and it also speaks about the general
robustness of the Shewhart control chart to deviation from normality.

Traditional Shewhart chart ignores skewness of the plotted statistic. Sometimes the skewness is too large to be ignored. Ignoring skewness affects the performance of the control chart and results in high false alarm rate. Many such circumstances have been extensively studied and reported in literature. This has motivated the various control schemes that take account of non-normality of the monitoring statistic. Haim Shore[2000] have developed control chart for N for (M/M/s):($\infty$/FCFS) queuing models. He has developed a general framework for constructing Shewhart like control charts for attributes based on fitting a quantile function that preserves all first three moments of the plotted statistic.

The formulae for calculating the control limits are also based on these three moments. To make these control limits more accurate, the skewness measure used in the calculation is inflated by 44%. This inflation rate gives more accurate control limits for diversely shaped attribute distribution like binomial, the Poisson, the Geometric, the negative binomial. Comprehensive numerical assessment of this new approach with regard to binomial, the Poisson, the Geometric, the negative binomial is carried out and it is shown that this new approach is very effective, convenient to
use and most importantly there is no major departure from Shewhart control charts.

When it is possible to specify standard values for the process mean and standard deviation, we may use these standards to establish the control chart. Then the parameters of the chart are

\[ UCL = \mu + A\sigma \]
\[ CL = \mu \]
\[ LCL = \mu - A\sigma \]

where the quantity \( A = \frac{3}{\sqrt{a}} \), say, is a constant that depends on \( a \).

Using expressions (6) and (9), the control limits for ‘Power Supply Model’ are calculated as follows:

\[ UCL = \left( \frac{\lambda}{\mu + \lambda} \right) a + A \sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)} \]

\[ UCL = \frac{\rho}{(1 + \rho)} + A \frac{\sqrt{a\rho}}{(1 + \rho)} \]

----------------------------------------(14)

\[ CL = \left( \frac{\lambda}{\mu + \lambda} \right) a \]
\[
LCL = \left( \frac{\lambda}{\mu + \lambda} \right) a - A \sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)}
\]

\[
= \frac{\rho}{(1 + \rho)} - A \frac{\sqrt{\alpha \rho}}{(1 + \rho)}
\]

where \( \frac{\lambda}{\mu} \) is traffic intensity.

4.5. **CONSTRUCTION OF CONTROL CHART \( C_3 \) FOR RANDOM VARIABLE \( N \)**

Before constructing the chart, it is essential to notice that the probability function of a geometric distribution is a monotonously decreasing function of \( n \) and since control limits for queueing system are to be obtained, there is no meaning in defining lower control limit (LCL) for this distribution. Also upper control limit (UCL) obviously makes sense since it will enable monitoring improbable high value of \( N \).

5.1 **CONTROL CHART \( C_3 \)**

In the Power Supply queueing model, the random variable \( N \)
denote the number of customers in the system both waiting and in service. The steady state distribution of random variable \( N \) is given by expression (1). In order to obtain control limits using Shore’s method (2000), the first three moments of the random variable \( N \) given by expressions (6), (8) and (12) respectively, are needed.

The expression (13) shows that skewness of the distribution of \( N \) depends on \( \lambda \), \( \mu \) and \( a \). Taking into account the mean, standard deviation and skewness of the underlying geometric distribution of \( N \) the expression for the new control limits for the number of customers in ‘Power Supply Model’ using Shore’s method (2000) are as follows:

**Control Limits for Control Chart C3, when \( S_K < 0.5 \)**

\[
UCL = E(N) + 3\sqrt{V(N)} + 1.324\sqrt{V(N)} S_K - \frac{1}{2} \\
CL = E(N) \\
LCL = E(N) - 3\sqrt{V(N)} + 1.324\sqrt{V(N)} S_K + \frac{1}{2}
\]

**Control Limits for Control Chart C3, when \( S_K > 0.5 \)**

\[
UCL = E(N) + 3.642\sqrt{V(N)} + 0.9146\sqrt{V(N)} S_K - \frac{1}{2} \\
CL = E(N) \\
LCL = E(N) - 3.642\sqrt{V(N)} + (1.40)(0.9146)\sqrt{V(N)} S_K + \frac{1}{2}
\]
Control Limits for Control Chart C3 for ‘Power Supply model’ when $S_k < 0.5$

\[
UCL = E(N) + 3 \sqrt{V(N)} + 1.324 \sqrt{V(N)} S_k - \frac{1}{2}
\]

\[
= \left[ \left( \frac{\lambda}{\mu + \lambda} \right) a \right] + 3 \sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)}
\]

\[
+ 1.324 \sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)} \left[ \frac{\mu - \lambda}{\mu + \lambda} \right] - \frac{1}{2}
\]

\[
UCL = \frac{\rho}{(1 + \rho)} a + 3 \left( \frac{\sqrt{a \rho}}{(1 + \rho)} \right) + 1.324 \frac{\sqrt{a \rho}}{(1 + \rho)} \times \frac{(1 - \rho)}{\sqrt{a \rho}} - \frac{1}{2}
\]

\[
UCL = \frac{1}{(1 + \rho)} \left[ a \rho + 3 \sqrt{a \rho} + 1.324 (1 - \rho) \right] - \frac{1}{2}
\]

\[\text{------------------------------------(17)}\]

\[CL = E(N)\]

\[
= \left[ \left( \frac{\lambda}{\mu + \lambda} \right) a \right]
\]
\[
LCL = E(N) - 3 \sqrt{V(N)} + 1.324 \sqrt{V(N)} S_K + \frac{1}{2}
\]

\[
LCL = \left[ \left( \frac{\lambda}{\mu + \lambda} \right) a \right] \\
-3 \sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)} \\
+1.324 \sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)} \left[ \frac{\mu - \lambda}{\lambda} \left( \frac{\mu}{\mu + \lambda} \right) \right] + \frac{1}{2}
\]

\[
LCL = \frac{\lambda}{\mu} a - 3 \left( \frac{\sqrt{a \rho}}{(1 + \rho)} \right) + 1.324 \frac{\sqrt{a \rho}}{(1 + \rho)} \times \frac{(1 - \rho)}{\sqrt{a \rho}} + \frac{1}{2}
\]

Control Limits for Control Chart C3 for ‘Power Supply model’ when \( S_K > 0.5 \)

\[
UCL = E(N) + 3.642 \sqrt{V(N)} + 0.9146 \sqrt{V(N)} S_K - \frac{1}{2}
\]
\[
= \left[ \left( \frac{\lambda}{\mu + \lambda} \right) a \right]
\]
\[
+ 3.642 \sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)}
\]
\[
+ 0.9146 \sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)} \left[ \frac{\mu}{\sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)}} \right] \frac{1}{2}
\]

\[
UCL = \frac{\rho}{(1 + \rho)} a + 3.642 \left( \frac{\sqrt{a \rho}}{(1 + \rho)} \right) + 0.9146 \frac{\sqrt{a \rho}}{(1 + \rho)} \times \frac{(1 - \rho)}{\sqrt{a \rho}} - \frac{1}{2}
\]

\[
UCL = \frac{1}{(1 + \rho)} \left[ a \rho + 3.642 \sqrt{a \rho} + 0.9146(1 - \rho) \right] - \frac{1}{2}
\]

\[
CL = E(N)
\]
\[
= \left[ \left( \frac{\lambda}{\mu + \lambda} \right) a \right]
\]
\[
= \frac{a \rho}{(1 + \rho)}
\]

\[
LCL = E(N) - 3.642 \sqrt{V(N)} + (1.40)(0.9196) \sqrt{V(N)} S_K + \frac{1}{2}
\]
\[
= \left[ \left( \frac{\lambda}{\mu + \lambda} \right) a \right]
\]
\[-3.642 \sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)}\]

\[+(1.40)(0.9196) \sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)} \left[ \frac{\mu}{\sqrt{a \left( \frac{\lambda}{\mu + \lambda} \right) \left( \frac{\mu}{\mu + \lambda} \right)}} \right] + \frac{1}{2}\]

\[= \frac{\rho}{1 + \rho} a - 3.642 \left( \frac{\sqrt{a \rho}}{1 + \rho} \right) + (1.40)(0.9196) \frac{\sqrt{a \rho}}{1 + \rho} \times \left(1 - \frac{\rho}{1 - \rho} + \frac{1}{2}\right)\]

\[LCL = \frac{1}{1 + \rho} \left[ a \rho - 3.642 \sqrt{a \rho} + (1.40)(0.9196)(1 - \rho) \right] + \frac{1}{2}\]

\[\text{------------------------------------(22)}\]

where \( \rho = \frac{\lambda}{\mu} \), traffic intensity.

4.5 **Comparison of Control Charts:**

**Table 1.**

Mean, standard deviation, Upper control limit for the random variable for different values of \( \rho \) using Shewhart control charts C1 with A=1, a=1 with L=3

<table>
<thead>
<tr>
<th>S.NO.</th>
<th>( \lambda )</th>
<th>( \mu )</th>
<th>( \rho )</th>
<th>( \sigma )</th>
<th>CL</th>
<th>UCL</th>
<th>Skewness</th>
<th>FAR</th>
<th>ARL</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>20</td>
<td>0.05</td>
<td>0.212959</td>
<td>0.047619</td>
<td>0.260578</td>
<td>4.248529</td>
<td>0.078359</td>
<td>12.76178</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>20</td>
<td>0.1</td>
<td>0.28748</td>
<td>0.090909</td>
<td>0.378389</td>
<td>2.84605</td>
<td>0.074983</td>
<td>13.33636</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>20</td>
<td>0.15</td>
<td>0.336781</td>
<td>0.130435</td>
<td>0.467216</td>
<td>2.194691</td>
<td>0.06556</td>
<td>15.2532</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>20</td>
<td>0.2</td>
<td>0.372678</td>
<td>0.166667</td>
<td>0.539345</td>
<td>1.788854</td>
<td>0.057944</td>
<td>17.25804</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>20</td>
<td>0.25</td>
<td>0.4</td>
<td>0.2</td>
<td>0.6</td>
<td>1.5</td>
<td>0.049412</td>
<td>20.238</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>20</td>
<td>0.3</td>
<td>0.421325</td>
<td>0.230769</td>
<td>0.652094</td>
<td>1.278019</td>
<td>0.04339</td>
<td>23.04678</td>
</tr>
</tbody>
</table>
### Table 2

Comparison of performance measure FAR for different values of $\rho$ for control charts C1, C2 and C3 with $L=3$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Control chart C1</th>
<th>Control chart C2</th>
<th>Control chart C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.078359</td>
<td>0.038645</td>
<td>0.008803</td>
</tr>
<tr>
<td>0.1</td>
<td>0.074983</td>
<td>0.029158</td>
<td>0.005205</td>
</tr>
<tr>
<td>0.15</td>
<td>0.06556</td>
<td>0.025966</td>
<td>0.003562</td>
</tr>
<tr>
<td>0.2</td>
<td>0.057944</td>
<td>0.024458</td>
<td>0.002604</td>
</tr>
<tr>
<td>0.25</td>
<td>0.049412</td>
<td>0.0236639</td>
<td>0.001972</td>
</tr>
<tr>
<td>0.3</td>
<td>0.04339</td>
<td>0.023165</td>
<td>0.001523</td>
</tr>
<tr>
<td>0.35</td>
<td>0.03989</td>
<td>0.023007</td>
<td>0.001344</td>
</tr>
<tr>
<td>0.4</td>
<td>0.032007</td>
<td>0.022886</td>
<td>0.001187</td>
</tr>
<tr>
<td>0.45</td>
<td>0.029222</td>
<td>0.022794</td>
<td>0.001049</td>
</tr>
<tr>
<td>0.5</td>
<td>0.026052</td>
<td>0.022726</td>
<td>0.000926</td>
</tr>
<tr>
<td>0.55</td>
<td>0.024888</td>
<td>0.022677</td>
<td>0.000816</td>
</tr>
<tr>
<td>0.6</td>
<td>0.01935</td>
<td>0.022647</td>
<td>0.000718</td>
</tr>
<tr>
<td>0.65</td>
<td>0.019207</td>
<td>0.022638</td>
<td>0.000699</td>
</tr>
<tr>
<td>0.7</td>
<td>0.01901</td>
<td>0.022633</td>
<td>0.000681</td>
</tr>
</tbody>
</table>
TABLE 3:
Comparison of performance measure ARL for different values of $\rho$ for control charts C1, C2 and C3 with L=3

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Control chart C1</th>
<th>Control chart C2</th>
<th>Control chart C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>12.76177593</td>
<td>25.87656877</td>
<td>113.5976372</td>
</tr>
<tr>
<td>0.1</td>
<td>13.33635624</td>
<td>34.29590507</td>
<td>192.1229587</td>
</tr>
<tr>
<td>0.15</td>
<td>15.25320317</td>
<td>38.51190018</td>
<td>280.7411567</td>
</tr>
<tr>
<td>0.2</td>
<td>17.25804225</td>
<td>40.88641753</td>
<td>384.0245776</td>
</tr>
<tr>
<td>0.25</td>
<td>20.23799887</td>
<td>42.2584612</td>
<td>507.0993915</td>
</tr>
<tr>
<td>0.3</td>
<td>23.04678497</td>
<td>43.16857328</td>
<td>656.5988181</td>
</tr>
<tr>
<td>0.35</td>
<td>25.06893958</td>
<td>43.46503238</td>
<td>744.047619</td>
</tr>
<tr>
<td>0.4</td>
<td>31.24316556</td>
<td>43.69483527</td>
<td>842.4599832</td>
</tr>
<tr>
<td>0.45</td>
<td>34.22079255</td>
<td>43.87119417</td>
<td>953.2888465</td>
</tr>
<tr>
<td>0.5</td>
<td>38.38476892</td>
<td>44.00246414</td>
<td>1079.913607</td>
</tr>
<tr>
<td>0.55</td>
<td>40.18000643</td>
<td>44.09754377</td>
<td>1225.490196</td>
</tr>
<tr>
<td>0.6</td>
<td>51.67958656</td>
<td>44.15595885</td>
<td>1392.75766</td>
</tr>
<tr>
<td>0.65</td>
<td>52.06435154</td>
<td>44.17351356</td>
<td>1430.615165</td>
</tr>
<tr>
<td>0.7</td>
<td>52.60389269</td>
<td>44.18327221</td>
<td>1468.428781</td>
</tr>
<tr>
<td>0.75</td>
<td>52.63989051</td>
<td>44.19108224</td>
<td>1508.295626</td>
</tr>
<tr>
<td>0.8</td>
<td>52.92685509</td>
<td>44.19889503</td>
<td>1547.987616</td>
</tr>
<tr>
<td>0.85</td>
<td>53.20847079</td>
<td>44.19889503</td>
<td>1592.356688</td>
</tr>
<tr>
<td>0.9</td>
<td>53.49023803</td>
<td>44.21061939</td>
<td>1633.986928</td>
</tr>
<tr>
<td>0.95</td>
<td>53.77211378</td>
<td>44.21452889</td>
<td>1680.672269</td>
</tr>
</tbody>
</table>

Comparison of control charts C1, C2 and C3 and conclusion:

(i) Table 1 shows as $\rho$ increases, false alarm rate decreases and values of control limit and ARL increases.

(ii) It can be observed from table 2 and 3 that the value of UCL for control chart C3 is very high as compared to control chart C1 and C2. This is because
skewness of the underlying distribution of $N$ is taken into consideration while obtaining control limits.

(iii) Comparing control charts C1, C2 and C3, it is observed that in all the three control charts as $\rho$ increases false alarm rates decreases.
REFERENCES


