Chapter 2

Entire and meromorphic solutions of difference equations

The content of the section 1 in this chapter is published in Fasciculi Mathematici, 56(2016), 43-59 and the content of the section 2 in this chapter is published in Mathematical Sciences International Research Journal, 4(2) (2015), 263-266.
2.1 Entire and meromorphic solutions of linear difference equations

2.1.1 Introduction

In this chapter, we investigate the existence of finite order entire and meromorphic solutions of linear difference equation of the form

\[ f^n(z) + p(z)f^{n-2}(z) + L(z, f) = h(z), \quad L(z, f) \neq 0 \]  

where

\[ L(z, f) = a_0 f(z) + a_1 f(z + c_1) + \cdots + a_k f(z + c_k) \]

is linear difference polynomial in \( f(z) \) with small functions as the coefficients and \( c_i \) are constants \( i = 1, 2, \cdots, k \), \( p(z) \) is non-zero polynomial and \( h(z) \) is a meromorphic function of finite order. We also investigate the existence of entire solution of linear difference equation of the form

\[ f^n(z) + p(z)L(z, f) = r(z)e^{q(z)} \]

where \( p(z) \neq 0 \), \( r(z) \) and \( q(z) \) are polynomials.

2.1.2 Preliminaries

The study of the existence and uniqueness of finite order entire solutions of non-linear differential equation of the form

\[ L(f) - p(z)f^n(z) = h(z) \]
was started by C. C. Yang [72] in 2001, where \( L(f) \) is a linear differential polynomial in \( f(z) \) with polynomial coefficients, \( \rho(z) \) is a non-vanishing polynomial, \( h(z) \) is an entire function and \( n \geq 3 \) is an integer.

Later on, In 2010, C. C. Yang and I. Laine [77] proved the following theorem.

**Theorem 2.1.1.** [77]

*Let \( n \geq 4 \) be an integer, \( M(z, f) \) be a linear differential-difference polynomial of \( f(z) \), not vanishing identically, and \( h(z) \) be a meromorphic function of finite order. Then the differential-difference equation

\[
f^n + M(z, f) = h(z),
\]

possesses atmost one admissible transcendental entire solution of finite order such that all coefficients of \( M(z, f) \) are small functions of \( f(z) \). If such a solution \( f(z) \) exists, then \( f(z) \) is of the same order as \( h(z) \).*

X. Qi and L. Yang [67] in 2013 proved the following result for the existence of finite order meromorphic solution of the difference equation of the form

\[
f^n(z) + L(z, f) = h(z) \tag{2.1.4}
\]

where \( h(z) \) and \( L(z, f) \) as defined in (2.1.1) and (2.1.2).

**Theorem 2.1.2.** [67]

*If \( f(z) \) is a finite order meromorphic solution of the difference equation (2.1.4) satisfying \( N(r, f) = S(r, f) \) and \( n \geq 4 \) be an integer, then one of the following statements holds:
(1) Equation (2.1.4) has \( f(z) \) as its unique transcendental meromorphic solution of finite order such that \( N(r, f) = S(r, f) \);

(2) Equation (2.1.4) has exactly \( n \) transcendental meromorphic solutions \( f_j (j = 1, 2, \cdots, n) \) of finite order such that \( N(r, f_j) = S(r, f_j) \).

Later, in 2014, X. Qi, J. Dou and L. Yang [64] obtained the following result for the non-linear difference equation of the form

\[
f(z)^n + p(z) (\Delta_c f)^m = r(z)e^{g(z)},
\]

(2.1.5)

where \( \Delta_c(f) = f(z + c) - f(z) \) and \( c \) is a non-zero constant.

**Theorem 2.1.3. [64]**

Consider the non-linear difference equation of the form (2.1.5), where \( p(z) \neq 0 \), \( q(z), r(z) \) are polynomials, \( n \) and \( m \) are positive integers. Suppose that \( f(z) \) is a transcendental entire function of finite order, not of period \( c \). If \( n > m \), then \( f(z) \) cannot be a solution of (2.1.5).

### 2.1.3 Main results

Main results of this section for linear difference equation of the form (2.1.1) and (2.1.3) are as follows.

**Theorem 2.1.4.**

Let \( n \geq 4 \) be an integer, \( L(z, y) = a_0 y(z) + a_1 y(z + c_1) + \cdots + a_k y(z + c_k) \) be a non-zero linear difference polynomial of \( y(z) \), \( h(z) \) be a meromorphic function of finite
order and $p(z)$ be a non-zero polynomial. Then, there exists at most one finite order transcendental entire function $f(z)$ such that (2.1.1) and such that all coefficients $a_{\lambda}$ of $I(z, y)$ are small functions of $f(z)$ and $c_{i}, i = 1, 2, \cdots, k$ are constants. If such solution $f(z)$ exists then $f(z)$ has same order as $h(z)$.

Next, we consider Theorem 2.1.4 for meromorphic function.

**Theorem 2.1.5.**

Let $n \geq 4$ be an integer, $I(z, y)$ be as in Theorem 2.1.4, $h(z)$ be a meromorphic function of finite order and $p(z)$ be a non-zero polynomial. If there exists a finite order transcendental meromorphic solution $f(z)$ of (2.1.1) satisfying $N(r, f) = S(r, f)$, then $f(z)$ is an unique solution.

**Theorem 2.1.6.**

Let $n > 1$ be an integer. Let $I(z, y)$ be as in Theorem 2.1.4. Consider the linear difference equation of the form (2.1.3). Then a finite order transcendental entire function $f(z)$ cannot be a solution of (2.1.3).

### 2.1.4 Proof of the main results

**Proof of the Theorem 2.1.4:**

We first prove that $\rho(h) = \rho(f)$ for all entire solutions of finite order of (2.1.1).

Since the inequality $\rho(h) \leq \rho(f)$ trivially holds, now suppose that $\rho(h) < \sigma < \rho(f) = \rho$. 

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(2.1.1) can be written as
\[
f^{n-1} = \frac{h}{f} - p(z)f^{n-3} - \frac{L(z, f)}{f}
\]  
(2.1.6)

By the hypothesis, \( f(z) \) is an entire function, we have
\[
(n - 1)T(r, f) = (n - 1)m(r, f) + (n - 1)N(r, f) = (n - 1)m(r, f)
\]  
(2.1.7)

Using (2.1.6), (2.1.7) reduces to
\[
(n - 1)T(r, f) \leq m \left( r, \frac{h}{f} \right) + m \left( r, p(z)f^{n-3} \right) + m \left( r, \frac{L(z, f)}{f} \right) + S(r, f)
\]

From Lemma 1.5.2, we deduce
\[
(n - 1)T(r, f) \leq m(r, h) + T(r, f) + (n - 3)T(r, f) + O \left( r^{\sigma(f)-1+\epsilon} \right) + S(r, f)
\]
\[
\leq T(r, h) + T(r, f) + (n - 3)T(r, f) + O \left( r^{\sigma(f)-1+\epsilon} \right) + S(r, f)
\]
i.e.,
\[
T(r, f) \leq T(r, h) + O \left( r^{\sigma(f)-1+\epsilon} \right) + S(r, f)
\]  
(2.1.8)

Since \( \rho(h) < \sigma < \rho(f) = \rho \), by definition of order, we have \( T(r, h) < r^\sigma \).

Thus, (2.1.8) reduces to
\[
T(r, f) \leq r^\sigma + O \left( r^{\sigma(f)-1+\epsilon} \right) + S(r, f)
\]
\[
\leq r^{\sigma+\epsilon} + O \left( r^{\sigma(f)-1+2\epsilon} \right) + S(r, f)
\]

for all \( r \) sufficiently large, outside of an exceptional set of finite logarithmic measure, provided \( \epsilon \) has been chosen small enough and removing the exceptional set, we get
\[
\rho(f) \leq \max \{ \rho(f) - 1 + 2\epsilon, \sigma + \epsilon \} < \rho(f)
\]
i.e., \( \rho(f) < \rho(f) \),
which is a contradiction. Hence, we have proved that \( \rho(f) = \rho = \rho(h) \).

Next, we prove that (2.1.1) possesses at most one admissible transcendental entire solution of finite order.

Now, assume that \( f(z) \) and \( g(z) \) are two distinct finite order transcendental entire solutions of (2.1.1).

Thus, we have

\[
\begin{align*}
  f^n + p(z)f^{n-2} + L(z, f) &= g^n + p(z)g^{n-2} + L(z, g) \\
  \text{(2.1.9)}
\end{align*}
\]

Clearly, \( \rho(f) = \rho(g) \). Since the difference polynomial \( L \) is linear, (2.1.9) can be written as

\[
(f^n - g^n) + p(z) (f^{n-2} - g^{n-2}) = L(z, g) - L(z, f) = L(z, g - f) \quad \text{(2.1.10)}
\]

Let

\[
F = \frac{f^n - g^n}{f - g} + p(z) \frac{f^{n-2} - g^{n-2}}{f - g} \\
= \prod_{\eta=1}^{n-1} (f - \eta g) + p(z) \prod_{\gamma=1}^{n-2} (f - \gamma_m g) \quad \text{(2.1.11)}
\]

is an entire function, here \( \eta_1, \eta_2, \cdots, \eta_{n-1} \) are the distinct roots \( \neq 1 \) of the equation \( z^n = 1 \) and \( \gamma_1, \gamma_2, \cdots, \gamma_{n-3} \) are distinct roots \( \neq 1 \) of the equation \( z^{n-2} = 1 \).

From (2.1.10), (2.1.11) and Lemma 1.5.2, we obtain

\[
\begin{align*}
  T(r, F) &= m(r, F) + N(r, F) = m(r, F) = m \left( r, \frac{L(z, g - f)}{f - g} \right) \\
  &= O \left( r^{\rho(f-g)-1+\epsilon} \right) + S(r, f) + S(r, g) \\
  \leq& \quad O \left( r^{\rho(f)-1+\epsilon} \right) + S(r, f) \\
  \text{i.e., } T(r, F) &= S_\rho(r, f) \quad \text{(2.1.12)}
\end{align*}
\]
where $c > 0$ is arbitrary and sufficiently small.

From (2.1.12), we deduce

$$N \left( r, \frac{1}{f} \right) = S_\rho(r, f)$$

Hence

$$N \left( r, \frac{1}{f - \eta f} \right) = S_\rho(r, f) \quad \text{and} \quad N \left( r, \frac{1}{f - \gamma m f} \right) = S_\rho(r, f)$$

holds for all $\rho = 1, 2, \ldots, n - 1$ and $m = 1, 2, \ldots, n - 3$.

Since

$$\frac{1}{f - \eta f} = \frac{1}{g} \frac{1}{f - \eta f g} \quad \text{and} \quad \frac{1}{f - \gamma m f} = \frac{1}{g} \frac{1}{f - \gamma m g},$$

we obtain

$$N \left( r, \frac{1}{g - \eta} \right) = S_\rho(r, f) \quad \text{and} \quad N \left( r, \frac{1}{g - \gamma m} \right) = S_\rho(r, f)$$

holds for all $\rho = 1, 2, \ldots, n - 1$ and $m = 1, 2, \ldots, n - 3$.

Using $n \geq 4$ and applying the Second fundamental theorem for $\frac{f}{g}$, we get

$$(n - 3)T \left( r, \frac{f}{g} \right) \leq \sum_{n=1}^{n-1} N \left( r, \frac{1}{g - \eta} \right) + S \left( r, \frac{f}{g} \right)$$

$$\leq S_\rho(r, f) + S \left( r, \frac{f}{g} \right)$$

i.e.,

$$T \left( r, \frac{f}{g} \right) = S_\rho(r, f) \quad (2.1.13)$$

Now, consider

$$T(r, f) = T \left( r, \frac{f}{g} \right)$$

$$\leq T \left( r, \frac{f}{g} \right) + T(r, g)$$

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From (2.1.13), the above inequality reduces

\[ T(r, f) \leq T(r, g) + S_p(r, f) \]
i.e.,

\[ T(r, f) = T(r, g) + S_p(r, f) \] (2.1.14)

From (2.1.11), we have

\[ F = g^{n-1} \prod_{p=1}^{n-1} \left( \frac{f}{g} - \eta_p \right) + p(z)g^{n-3} \prod_{m=1}^{n-3} \left( \frac{f}{g} - \gamma_m \right) \]
i.e.,

\[ g^{n-1} = \frac{F}{\prod_{p=1}^{n-1} \left( \frac{f}{g} - \eta_p \right)} - \frac{p(z)g^{n-3} \prod_{m=1}^{n-3} \left( \frac{f}{g} - \gamma_m \right)}{\prod_{p=1}^{n-1} \left( \frac{f}{g} - \eta_p \right)} \] (2.1.15)

provided \( \frac{f}{g} \) is not identically equal to \( \eta_p \) and \( \gamma_m \), for \( p = 1, 2, \ldots, n-1 \) and \( m = 1, 2, \ldots, n-3 \).

From (2.1.14) and (2.1.15), we obtain

\[ (n-1)T(r, f) = (n-1)T(r, g) + S_p(r, f) \]
\[ \leq T(r, F) + T \left( r, \prod_{p=1}^{n-1} \left( \frac{f}{g} - \eta_p \right)^{-1} \right) + T(r, p(z)) \]
\[ + T \left( r, \prod_{m=1}^{n-3} \left( \frac{f}{g} - \gamma_m \right) \right) + (n-3)T(r, g) \]
\[ + T \left( r, \prod_{p=1}^{n-1} \left( \frac{f}{g} - \eta_p \right)^{-1} \right) + S_p(r, f) \]
i.e.,

\[ (n-1)T(r, f) \leq (n-3)T(r, g) + S_p(r, f) + S(r, f) \] (2.1.16)
Since $\rho(f) = \rho(g)$ implies $T(r, f) = T(r, g)$.

Equation (2.1.16) reduces

$$(n-1)T(r, f) \leq (n-3)T(r, f) + S_\rho(r, f) + S(r, f)$$

i.e.,

$$T(r, f) = S_\rho(r, f)$$

which is a contradiction.

Therefore, we must have $\frac{f}{g} = \eta_p$ for some $p = 1, 2, \cdots, n-1$ and $\frac{f}{g} = \gamma_m$ for some $m = 1, 2, \cdots, n-3$, which implies $f = \eta_p f$ and $f = \gamma_m g$ which again implies $f^n + \rho(z)g^{n-2} = f^n + \rho(z)g^{n-2}$ and $L(z, f) = L(z, g)$.

Since $f = \eta_p f$ and by the linearity of the difference polynomial $L$, we get $L(z, f) = \eta_p L(z, g)$, since $\eta_p \neq 1$, we get a contradiction to $L(z, f) = L(z, g)$.

Hence (2.1.1) possesses at most one admissible transcendental entire solution of finite order such that all coefficients of $L(z, f)$ are small functions of $f(z)$.

Hence the proof of Theorem 2.1.4.

**Proof of the Theorem 2.1.5:**

Suppose $f_1(z)$ and $f_2(z)$ are two distinct finite order transcendental meromorphic solutions of (2.1.1) such that $N(r, f_i) = S(r, f_i)$ ($i = 1, 2$).

From (2.1.1), we obtain

$$G(z) = \frac{f_1^n - f_2^n + \rho(z) \left( f_1^{n-2} - f_2^{n-2} \right)}{f_1 - f_2} = \frac{L(z, f_2) - L(z, f_1)}{f_1 - f_2} = \frac{L(z, f_1) - L(z, f_2)}{f_2 - f_1}$$

i.e.,

$$G(z) = \prod_{p=1}^{n-1} \left( f_1 - \eta_p f_2 \right) + \rho(z) \prod_{m=1}^{n-3} \left( f_1 - \gamma_m f_2 \right)$$

(2.1.17)
where,

\[ G(z) = (f_1 - \eta_1 f_2) \cdot (f_1 - \eta_2 f_2) \cdots (f_1 - \eta_{n-1} f_2) + \rho(z) \cdot (f_1 - \gamma_1 f_2) \cdot (f_1 - \gamma_2 f_2) \cdots (f_1 - \gamma_{n-3} f_2). \]

Here \( \eta_p \neq 1(p = 1, 2, \cdots, n - 1) \) are the distinct \( n^{th} \) roots of the unity and \( \gamma_m \neq 1(m = 1, 2, \cdots, n - 3) \) are the distinct \( (n - 2)^{th} \) roots of the unity.

Using (2.1.17) and Lemma 1.5.2, we obtain

\[
m(r, G) = m \left( r, \frac{L(z, f_1) - L(z, f_2)}{f_2 - f_1} \right) = m \left( r, \frac{L(z, f_1 - f_2)}{f_2 - f_1} \right)
\]

i.e.,

\[ m(r, G) = S(r, f_1) + S(r, f_2) \]

Since by hypothesis \( N(r, f_i) = S(r, f_i)(i = 1, 2) \), it follows that \( N(r, G) = S(r, f_1) + S(r, f_2) \).

Hence, we get

\[ T(r, G) = m(r, G) + N(r, G) \]

\[ = S(r, f_1) + S(r, f_2) \] \hspace{1cm} (2.1.18)

Now, we will discuss the following two cases for \( G(z) \).

**Case 1:** If \( G(z) \equiv 0 \)

From (2.1.17), we get

\[
L(z, f_1) - L(z, f_2) = 0
\]

\[
L(z, f_1) = L(z, f_2)
\]
\[ a_0 f_1(z) + a_1(z) f_1(z + c_1) + \cdots + a_k(z) f_1(z + c_k) = a_0 f_2(z) + a_1(z) f_2(z + c_1) + \cdots + a_k(z) f_2(z + c_k) \]

(2.1.19)

Let \( h = \frac{f_1}{f_2} \), then substituting \( f_1 = h f_2 \), equation (2.1.19) can be written as

\[ a_0(z) f_2(z)(h(z) - 1) + a_1(z) f_2(z + c_1) (h(z + c_1) - 1) + \cdots + a_k(z) f_2(z + c_k) (h(z + c_k) - 1) = 0 \]

which implies \( h = 1 \) that is, \( f_1 = f_2 \). Thus (2.1.1) has an unique solution.

**Case 2:** If \( G(z) \neq 0 \)

Consider,

\[ G(z) = \left( f_1 - \gamma_1 f_2 \right) \left( f_1 - \gamma_2 f_2 \right) \cdots \left( f_1 - \gamma_{n-1} f_2 \right) + p(z) \left( f_1 - \gamma_1 f_2 \right) \]

\[ \left( f_1 - \gamma_2 f_2 \right) \cdots \left( f_1 - \gamma_{n-1} f_2 \right) \]

i.e.,

\[ G(z) = f_2^{n-1} Q_1 \left( \frac{f_1}{f_2} \right) + p(z) f_2^{n-3} Q_2 \left( \frac{f_1}{f_2} \right) \]

(2.1.20)

where \( Q_1 \left( \frac{f_1}{f_2} \right) \) is a polynomial in \( \frac{f_1}{f_2} \) of degree \( n - 1 \) and \( Q_2 \left( \frac{f_1}{f_2} \right) \) is a polynomial in \( \frac{f_1}{f_2} \) of degree \( n - 3 \) with constant coefficients. (2.1.20) can be written as

\[ G(z) = f_2^{n-1} \left[ Q_1 \left( \frac{f_1}{f_2} \right) + \frac{p(z)}{f_2^2} Q_2 \left( \frac{f_1}{f_2} \right) \right] \]

\[ \frac{G(z)}{f_2^{n-1}} = \left[ Q_1 \left( \frac{f_1}{f_2} \right) + \frac{p(z)}{f_2^2} Q_2 \left( \frac{f_1}{f_2} \right) \right] \]

i.e.,

\[ T \left( r, Q_1 \left( \frac{f_1}{f_2} \right) + \frac{p(z)}{f_2^2} Q_2 \left( \frac{f_1}{f_2} \right) \right) = T \left( r, \frac{G(z)}{f_2^{n-1}} \right) \]

(2.1.21)
Using (2.1.18), (2.1.21) deduces to

$$(2n - 4)T \left( r, \frac{f_1}{f_2} \right) = (n - 3)T (r, f_2) + S (r, f_1) + S (r, f_2)$$  \hspace{1cm} (2.1.22)

Similarly, we can write

$$(2n - 4)T \left( r, \frac{f_2}{f_1} \right) = (n - 3)T (r, f_1) + S (r, f_1) + S (r, f_2)$$  \hspace{1cm} (2.1.23)

From (2.1.22) and (2.1.23), we obtain

$$T (r, f_1) + S (r, f_1) = T (r, f_2) + S (r, f_2)$$  \hspace{1cm} (2.1.24)

Since $\rho(f_1) = \rho(f_2)$, (2.1.24) reduces to

$$S (r, f_1) = S (r, f_2)$$  \hspace{1cm} (2.1.25)

Using (2.1.25), (2.1.22) reduces

$$2(n - 2)T \left( r, \frac{f_2}{f_1} \right) = (n - 3)T (r, f_1) + S (r, f_2)$$

$$S \left( r, \frac{f_2}{f_1} \right) = S (r, f_2)$$

From (2.1.18) and First Fundamental Theorem 1.2.14, we can write

$$N \left( r, \frac{1}{G} \right) = S_2(r, f)$$

$$\sum_{p=1}^{n-1} N \left( r, \frac{1}{f_1 - \eta_p f_2} \right) = S_2(r, f) \quad \text{and} \quad \sum_{m=1}^{n-3} N \left( r, \frac{1}{f_1 - \gamma_m f_2} \right) = S_2(r, f)$$

holds for all $p = 1, 2, \ldots, n - 1$ and $m = 1, 2, \ldots, n - 3$.

Since $\frac{1}{f_1 - \eta_p} = \frac{f_2}{f_1 - \eta_p f_2}$ and $\frac{1}{f_1 - \gamma_m} = \frac{f_2}{f_1 - \gamma_m f_2}$, we obtain
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\[
N \left( r, \frac{1}{f_1 f_2 - t_{lp}} \right) = S_2(r, f) \quad \text{and} \quad N \left( r, \frac{1}{f_1 f_2 - \gamma_m} \right) = S_2(r, f)
\]  

(2.1.26)

holds for all \( \rho = 1, 2, \cdots, n - 1 \) and \( m = 1, 2, \cdots, n - 3 \).

From (2.1.26) and the Second Fundamental Theorem 1.2.15, we obtain

\[
(n - 3) T \left( r, \frac{f_1}{f_2} \right) \leq \sum_{p=1}^{n-1} N \left( r, \frac{1}{f_1 f_2 - t_{lp}} \right)
\]

\[
\leq S \left( r, f_2 \right) = S \left( r, \frac{f_1}{f_2} \right)
\]

i.e.,

\[
(n - 3) T \left( r, \frac{f_1}{f_2} \right) \leq S \left( r, \frac{f_1}{f_2} \right)
\]

(2.1.27)

which is a contradiction to \( n \geq 4 \).

Thus (2.1.1) has \( f(z) \) as its unique transcendental meromorphic solution with finite order such that \( N(r, f) = S(r, f) \).

Hence the proof of Theorem 2.1.5.

**Proof of the Theorem 2.1.6:**

First we consider two cases for \( q(z) \) and \( r(z) \).

**Case 1:** If \( q(z) \) is a constant or \( r(z) = 0 \).

Then (2.1.3) can be reduced to \( f^n + p(z) I(z, f) = O(z) \), where \( O(z) \) is a polynomial.

\[
f^n = O(z) - p(z) \frac{I(z, f)}{f(z)} \cdot f(z)
\]

\[
n T(r, f) \leq T(r, O(z)) + T(r, p(z)) + T \left( r, \frac{I(z, f)}{f(z)} \right) + T(r, f) + S(r, f)
\]

\[
\leq T(r, f) + S(r, f)
\]

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i.e.,

\[(n - 1)T(r, f) \leq S(r, f)\]

which is a contradiction to \(n > 1\).

Hence, if \(q(z)\) is a constant or \(r(z) \equiv 0\) then transcendental entire function \(f(z)\) of finite order cannot be solution of (2.1.3).

**Case 2:** If \(q(z)\) is a non-constant polynomial and \(r(z) \neq 0\). Assume that transcendental entire function \(f(z)\) of finite order is a solution of (2.1.3).

Differentiating (2.1.3), we get

\[n f^{n-1} f' + p(z) L'(z, f) + p'(z) L(z, f) = \left[r(z) q'(z) + r'(z)\right] e^{g(z)} \]  \hspace{1cm} (2.1.28)

From (2.1.28) and (2.1.3), we get

\[\frac{n f^{n-1} f' + p(z) L'(z, f) + p'(z) L(z, f)}{f^n + p(z) L(z, f)} = \frac{\left[r(z) q'(z) + r'(z)\right] e^{g(z)}}{r(z) e^{g(z)}}\]

\[n f^{n-1} f' + p(z) L'(z, f) + p'(z) L(z, f) = \left[q'(z) + \frac{r'(z)}{r(z)}\right] f(z)^n + \left[q'(z) + \frac{r'(z)}{r(z)}\right] p(z) L(z, f)\]

i.e.,

\[f^{n-1} \left[n f' - \left(q'(z) + \frac{r'(z)}{r(z)}\right) f\right] = \left(q'(z) + \frac{r'(z)}{r(z)}\right) p(z) L(z, f)\]

\[- p(z) L'(z, f) - p'(z) L(z, f)\]  \hspace{1cm} (2.1.29)

If \(n f' - \left(q'(z) + \frac{r'(z)}{r(z)}\right) f \equiv 0\), then integrating and simplifying, we get \(f(z)^n = B r(z) e^{g(z)}\) implies \(f(z) = g(z) e^{\frac{g(z)}{n}}\) where \(g(z)^n = B r(z)\), \(B\) is a non-zero constant.

Thus, (2.1.3) can be written as

\[(B - 1) r(z) e^{g(z)} + p(z) L(z, f) \equiv 0\]  \hspace{1cm} (2.1.30)
Note that if \( B = 1 \), then \( I(z, f) \equiv 0 \), which contradicts the hypothesis. Thus, \( B \neq 1 \), substituting \( f(z) = g(z)e^{\frac{a(z)}{n}} \) in \( I(z, f) \) and considering \( h(z) = e^{\frac{a(z)}{n}} \), then \( I(z, f) \) can be expressed as

\[
I(z, f) = a_0 g(z) h(z) + \sum_{s=1}^{k} a_s(z) g(z + c_s) h(z + c_s)
\]

Using Lemma 1.5.1, we deduce

\[
T(r, I(z, f)) \leq T(r, h(z)) + S(r, h(z)) \tag{2.1.31}
\]

From (2.1.30), we get

\[
T(r, (1 - B)r(z)c^{a(z)}) = T(r, p(z)I(z, f)) \leq T(r, p(z)) + T(r, I(z, f)) \tag{2.1.32}
\]

Since polynomial \( p(z) \) is small function with respect to transcendental entire function \( h(z) \), we have \( T(r, p(z)) = S(r, h) \).

Using (2.1.31), (2.1.32) reduces to

\[
(n - 1)T(r, h(z)) \leq S(r, h)
\]

which is contradiction to \( n > 1 \). Hence \( n f' - \left( q' + \frac{r'(z)}{r(z)} \right) f \neq 0 \).

Now, we consider following two subcases for \( n \).

**Subcase 1:** If \( n > 2 \)

(2.1.29) can be deduced to the following equations

\[
f^{n-2} \left[ n f' - \left( q' + \frac{r'(z)}{r(z)} \right) f \right] = \left( q(z) + \frac{r'(z)}{r(z)} \right) p(z) \frac{I(z, f)}{f} - p(z) \frac{I'(z, f)}{f} - \frac{p'(z)}{f} I(z, f) \tag{2.1.33}
\]
and

\[ f^{n-3} \left[ f \left( n f' - \left( q'(z) + \frac{r'(z)}{r(z)} \right) f \right) \right] = \left( q'(z) + \frac{r'(z)}{r(z)} \right) \rho(z) \frac{L(z, f)}{f} \]

\( - \rho(z) \frac{L'(z, f)}{f} - \rho'(z) \frac{L(z, f)}{f} \)  

(2.1.34)

Applying Lemma 1.5.1, Lemma 1.5.8, Remark 1.5.9 and the Lemma on the logarithmic derivative to (2.1.33) and (2.1.34), we get

\[ m \left( r, n f' - \left( q'(z) + \frac{r'(z)}{r(z)} \right) f \right) = S(r, f) \]

and

\[ m \left( r, f \left( n f' - \left( q'(z) + \frac{r'(z)}{r(z)} \right) f \right) \right) = S(r, f) \]

Thus,

\[ T \left( r, n f' - \left( q'(z) + \frac{r'(z)}{r(z)} \right) f \right) = m \left( r, n f' - \left( q'(z) + \frac{r'(z)}{r(z)} \right) f \right) \]

\[ = S(r, f) \]  

(2.1.35)

and

\[ T \left( r, f \left( n f' - \left( q'(z) + \frac{r'(z)}{r(z)} \right) f \right) \right) = m \left( r, f \left( n f' - \left( q'(z) + \frac{r'(z)}{r(z)} \right) f \right) \right) \]

\[ = S(r, f) \]  

(2.1.36)

By the First Fundamental Theorem, we have

\[ T(r, f) = T \left( r, \frac{f \left( n f' - \left( q'(z) + \frac{r'(z)}{r(z)} \right) f \right)}{n f' - \left( q'(z) + \frac{r'(z)}{r(z)} \right) f} \right) \]

\[ \leq T \left( r, f \left( n f' - \left( q'(z) + \frac{r'(z)}{r(z)} \right) f \right) \right) \]

\[ + T \left( r, n f' - \left( q'(z) + \frac{r'(z)}{r(z)} \right) f \right) + S(r, f) \]  

(2.1.37)
From (2.1.35) and (2.1.36), (2.1.37) reduces to

$$T(r, f) = S(r, f)$$

a contradiction.

**Subcase 2: If** \( n = 2 \)

(2.1.3) and (2.1.29) reduce

$$f^2 + p(z)L(z, f) = r(z)e^{q(z)} \quad (2.1.38)$$

$$f \left[ 2f'(z) - \left( q'(z) + \frac{r'(z)}{r(z)} \right) f \right] = \left( q'(z) + \frac{r'(z)}{r(z)} \right)p(z)L(z, f) - p(z)L'(z, f)$$

$$- p'(z)L(z, f) \quad (2.1.39)$$

Let \( G(z) = 2f' - \left( q'(z) + \frac{r'(z)}{r(z)} \right) f. \)

Then, (2.1.39) reduces to

$$G(z) = \left( q'(z) + \frac{r'(z)}{r(z)} \right)p(z)\frac{L(z, f)}{f} - p(z)\frac{L'(z, f)}{f} - p'(z)\frac{L(z, f)}{f} \quad (2.1.40)$$

Since \( f(z) \) is an entire function, Lemma 1.5.1 and the Lemma on the logarithmic derivative, (2.1.40) deduces to

$$T(r, G(z)) = m(r, G(z)) + S(r, f) = S(r, f) \quad (2.1.41)$$

Differentiating \( G(z) \), we get

$$2f'' - \left( q'(z) + \frac{r'(z)}{r(z)} \right)' - \left( q'(z) + \frac{r'(z)}{r(z)} \right)' f' = \frac{G'}{G}G'$$

$$2f'' - \left( q'(z) + \frac{r'(z)}{r(z)} + 2\frac{G'(z)}{G(z)} \right)' f'$$

$$- \left( q''(z) - \frac{G'(z)}{G(z)} q'(z) + \left( \frac{r'(z)}{r(z)} \right)'' - \frac{G'(z)}{G(z)} \left( \frac{r'(z)}{r(z)} \right)' \right) f = 0$$
i.e.,

\[
2 \left( \left( \frac{f'}{f} \right)' + \left( \frac{f'}{f} \right)^2 \right) - \left( q'(z) + \frac{r'(z)}{r(z)} + 2 \frac{G'(z)}{G(z)} \right) \frac{f'}{f} \\
- \left( q''(z) - q'(z) \frac{G'(z)}{G(z)} + \left( \frac{r'(z)}{r(z)} \right)' - \frac{G'(z)}{G(z)} \cdot \frac{r'(z)}{r(z)} \right) = 0
\]

(2.1.42)

Suppose that \( z_0 \) is a zero of \( f(z) \) with multiplicity \( p \). If \( z_0 \) is a zero of \( r(z) \) as well, then the contribution of \( z_0 \) to \( N \left( r, r, \frac{1}{f} \right) \) is \( S(r, f) \). Assume that \( z_0 \) is not a zero of \( r(z) \), we now discuss the following two Subcases of Case 2:

**Subcase A:** Suppose \( z_0 \) is a zero of \( G(z) \) with multiplicity \( k \). From (2.1.42), we obtain that \( p = 1 + k \leq 2k \), by (2.1.41) implies that the contribution of \( z_0 \) to \( N \left( r, r, \frac{1}{f} \right) \) is \( S(r, f) \).

**Subcase B:** Suppose \( z_0 \) is not a zero of \( G(z) \). By (2.1.42), we obtain \( p^2 - p = 0 \), then such a zero of \( f(z) \) must be simple and we notice that \( q' + \frac{r'}{r} + 2 \frac{G'}{G} \) must vanish at \( z_0 \). Thus, by (2.1.41) implies that the contribution of \( z_0 \) to \( N \left( r, r, \frac{1}{f} \right) \) is \( S(r, f) \). Hence, \( N \left( r, r, \frac{1}{f} \right) = S(r, f) \). Thus, by Hadamard’s factorization theorem 1.2.5, \( f(z) \) can be expressed as \( f(z) = B(z)e^{d(z)} \), where \( B(z) \) is an entire function satisfying \( N \left( r, r, \frac{1}{B(z)} \right) = S(r, f) \) and \( d(z) \) is non-constant polynomial.

Substituting \( f(z) \) in (2.1.38), we get

\[
B(z)^2 e^{2d(z)} + p(z)a(z)B(z)e^{d(z)} + f(z) \sum_{j=1}^{k} a_j(z)B(z + c_j) e^{d(z + c_j)} = r(z)e^{q(z)}
\]

i.e.,

\[
\frac{B(z)e^{d(z)}}{p(z)a(z)} + \frac{r(z)e^{q(z)}}{f(z)a(z)B(z)e^{d(z)}} - \sum_{j=1}^{k} a_j(z)B(z + c_j) e^{d(z + c_j)} = 1
\]

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i.e.,

\[ f_1 + f_2 + f_3 = 1 \]

where

\[ f_1 = -\frac{B(z)e^{d(z)}}{p(z)a_0(z)}, \quad f_2 = \frac{r(z)e^{g(z)}}{p(z)a_0(z)B(z)e^{d(z)}}, \]

and

\[ f_3 = -\frac{\sum_{j=1}^{k} a_j(z)B(z+c_j)e^{d(z+c_j)}}{a_0(z)B(z)e^{d(z)}} \]

Note that \( f_1 = \frac{B(z)e^{d(z)}}{p(z)a_0(z)} \) is not a constant and we deduce that

\[ \sum_{l=1}^{3} N \left( r, \frac{1}{f_l} \right) + \sum_{l=1}^{3} N(r, f_l) \leq S(r, f) < (\lambda + o(1))T(r) \]

Thus, by Lemma 1.4.11, we get either \( f_2(z) \equiv 1 \) or \( f_3(z) \equiv 1 \).

If \( f_2(z) \equiv 1 \), then by (2.1.38), we deduce \( T(r, f) \leq S(r, f) \), which is a contradiction.

If \( f_3(z) \equiv 1 \), then we get \( l(z, f) \equiv 0 \), by hypothesis, we again get a contradiction.

Thus, finite order transcendental entire function cannot be a solution of (2.1.3).

Hence the proof of Theorem 2.1.6.

2.2 Transcendental entire solutions of certain type of difference equations

2.2.1 Introduction

We define

\[ P(z, f) = \sum_{\lambda \in I} a_{\lambda}(z)f(z)^{i_{\lambda, 0}}f(z + c_{1})^{i_{\lambda, 1}} \cdots f(z + c_{k})^{i_{\lambda, k}} \tag{2.2.1} \]
be a difference polynomial in \( f(z) \) and its shifts and \( d(P) = \max_{\lambda \in I} \) is its degree where \( I \) is a finite set of the index \( \lambda = \{ \lambda_0, \cdots, \lambda_k \} \) and \( a_\lambda(z) \) are meromorphic coefficients being small with respect to \( f(z) \) and \( f(z)^{\lambda_0} f(z + c_1)^{\lambda_1} \cdots f(z + c_k)^{\lambda_k} \) is monomial in \( f(z) \) and its shift \( f(z+c_1), \cdots, f(z+c_k) \) where \( c_1, \cdots, c_k \) are distinct non-zero complex constants and \( d(\lambda) = \lambda_0 + \cdots + \lambda_k \) is its degree.

In this section, an investigation is continued to find the deficiency of zeros of transcendental entire solution of differential-difference equation of the form

\[
f^n(z) + q(z)P(z, f) = P_1(z)e^{\alpha_1 z} + P_2(z)e^{\alpha_2 z},
\]

where \( P(z, f) \) is as defined in (2.2.1) with degree \( d \leq n - 2 \), \( P_1(z), P_2(z) \) and \( q(z) \) be three non-zero polynomials such that the degree of \( P_1(z) \) and \( P_2(z) \) is greater than or equal to degree of \( q(z) \) and \( \alpha_1, \alpha_2 \) be two non-zero constants.

### 2.2.2 Preliminaries

In 2014, N. Liu, W. Lu, T. Shen, C. Yang [58], obtained the following theorem.

**Theorem 2.2.1. [58]**

Let \( n \geq 4 \) be an integer, \( q(z) \) be a polynomial, and \( p_1, p_2, \alpha_1, \alpha_2 \) be non-zero constants such that \( \alpha_1 \neq \alpha_2 \). If there exists some entire solution \( f(z) \) of finite order to (2.2.2) below

\[
f^n(z) + q(z)\Delta f(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},
\]

then \( q(z) \) is a constant and one of the following relations holds:

1. \( f(z) = c_1 e^{\omega_1 z} \) and \( c_1 \left( e^{\omega_1} - 1 \right) q = p_2, \alpha_1 = n \alpha_2. \)
(2) \( f(z) = c_2 e^{c_1^2 z} \) and \( c_2 \left( e^{c_1^2} - 1 \right) q = p_1, \alpha_2 = p_2, \alpha_1. \)

where \( c_1, c_2 \) are constants satisfying \( c_1^n = p_1, c_2^n = p_2. \)

Next, in 2014, X. Q. Qi, J. Dou and L. Z. Yang [64], proved the following theorem.

**Theorem 2.2.2. [64]**

Let \( p(z) \neq 0, q(z), r(z) \) be polynomials, \( n \) and \( m \) be positive integers satisfying \( n > m. \) Let \( f(z) \) be finite order entire solution of

\[
f(z)^n + p(z)e^{q(z)}(\Delta_e f)^m = r(z)
\]

then \( f(f) = \deg q(z). \)

### 2.2.3 Main results

In this section, we prove the following theorem for difference equation of the form (2.2.1) as follows.

**Theorem 2.2.3.**

Let \( n \geq 3 \) be an integer and \( P(z, f) \) be a difference polynomial in \( f(z) \) of finite order and its shift as defined in (2.2.1) of degree \( d \leq n - 2 \). Let \( P_1(z), P_2(z) \) and \( q(z) \) be three non-zero polynomials such that the degree of \( P_1(z) \) and \( P_2(z) \) is greater than or equal to degree of \( q(z) \). Let \( \alpha_1, \alpha_2 \) be two non-zero constants such that \( \frac{\alpha_1}{\alpha_2} \neq \left( \frac{d}{n} \right)^{\pm1}, 1. \) Then any finite order transcendental entire solution \( f(z) \) of the equation

\[
f^n(z) + q(z)P(z, f) = P_1(z)e^{\alpha_1 z} + P_2(z)e^{\alpha_2 z}
\]
satisfies $\Theta(0,f) = 0$.

The following theorem is obtained by replacing $(\Delta_c f)^m$ in (2.2.3) by more general linear difference polynomial in $f(z)$ and its shifts as defined in (1.3.2).

**Theorem 2.2.4.**

Let $n > 1$ be an integer and $f(z)$ be finite order entire solution of

$$f(z)^n + P(z)e^{q(z)}L(z,f) = r(z)$$  \hfill (2.2.5)

where $L(z,f)$ is as defined in (1.3.2) and not vanishing identically, $q(z)$, $r(z)$ and $P(z) \neq 0$ are polynomials, then $\rho(f) = \text{deg} q(z)$.

### 2.2.4 Proof of the main results

**Proof of the Theorem 2.2.3:**

Suppose $f(z)$ is a transcendental entire solution of (2.2.4) with $\Theta(0,f) > 0$.

Differentiating (2.2.4) on both sides, we get

$$nf^{n-1}f' + q(z)P(z,f) + q(z)P'(z,f) = \left(P'_1 + P_1\alpha_1\right)e^{\alpha_1 z} + \left(P'_2 + P_2\alpha_2\right)e^{\alpha_2 z}$$ \hfill (2.2.6)

Eliminating $e^{\alpha_1 z}$ and $e^{\alpha_2 z}$ from (2.2.4) and (2.2.6), we get

$$
\left(P'_1 + P_1\alpha_1\right)f^n(z) - nP_1 f^{n-1}f' + W_1(z,f) = \eta_1 e^{\alpha_2 z} \\
\left(P'_2 + P_2\alpha_2\right)f^n(z) - nP_2 f^{n-1}f' + W_2(z,f) = \eta_2 e^{\alpha_1 z}
$$ \hfill (2.2.7)

where

$$W_1(z,f) = q(z)\left(P'_1 + P_1\alpha_1\right)P(z,f) - P_1 q'(z)P(z,f) - P_2 q(z)P'(z,f)$$ \hfill (2.2.9)
\[ W_2(z, f) = q(z) \left( P_2' + P_2 \alpha_2 \right) P(z, f) - P_2 q(z) P(z, f) - P_2 q(z) P'(z, f) \]  
\[ (2.2.10) \]

\[ \eta_1 = P_2 \left( P_1' + P_1 \alpha_1 \right) - P_1 \left( P_2' + P_2 \alpha_2 \right) \]  
\[ (2.2.11) \]

and

\[ \eta_2 = P_1 \left( P_2' + P_2 \alpha_2 \right) - P_2 \left( P_1' + P_1 \alpha_1 \right) \]  
\[ (2.2.12) \]

Differentiating (2.2.8), we obtain

\[ \left( P_2' + P_2 \alpha_2 \right)' f^n + n P_2 \alpha_2 f^{n-1} f' - n P_2 f^{n-1} f'' - n(n - 1) P_2 f^{n-2} \left( f' \right)^2 \]  
\[ + W_2'(z, f) = \left( \eta_2' + \eta_2 \alpha_1 \right) e^{\alpha z} \]  
\[ (2.2.13) \]

Eliminating \( e^{\alpha z} \) from (2.2.8) and (2.2.13), we get

\[ \eta_3 f^n - \eta_4 n P_2 f^{n-1} f' + n(n - 1) P_2 \eta_2 f^{n-1} \left( f' \right)^2 \]  
\[ + n P_2 \eta_2 f^{n-1} f'' = \eta_2 W_2'(z, f) - \left( \eta_2' + \eta_2 \alpha_1 \right) W_2(z, f) \]  
\[ (2.2.14) \]

where, \( \eta_3 = \left( P_2' + P_2 \alpha_2 \right) \left( \eta_2' + \eta_2 \alpha_1 \right) - \left( P_1' + P_2 \alpha_2 \right) \eta_2 \)  
\[ (2.2.15) \]

and \( \eta_4 = \left( \eta_2' + \eta_2 \alpha_1 \right) + \eta_2 \alpha_2 \)  
\[ (2.2.16) \]

(2.2.14) can be written as

\[ f^{n-2} \left[ \eta_3 f^2 - \eta_4 n P_2 f' f' + n(n - 1) P_2 \eta_2 \left( f' \right)^2 + n P_2 \eta_2 f f'' \right] \]  
\[ = \eta_2 W_2'(z, f) - \left( \eta_2' + \eta_2 \alpha_1 \right) W_2(z, f) \]  
\[ (2.2.17) \]

Let

\[ \psi = \eta_3 f^2(z) - \eta_4 n P_2 f' f' + n(n - 1) P_2 \eta_2 \left( f' \right)^2 + n P_2 \eta_2 f f'' \]  
\[ (2.2.18) \]
and 
\[ U(z, f) = \eta_2 W'_2(z, f) - \left( \eta'_2 + \eta_2 \alpha_1 \right) W_2(z, f) \] (2.2.19)

(2.2.17) reduces to
\[ f^{n-2} \psi = U(z, f) \]

Since \( P(z, f) \) is a polynomial of degree \( n - 2 (n \geq 3) \) and by Lemma 1.4.15, we get \( m(r, \psi) = S(r, f) \) and since \( \psi \) is an entire function implies \( T(r, \psi) = m(r, \psi) = S(r, f) \).

Next, (2.2.18) can be reduced to
\[ \psi = f^2 B(z) \] (2.2.20)

where,
\[ B(z) = \eta_3 - \eta_4 \eta_2 \left( \frac{f'}{f} \right) + m(n - 1) \eta_2 \left( \frac{f'}{f} \right)^2 + n \eta_2 \eta_2 \left( \frac{f''}{f} \right) \] (2.2.21)

From (2.2.21) and by the Lemma of Logarithmic derivative, we deduce \( m(r, B(z)) = S(r, f) \).

If \( \psi \neq 0 \) then \( B(z) \neq 0 \).

From (2.2.20) and for any small \( \epsilon > 0 \), we have
\[ 2T(r, f) = T\left( r, f^2 \right) \leq m(r, f) \leq m(r, \psi) + m\left( r, \frac{1}{B(z)} \right) + S(r, f) \leq S(r, f) + T(r, B(z)) \leq 2N\left( r, \frac{1}{f} \right) + S(r, f) \leq 2 (1 - \Theta(0, f) + \epsilon) T(r, f) + S(r, f) \]

since \( f(z) \) is transcendental entire function and \( \Theta(0, f) > \epsilon > 0 \), above inequality leads to a contradiction.
Hence $\psi \equiv 0$ implies $B(z) \equiv 0$ and $U(z, f) = 0$.

Next, we consider two cases for $W_2(z, f)$.

**Case 1:** If $W_2(z, f) \neq 0$

Then, $\eta_2 W'_2(z, f) - (\eta'_2 + \eta_2 \alpha_1) W_2(z, f) = 0$

Integrating, we get

$$W_2(z, f) = \eta_2 k_1 e^{\alpha_1 z}, \quad k_1 \neq 0 \quad (2.2.22)$$

Substituting (2.2.22) in (2.2.8), we get

$$f^{m-1} \phi = \left(\frac{1}{k_1} - 1\right) W_2(z, f), \text{ where } \phi = \left(P'_2 + P_2 \alpha_2\right) f - n P_2 f' \quad (2.2.23)$$

Suppose $\phi \neq 0$, then, by (2.2.23) and by Lemma 1.4.15, we get $m(r, \phi) = S(r, f)$, implies $T(r, \phi) = S(r, f)$ and (2.2.23) can be written as

$$f^{m-2} (f \phi) = \left(\frac{1}{k_1} - 1\right) W_2(z, f) \quad (2.2.24)$$

Applying Lemma 1.4.15 to (2.2.24), we get $m(r, f \phi) = S(r, f)$, which implies $T(r, f \phi) = S(r, f)$.

Thus, $T(r, f) = m(r, f) = m \left(r, \frac{f \phi}{\phi}\right) \leq S(r, f)$, which is a contradiction. Hence $\phi \equiv 0$ and from (2.2.24), we get $k_1 = 1$.

Substituting $k_1 = 1$ in (2.2.22), we get

$$W_2(z, f) = \eta_2 e^{\alpha_1 z}$$

Now, consider $\phi = (P'_2 + P_2 \alpha_2) f - n P_2 f' = 0$.

Integrating $\phi$, we deduce

$$f^n = P_2 k_2 e^{\alpha_2 z} (k_2 \neq 0)$$
From (2.2.4) and applying same method as above, we deduce $l_2 = 1$, which implies

$$f^n = P_2 e^{\alpha_2 z}$$

Substituting $f^n = P_2 e^{\alpha_2 z}$ in (2.2.4), we get $P(z, f) = \frac{p_2}{q(z)} e^{\alpha_1 z}$ that is, $f = (P_2)^{\frac{1}{n}} e^{\frac{\alpha_1}{n} z}$ and $P(z, f) = h e^{\frac{\alpha_1}{n} z}$.

Since, degree of $q(z)$ is less than or equal to degree of $P_1(z)$ implies $h e^{\frac{\alpha_1}{n} z}$ is a polynomial of $e^{\frac{\alpha_1}{n} z}$ with degree $d$ and its co-efficients are small functions of $h e^{\frac{\alpha_1}{n} z}$.

Hence, by Lemma 1.4.13, we have, $\frac{d \alpha_1}{n} = \alpha_2$, which implies $\frac{\alpha_1}{\alpha_2} = \frac{n}{d}$, which is not possible.

**Case 2:** If $W_2(z, f) = 0$

From (2.2.10), we get

$$q(z) \left(P_2' + P_2 \alpha_2 \right) P(z, f) - P_2 q'(z) P(z, f) - P_2 q(z) P'(z, f) = 0 \quad (2.2.25)$$

If $P(z, f) = 0$, then (2.2.4) reduces to

$$f^n = P_1(z) e^{\alpha_1 z} + P_2(z) e^{\alpha_2 z}$$

\[i.e., \quad \frac{1}{P_1} \left( \frac{e^{-\frac{\alpha_1}{m} z}}{m} \right)^n + \left( \frac{P_2}{P_1} \right) \left( e^{\frac{\alpha_2}{m} z} \right)^m = 1 \text{ where, } m \text{ is a positive integer}\]

By Lemma 1.4.12, we obtain $\alpha_1 = \alpha_2$, which is a contradiction. Hence $P(z, f) \neq 0$.

Next integrating (2.2.25), we get $P(z, f) = \frac{p_3 k_3}{q(z)} e^{\alpha_2 z}$, $k_3 \neq 0$.

Substituting value of $P(z, f)$ in (2.2.4), we obtain

$$f^n + (k_3 - 1) P_2 e^{\alpha_2 z} = P_1 e^{\alpha_1 z} \quad (2.2.26)$$

Similarly as above method, we get $k_3 = 1$ that is $P(z, f) = \frac{P_2}{q(z)} e^{\alpha_2 z}$.

Substituting $k_3 = 1$ in (2.2.26), we get

$$f(z) = \left( P_1 \right)^{\frac{1}{n}} e^{\frac{\alpha_1}{n} z}$$

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By the same argument as in Case 1, we have \[ \frac{\alpha_2}{n} = \alpha_1, \] that is \[ \frac{\alpha_1}{\alpha_2} = \frac{d}{n}, \] which is a contradiction.

Hence the assumption \( \Theta(0, f) > 0 \) for an entire solution (2.2.4) is wrong. Thus any finite order transcendental entire solution \( f(z) \) of (2.2.4) satisfies \( \Theta(0, f) = 0 \).

Hence the proof of Theorem 2.2.3.

**Proof of the Theorem 2.2.4:**

(2.2.5) can be written as

\[
\begin{align*}
    f^n &= r(z) - P(z)c^{g(z)}L(z, f) \\
    nT(r, f) &= T(r, r(z) - P(z)c^{g(z)}L(z, f)) \\
    &\leq T(r, c^{g(z)}) + T(r, L(z, f)) + S(r, f) \\
    &\leq T(r, c^{g(z)}) + T\left(r, \frac{L(z, f)}{f}\right) \\
    &\quad + T(r, f) + S(r, f)
\end{align*}
\]

By Lemma 1.5.1, we get

\[
\begin{align*}
    nT(r, f) &\leq T(r, c^{g(z)}) + T(r, f) + S(r, f) \\
    \text{i.e.,} \quad (n - 1)T(r, f) &\leq T(r, c^{g(z)}) + S(r, f)
\end{align*}
\]

From the assumption that \( n > 1 \) and the above inequality, we conclude \( \rho(f) \leq \deg q(z) \).

Now we show that \( \rho(f) = \deg q(z) \).

Suppose to the contrary that \( \rho(f) < \deg q(z) \). Then \( \rho(f^n + P(z)c^{g(z)}L(z, f)) = \)
\[ \deg q(z) > 0 \text{ from Remark 1.5.4 and } \rho(r(z)) = 0. \text{ This is contradiction to (2.2.5).} \]

Thus \( \rho(f) = \deg q(z) \).

Hence the proof of Theorem 2.2.4