Chapter 8

Comparative study of Haar Wavelet and Hosoya polynomial for the numerical solution of Fredholm integral equations
8.1 Introduction

In graph theory, as in discrete mathematics in general, not only the existence, but also the counting of objects with some given properties, is of main interest. Each area introduces its own special terms for shared concepts in discrete mathematics. The only way to keep from reinventing the wheel from area to area is to know the precise mathematical ideas behind the concepts being applied by these various fields.

Integral equations have motivated a large amount of research work in recent years. Integral equations find its applications in various fields of mathematics, science and technology has been studied extensively both at the theoretical and practical level. In particular, integral equations arise in fluid mechanics, biological models, solid state physics, kinetics in chemistry etc. In most of the cases, it is difficult to solve them, especially analytically [129]. Analytical solutions of integral equations, however, either does not exist or are difficult to find. It is precisely due to this fact that several numerical methods have been developed for finding solutions of integral equations.

Consider the Fredholm integral equation:

\[ y(x) = f(x) + \int_{0}^{1} k(x, t)y(t)dt \quad 0 \leq x, t \leq 1 \]  \hspace{1cm} (8.1)

where \( f(x) \) and the kernels \( k(x, t) \) are assumed to be in \( L^2(R) \) on the interval \( 0 \leq x, t \leq 1 \). We assume that Eq.(8.1) has a unique solu-
tion $y(x)$ to be determined. There are several numerical methods for approximating the solution of Fredholm integral equations are known and many different basic functions have been used. Such as, Lepik et al.[80] applied the Haar Wavelets. Maleknejad et al. [90] applied a combination of Hybrid taylor and block-pulse functions, Rationalized haar wavelet [91].

Wavelet theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [7]. Since 1991 the various types of wavelet method have been applied for the numerical solution of different kinds of integral equations. The solutions are often quite complicated and the advantages of the wavelet method get lost. Therefore any kind of simplification is welcome. One possibility for it is to make use of the Haar wavelets, which are mathematically the simplest wavelets. In the previous work, system analysis via Haar wavelets was led by Chen and Hsiao [17], who first derived a Haar operational matrix for the integrals of the Haar function vector and put the applications for the Haar analysis into the dynamic systems.
cently, Haar wavelet method is applied for different type of problems. Namely, Bujurke et al. [13, 14] used the single term Haar wavelet series for the numerical solution of stiff systems from nonlinear dynamics and Sturm-Liouville problems. Shiralashetti et al. [120, 121] applied for the numerical solution of multi-term fractional differential equations and nonlinear Fredholm integral equations.

In the present work, a comparison of a Haar wavelet and Hosoya polynomial methods for the numerical solution of Fredholm integral equations is proposed.

8.2 Hosoya Polynomial

The Hosoya polynomial of a graph is a generating function about distance distributing, introduced by Hosoya [66] in 1988. For a connected graph $G$, the Hosoya polynomial denoted by $H(G, \lambda)$ is defined as

$$H(G, \lambda) = \sum_{k \geq 0} d(G, k) \lambda^k$$ (8.2)

where $d(G, k)$ is the number of pairs of vertices of $G$ that are at distance $k$ and $\lambda$ is the parameter.

The path on $n$ vertices (or of length $n - 1$) is the graph with $n$ vertices-say, $1, 2, \ldots, n$ and with $n - 1$ edges, such that vertices $i$ and $i + 1$ are adjacent, $i = 1, 2, \ldots, n - 1$. For any positive integer $n$ we
denote path as $P_n$, then Hosoya polynomial of path is:

The paths $P_1$, $P_2$ and $P_3$ are depicted in Fig. 8.1.

![Figure 8.1: Path graphs $P_1$, $P_2$ and $P_3$.](image)

The Hosoya polynomial of a path $P_n$ is:

$$H(P_n, \lambda) = n + (n-1)\lambda + (n-2)\lambda^2 + \cdots + [n-(n-2)]\lambda^{n-2} + [n-(n-1)]\lambda^{n-1}.$$  

In particular,

$$H(P_1, \lambda) = 1$$

$$H(P_2, \lambda) = \lambda + 2$$

$$H(P_3, \lambda) = \lambda^2 + 2\lambda + 3.$$  

### 8.3 Haar wavelet

The scaling function $h_1(x)$ for the family of the Haar wavelets is defined as

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0,1) \\ 0 & \text{otherwise} \end{cases} \quad (8.3)$$
The Haar wavelet family for $x \in [0, 1)$ is defined as,

$$h_i(x) = \begin{cases} 
1 & \text{for } x \in [\alpha, \beta) \\
-1 & \text{for } x \in [\beta, \gamma) \\
0 & \text{otherwise},
\end{cases} \quad (8.4)$$

where $\alpha = \frac{k}{m}, \beta = \frac{k + 0.5}{m}, \gamma = \frac{k + 1}{m}$.

In the above definition the integer, $m = 2^l, l = 0, 1, \ldots, J$, indicates the level of resolution and integer $k = 0, 1, \ldots, m - 1$ is the translation parameter. Maximum level of resolution is $J$. The index $i$ in Eqn. (8.7) is calculated using, $i = m + k + 1$. In case of minimal values $m = 1, k = 0$, then $i = 2$. The maximal value of $i$ is $N = 2^{J+1}$.

Let us define the collocation points $x_j = \frac{j - 0.5}{K}, j = 1, 2, \ldots, N$, discretize the Haar function $h_i(x)$ and the corresponding Haar coefficient matrix $H(i, j) = (h_i(x_j))$, which has the dimension $N \times N$.

As Haar Wavelets are orthogonal; this mean that any square integrable function over $[0, 1]$ can be expanded into Haar wavelets series as:

$$u(x) = \sum_{i=1}^{\infty} a_i h_i(x) \quad (8.5)$$
where $a_i$’s are Haar wavelet coefficients.

If $u(x)$ be piecewise constant, then sum can be terminated to finite term, that is

$$u(x) = \sum_{i=1}^{N} a_i h_i(x) = a^T H$$ \hspace{1cm} (8.6)

$$a^T = (a_1, a_2, ..., a^N), \ H = (h_1(x), h_2(x), ..., h_N(x))^T.$$ \hspace{1cm} (8.7)

### 8.4 Method of Solution

#### 8.4.1 Haar Wavelet Method (HWM)

Let us consider the equation

$$u(x) - \int_0^1 K(x, t)u(t)dt = f(x), \quad x, t \in [0, 1]$$ \hspace{1cm} (8.8)

and seek the solution in the form

$$u(t) = \sum_{i=1}^{N} a_i h_i(t)$$ \hspace{1cm} (8.9)

where $a_i$’s are Haar wavelet coefficients.

Substituting Eq. (8.8) in (8.9), we get

$$\sum_{i=1}^{N} a_i h_i(x) - \sum_{i=1}^{N} a_i G_i(x) = f(x),$$ \hspace{1cm} (8.10)

where, $G_i(x) = \int_0^1 K(x, t)h_i(t)dt$
Eq. (8.10) implies,

\[ \sum_{i=1}^{N} [a_i h_i(x) - a_i G_i(x)] = f(x_l), \quad l = 1, 2, \ldots, N \]  

(8.11)

Matrix form of Eqn. (8.9) and Eqn. (8.11) is as follows,

\[ u = aH, \]  

(8.12)

\[ a(H - G) = f, \]  

(8.13)

where, \( u = (u(t_l)) \), \( F = f(f(x_l)) \), \( G = (G_i(x_l)) \).

In the present case,

\[ G_i(x) = \int_{0}^{1} K(x, t) h_i(t) dt = \begin{cases} 
   x + 0.5 & \text{for } i = 1 \\
   \frac{-1}{4m^2} & \text{for } i > 1 
\end{cases} \]  

(8.14)

where \( K(x, t) = x + t \).

### 8.4.2 Hosoya Polynomial Method (HPM)

A approximation function \( f(x) \in L^2[0, 1] \) is expanded as:

\[ f(x) = \sum_{i=1}^{n} c_i H(P_i, x) = C^T H_P(x), \]  

(8.15)

where \( C \) and \( H_P(x) \) are \( n \times 1 \) matrices given by:

\[ C = [c_1, c_2, \ldots, c_n]^T, \]  

(8.16)
Comparative study of Haar Wavelet and Hosoya polynomial for the numerical solution of Fredholm integral equations

\[ H_P(x) = [H(P_1, x), H(P_2, x), \ldots, H(P_n, x)]^T. \quad (8.17) \]

Consider the Fredholm integral equation,

\[ y(x) = f(x) + \int_0^1 K(x, t)y(t)dt, \quad 0 \leq x, t \leq 1 \quad (8.18) \]

To solve Eq. (8.18), the procedure is as follows:

Step 1: We first approximate \( y(x) \) as truncated series defined in Eq. (8.15). That is,

\[ y(x) = C^T H_P(x) \quad (8.19) \]

where \( C \) and \( H_P(x) \) are defined similarly to Eqs. (8.16) and (8.17).

Step 2: Then substituting Eq. (8.19) in Eq. (8.18), we get,

\[ C^T H_P(x) = f(x) + \int_0^1 K(x, t)[C^T H_P(t)]dt. \quad (8.20) \]

Step 3: Substituting the collocation point \( x_i = \frac{i-0.5}{n}, i = 1, 2, \ldots, n \) in Eq. (8.20), we obtain,

\[ C^T H_P(x_i) = f(x_i) + C^T \int_0^1 K(x_i, t)H_P(t)dt \quad (8.21) \]

\[ C^T(H_P(x_i) - Z) = f, \text{ where } Z = \int_0^1 K(x_i, t)H_P(t)dt. \]

Step 4: Now, we get the system of algebraic equations with unknown coefficients.

\[ C^T K = f, \text{ where } K = (H_P(x_i) - Z). \]
Solving the above system of equations, we get the Hosoya coefficients ‘C’ and then substituting these coefficients in Eq. (8.19), we get the required approximate solution of Eq. (8.18).

8.5 Numerical Examples

In this section, we demonstrate the capability of the method and error function is presented to verify the accuracy and efficiency of the following numerical results:

\[
\text{Error function} = \|y_e(x_i) - y_a(x_i)\|_{\infty} = \sqrt{\sum_{i=1}^{n} (y_e(x_i) - y_a(x_i))^2}
\]

where \(y_e\) and \(y_a\) are the exact and approximate solution respectively.

Example 8.5.1. Consider the Fredholm integral equation of the second kind [80],

\[
y(x) = x^2 + \int_{0}^{1} (x + t)g(t)dt, \quad 0 \leq x \leq 1. \tag{8.22}
\]

which has the exact solution \(y(x) = x^2 - 5x - (17/6)\). Solving the Eq. (8.22) by Hosoya polynomial method for \(n = 3\), we get the Hosoya coefficients \(C_1 = \frac{49}{6}, C_2 = -7, C_3 = 1\). Substituting these coefficients in Eq. (8.19) we get,

\[
y(x) = \frac{49}{6}H_P(x_1) + (-7)H_P(x_2) + (1)H_P(x_3)
\]

\[
y(x) = x^2 - 5x - (17/6)
\]
which is the analytic solution of this problem.

**Example 8.5.2.** Consider Fredholm integral equation of the second kind [92],

\[ y(x) = x^6 \log(x) + \int_0^1 (x + t)y(t)dt, \quad 0 \leq x \leq 1, \quad (8.23) \]

which has the exact solution \( y(x) = x^6 \log(x) + 0.3096x + 0.1752 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact solution</th>
<th>HWM ((N = 8))</th>
<th>Abs. Error (HWM)</th>
<th>HPM ((n = 8))</th>
<th>Abs. Error (HPM)</th>
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</table>
Example 8.5.3. Consider the Fredholm integral equation of the second kind \cite{90},

\[ y(x) = \exp(x) + \left(1 - \exp(1)\right)x - 1 + \int_{0}^{1} (x + t)y(t)\,dt, \quad 0 \leq x \leq 1, \quad (8.24) \]

which has the exact solution \( y(x) = \exp(x) \).

### Numerical result of Example 8.5.3

<table>
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<tr>
<th>( x )</th>
<th>Exact solution</th>
<th>HWM ((N = 8))</th>
<th>Abs. Error (HWM)</th>
<th>HPM ((n = 8))</th>
<th>Abs. Error (HPM)</th>
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### Maximum of Error Analysis.

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<tr>
<th>N</th>
<th>Example 1</th>
<th>Example 2</th>
<th>Example 3</th>
<th>n</th>
<th>Example 1</th>
<th>Example 3</th>
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Example 8.5.4. Consider the Fredholm integral equation of the second kind [110],

\[ y(x) = e^{2x + \frac{1}{3}} + \int_0^1 \frac{1}{3} e^{2x - \frac{5}{3} t} y(t) dt, \quad 0 \leq x \leq 1, \]  

(8.25)

which has the exact solution \( y(x) = e^{2x} \). Applying the proposed method to solve Eq. (8.25) for \( n = 8 \). We get the approximate solution \( y(x) \) as shown in Table 4. Error analysis is shown in Fig. 8.2.

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact solution</th>
<th>Method [110]</th>
<th>Abs. Error (Method [110])</th>
<th>HPM ((m = 128))</th>
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Comparative study of Haar Wavelet and Hosoya polynomials for the numerical solution of Fredholm integral equations

Example 8.5.5. Consider the Fredholm integral equation of the second kind [91],

\[ y(x) = e^x - \frac{e^{x+1} - 1}{x + 1} + \int_0^1 e^{xt} y(t) dt, \quad 0 \leq x \leq 1, \quad (8.26) \]

which has the exact solution \( y(x) = e^x \). Applying the proposed method to solve Eq. (8.26) for \( n = 8 \), we get the approximate solution \( y(x) \) as shown in Table 8.5. Error analysis is shown in Fig. 8.3.

Figure 8.2: Error analysis of Example 8.5.4.
Comparative study of Haar Wavelet and Hosoya polynomial for the numerical solution of Fredholm integral equations

Table 8.5: Numerical result of Example 8.5.5.

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<td>6.90e-07</td>
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</tr>
<tr>
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<td>2.71828</td>
<td>2.67690</td>
<td>4.13e-02</td>
<td>2.71828</td>
<td>1.10e-06</td>
</tr>
</tbody>
</table>

Error analysis of Example 8.5.5.
8.6 Conclusion

A new developed method (Hosoya polynomial method (HPM)) is applied for the numerical solution of Fredholm integral equations and compared with the simple Haar Wavelet Method (HWM)). The present method reduces an integral equation into a set of algebraic equations. For instance in Example 8.5.1, HPM gives higher accuracy with exact ones and with existing method (HWM). Subsequently other examples are also same in the nature and also tested several examples. The numerical result shows that, the accuracy improves with increasing the \( n \) number of vertices of polynomial for better accuracy. Error analysis justifies the effectiveness, validity and applicability with comparative study of a new developed method (HPM) with existing method (HWM).