Chapter 5

Uniqueness of linear polynomial of difference operator of entire functions sharing a small function
5.1 Introduction

In this chapter, an investigation has been done on the uniqueness problems of linear polynomial of difference operator of finite order entire function that share a small function \( a(z) \) CM and obtained results which extend the corresponding results obtained by B. Chen, Z. Chen, S. Li [16] and X. G. Qi and K. Liu [66]. We also extend the results of X. G. Qi, L. Z. Yang and K. Liu [68].

5.2 Preliminaries

In 2011, X. G. Qi and K. Liu [66] proved the following theorem for shifts of two entire functions sharing a small function.

Theorem 5.2.1. [66]

Suppose that \( f(z) \) and \( g(z) \) are two entire functions of finite order, and let \( a \) and \( b \) be distinct small functions related to \( f(z) \) and \( g(z) \) such that \( \delta(a) = \delta(a, f) + \delta(a, g) \).

1. If \( f(z + c_1) \) and \( g(z + c_2) \) share \( b \) CM, then exactly one of the following assertions holds

   (i) \( f(z) \equiv g(z + c), \) where \( c = c_2 - c_1. \)

   (ii) \( f(z + c_1) \equiv (a - b)c^h + a \) and \( g(z + c_2) \equiv (a - b)c^{-h} + a, \)

   where \( h(z) \) is an entire function.

In 2012, B. Chen, Z. Chen, S. Li[16] obtained the following results.
Theorem 5.2.2. [16]

Let \( f(z) \) and \( g(z) \) be entire functions of finite order, and \( a(z) \) and \( b(z) \) be small entire functions with respect to \( f(z) \) and \( g(z) \). Suppose that \( c \in \mathbb{C}\setminus\{0\} \) such that 
\[
\Delta_c(f - b) \cdot \Delta_c(g - b) \cdot (a - \Delta_c b) \neq 0. 
\]
If \( \Delta_c f(z) \) and \( \Delta_c g(z) \) share a CM, and 
\[
\delta(b) = \delta(b, f) + \delta(b, g) > 1, 
\]
then one of the following assertions holds:

(i) \( \Delta_c f(z) \equiv \Delta_c g(z) \);

(ii) \( \Delta_c f \equiv (\Delta_c b - a) e^{h(z)} + \Delta_c b, \ \Delta_c g \equiv (\Delta_c b - a) e^{-h(z)} + \Delta_c b, \) where 
\( h(z) \) is a polynomial.

Theorem 5.2.3. [16]

Let \( c_1, c_2 \in \mathbb{C}\setminus\{0\} \). Let \( f(z) \) and \( g(z) \) be entire functions of finite order \( \rho(f) \) and \( \rho(g) \), respectively. Suppose that \( a \) and \( b \) are distinct complex constants. If 
\( \Delta_{c_1} f(z) \) and \( \Delta_{c_2} g(z) \) share a CM, and \( \Delta_{c_1} f(z) - b \) and \( \Delta_{c_2} g(z) - b \) have at least 
\[
\max\{[\rho(f)], [\rho(g)], 1\} \text{ distinct common zeros of multiplicity } \geq 2, 
\]
then \( \Delta_{c_1} f(z) \equiv \Delta_{c_2} g(z) \).

In 2010, X. G. Qi, L. Z. Yang and K. Liu [68] considered the uniqueness problems on the meromorphic function \( f^n(z) \) sharing values with its shift and obtained the Theorems as follows.

Theorem 5.2.4. [68]

Let \( f(z) \) be a non-constant meromorphic function of finite order, \( n \geq 7 \) be an integer, and let \( F = f^n(z) \). If \( F \) and \( F(z + c) \) share \( a(z) \in S(f) - \{0\} \) and \( \infty \) CM, 
then \( f(z) = w f(z + c) \), for a constant \( w \) that satisfies \( w^n = 1 \).
Theorem 5.2.5. [68]

Let \( f(z) \) be a non-constant entire function of finite order, \( n \geq 5 \) be an integer, and let \( F = f^n(z) \). If \( F \) and \( F(z+c) \) share \( a(z) \in S(f) - \{0\} \) CM, then \( f(z) = w f(z+c) \), for a constant \( w \) that satisfies \( w^n = 1 \).

5.3 Main results

The following theorems are the main results of this chapter.

Theorem 5.3.1.

Let \( f(z) \) and \( g(z) \) be entire functions of finite order and let \( a(z) \) and \( b(z) \) be distinct small functions with respect to \( f(z) \) and \( g(z) \). Suppose \( \alpha_i (i = 1, 2, \ldots, k) \) are complex constants and \( c_i (i = 1, 2, \ldots, k) \) are non-zero complex constants. If

\[
I_z(. \Delta_{\alpha_i} f(z)) = \sum_{i=1}^{k} a_i \Delta_{\alpha_i} f(z) \text{ and } I_z(. \Delta_{\alpha_i} g(z)) = \sum_{i=1}^{k} c_i \Delta_{\alpha_i} g(z) \text{ share } a(z)
\]

CM such that

\[
I_z(. \Delta_{\alpha_i} \left(f(z) - b(z)\right)) \cdot I_z(. \Delta_{\alpha_i} \left(g(z) - b(z)\right)) \cdot (a(z) - I_z(. \Delta_{\alpha_i} b(z))) \neq 0 \text{ and } \delta(b) = \delta(b, f) + \delta(b, g) > 1,
\]

then one of the following assertions holds:

(i) \( I_z(. \Delta_{\alpha_i} f(z)) \equiv I_z(. \Delta_{\alpha_i} g(z)) \);

(ii) \( I_z(. \Delta_{\alpha_i} f(z)) \equiv (I_z(. \Delta_{\alpha_i} b(z)) - a(z)) e^{h(z)} + I_z(. \Delta_{\alpha_i} b(z)) \),

\[
I_z(. \Delta_{\alpha_i} g(z)) \equiv (I_z(. \Delta_{\alpha_i} b(z)) - a(z)) e^{-h(z)} + I_z(. \Delta_{\alpha_i} b(z))
\]

where \( h(z) \) is a polynomial and \( I_z(. \Delta_{\alpha_i} b(z)) = \sum_{i=1}^{k} a_i \left(b(z + c_i) - b(z)\right) \).

Theorem 5.3.2.

Let \( \alpha \) and \( b \) be distinct complex constants. Let \( I_z(. \Delta_{\alpha_i} f(z)) \) and \( I_z(. \Delta_{\alpha_i} g(z)) \) be
as defined in Theorem 5.3.1 share a CM and $L(z, \Delta_{c_i}f(z)) = b$ and $L(z, \Delta_{c_i}g(z)) = b$ have at least $\max\{\rho(f), \rho(g), 1\}$ distinct common zeros of multiplicities $\geq 2$, then $L(z, \Delta_{c_i}f(z)) \equiv L(z, \Delta_{c_i}g(z))$.

Next, we extend the Theorem 5.2.4 and Theorem 5.2.5 as follows.

**Theorem 5.3.3.**

Let $f(z)$ be a non-constant finite order meromorphic function and $n \geq 3k + 4$ be an integer and let $F = f^n(z)$, $L(z, F) = \sum_{i=1}^{k} a_i F(z + c_i)$, where $a_i(i = 1, 2, \ldots, k)$ are complex constants, $c_i(i = 1, 2, \ldots, k)$ are non-zero complex constants. If $F$ and $L(z, F)$ share $1$ and $\infty$ CM, then

$$F \equiv L(z, F).$$

In case of entire functions, using the same method as in the proof of Theorem 5.3.3, we get the following result.

**Theorem 5.3.4.**

Let $f(z)$ be a non-constant finite order entire function and $n \geq 2k + 3$ be an integer and let $F = f^n(z)$, $L(z, F) = \sum_{i=1}^{k} a_i F(z + c_i)$, where $a_i(i = 1, 2, \ldots, k)$ are complex constants, $c_i(i = 1, 2, \ldots, k)$ are non-zero complex constants. If $F$ and $L(z, F)$ share $1$ CM, then

$$F \equiv L(z, F).$$
5.4 Proof of the main results

Proof of the Theorem 5.3.1:

By hypothesis, \( \delta(b) = \delta(b, f) + \delta(b, g) > 1 \), we obtain \( \delta(b, f) > 0 \) and \( \delta(b, g) > 0 \).

We now consider a positive number \( \epsilon \) such that \( 0 < \epsilon < \min\{\frac{\delta(b, f)}{2}, \frac{\delta(b, g)}{2}, \frac{\delta(b)}{2}\} \).

By definition,

\[
\delta(b, f) = \lim_{r \to \infty} \frac{m \left( r, \frac{1}{f - b} \right)}{T(r, f)}
\]

implies \( (\delta(b, f) - \epsilon)T(r, f) \leq m \left( r, \frac{1}{f - b} \right) \) \hspace{1cm} (5.4.1)

Similarly, we get

\[
(\delta(b, g) - \epsilon)T(r, g) \leq m \left( r, \frac{1}{g - b} \right) \] \hspace{1cm} (5.4.2)

Now consider,

\[
m(r, L(z, \Delta_{\alpha} f(z))) = m \left( r, \sum_{i=1}^{k} \alpha_{i} \Delta_{\alpha_{i}} f(z) \right) \]

\[
\leq m \left( r, \sum_{i=1}^{k} \alpha_{i} \Delta_{\alpha_{i}} f(z) \right) \left( \frac{1}{f(z)} \right) + m(r, f(z)) + S(r, f) \] \hspace{1cm} (5.4.3)

From Lemma 1.5.1, (5.4.3) can be written as

\[
m(r, L(z, \Delta_{\alpha} f(z))) \leq m(r, f(z)) + S(r, f) \] \hspace{1cm} (5.4.4)

Since, \( f(z) \) is an entire function and from (5.4.4), we obtain

\[
T(r, L(z, \Delta_{\alpha} f(z))) = m(r, L(z, \Delta_{\alpha} f(z))) \]

\[
\leq m(r, f(z)) + S(r, f) \]

i.e.,

\[
T(r, L(z, \Delta_{\alpha} f(z))) \leq T(r, f(z)) + S(r, f) \] \hspace{1cm} (5.4.5)
Next consider,
\[
m\left(r, \frac{1}{f(z) - b(z)} \right) = m\left(r, \frac{L(z, \Delta f(z)) - L(z, \Delta b(z))}{f(z) - b(z)} \cdot \frac{1}{L(z, \Delta f(z)) - L(z, \Delta b(z))} \right)
\]
\[
\leq m\left(r, \frac{L(z, \Delta \Delta f(z)) - L(z, \Delta \Delta b(z))}{f(z) - b(z)} \right)
\]
\[
+ m\left(r, \frac{1}{L(z, \Delta f(z)) - L(z, \Delta b(z))} \right) + S(r, f)
\]
(5.4.6)

From Lemma 1.5.1, (5.4.6) can be written as
\[
m\left(r, \frac{1}{f(z) - b(z)} \right) \leq m\left(r, \frac{1}{L(z, \Delta \Delta f(z)) - L(z, \Delta \Delta b(z))} \right) + S(r, f) \quad (5.4.7)
\]

From (5.4.1) and (5.4.7), we obtain
\[
(\delta(b, f) - \epsilon) T(r, f) \leq m\left(r, \frac{1}{L(z, \Delta \Delta f(z)) - L(z, \Delta \Delta b(z))} \right) + S(r, f)
\]
\[
\leq T(r, L(z, \Delta \Delta f(z)) - L(z, \Delta \Delta b(z))) + S(r, f)
\]
i.e.,
\[
(\delta(b, f) - \epsilon) T(r, f) \leq T(r, L(z, \Delta \Delta f(z))) + S(r, f) \quad (5.4.8)
\]

Thus, from (5.4.5) and (5.4.8), we note that \( S(r, L(z, \Delta \Delta f(z))) = S(r, f) \)

Similarly, we obtain \( S(r, L(z, \Delta \Delta g(z))) = S(r, g) \).

From (5.4.1), (5.4.5) and (5.4.7), we get
\[
(\delta(b, f) - \epsilon) T(r, L(z, \Delta \Delta f(z))) \leq (\delta(b, f) - \epsilon) T(r, f) + S(r, f)
\]
\[
\leq m\left(r, \frac{1}{f(z) - b(z)} \right) + S(r, f)
\]
\[
\leq m\left(r, \frac{1}{L(z, \Delta \Delta f(z)) - L(z, \Delta \Delta b(z))} \right) + S(r, f)
\]
\[(\delta(b, f) - \epsilon) T(r, L(z, \Delta \alpha, f(z))) \leq T(r, L(z, \Delta \alpha, f(z)))
\]
\[-N \left( r, \frac{1}{L(z, \Delta \alpha, f(z)) - L(z, \Delta \alpha, b(z))} \right) + S(r, f) \]

i.e.,

\[N \left( r, \frac{1}{L(z, \Delta \alpha, f(z)) - L(z, \Delta \alpha, b(z))} \right) \leq (1 - \delta(b, f) + \epsilon) T(r, L(z, \Delta \alpha, f(z))), + S(r, f) \]

(5.4.9)

Similarly, we obtain

\[N \left( r, \frac{1}{L(z, \Delta \alpha, g(z)) - L(z, \Delta \alpha, b(z))} \right) \leq (1 - \delta(b, g) + \epsilon) T(r, L(z, \Delta \alpha, g(z))), + S(r, g) \]

(5.4.10)

Since, \(L(z, \Delta \alpha, f(z))\) and \(L(z, \Delta \alpha, g(z))\) share \(\alpha(z)\) CM, we have

\[\frac{L(z, \Delta \alpha, f(z)) - \alpha(z)}{L(z, \Delta \alpha, g(z)) - \alpha(z)} = e^{h(z)}, \quad \text{where } h(z) \text{ is a polynomial.} \quad (5.4.11)\]

(5.4.11) can be written as

\[\frac{L(z, \Delta \alpha, f(z)) - L(z, \Delta \alpha, b(z))}{\alpha(z) - L(z, \Delta \alpha, b(z))} e^{h(z)} + e^{h(z)} \left( \frac{L(z, \Delta \alpha, g(z)) - L(z, \Delta \alpha, b(z))}{\alpha(z) - L(z, \Delta \alpha, b(z))} e^{h(z)} \right) = 1 \]

(5.4.12)

i.e.,

\[F_1 + F_2 + F_3 = 1 \quad (5.4.13)\]

where \(F_1 = \frac{L(z, \Delta \alpha, f(z)) - L(z, \Delta \alpha, b(z))}{\alpha(z) - L(z, \Delta \alpha, b(z))} e^{h(z)}\), \(F_2 = e^{h(z)}\)

and

\[F_3 = -\left( \frac{L(z, \Delta \alpha, g(z)) - L(z, \Delta \alpha, b(z))}{\alpha(z) - L(z, \Delta \alpha, b(z))} e^{h(z)} \right) \]
Let $T(r) = \max_{1 \leq j \leq 3}\{T(r, F_j)\}$ and $S(r) = o(T(r))$.

From (5.4.12), we have

$$N(r, F_1) = N(r, F_3) \leq N\left(r, \frac{1}{a(z) - L(z, \Delta_e b(z))}\right) = S(r, f)$$

(5.4.14)

and

$$N(r, F_2) = 0, \quad N\left(r, \frac{1}{F_2}\right) = 0$$

(5.4.15)

From Lemma 1.4.14, we deduce

$$\sum_{j=1}^{3} N_2\left(r, \frac{1}{F_j}\right) + \sum_{j=1}^{3} N(r, F_j) \leq N_2\left(r, \frac{1}{F_1}\right) + N_2\left(r, \frac{1}{F_3}\right) + S(r, f) + S(r, g)$$

$$\leq N\left(r, \frac{1}{L(z, \Delta_e f(z)) - L(z, \Delta_e b(z))}\right) + N\left(r, \frac{1}{L(z, \Delta_e g(z)) - L(z, \Delta_e b(z))}\right) + S(r, f) + S(r, g)$$

(5.4.16)

From (5.4.9), (5.4.10) and (5.4.16), we obtain

$$\sum_{j=1}^{3} N_2\left(r, \frac{1}{F_j}\right) + \sum_{j=1}^{3} N(r, F_j) \leq (1 - \delta(b, f) + \epsilon)T(r, L(z, \Delta_e f(z)))$$

$$+ (1 - \delta(b, g) + \epsilon)T(r, L(z, \Delta_e g(z)))$$

$$+ S(r, f) + S(r, g)$$

$$\leq (2 - \delta(b) + 2\epsilon)T(r) + S(r)$$

i.e.,

$$\sum_{j=1}^{3} N_2\left(r, \frac{1}{F_j}\right) + \sum_{j=1}^{3} N(r, F_j) \leq (\lambda + o(1))T(r)$$

where $\lambda = 2 - \delta(b) < 1$. Thus, by Lemma 1.4.14, we get either $F_2 = 1$ or $F_3 = 1$.

If $F_2 = 1$ implies $c^{h(z)} = 1$. 

114
From (5.4.11), we get

\[ L(z, \Delta_{\epsilon_i} f(z)) = L(z, \Delta_{\epsilon_i} g(z)) \]

If \( F_3 = 1 \), implies

\[ L(z, \Delta_{\epsilon_i} g(z)) = [L(z, \Delta_{\epsilon_i} b(z)) - a(z)] e^{-h(z)} + L(z, \Delta_{\epsilon_i} b(z)) \]

Continuing in the same manner, we get

\[ L(z, \Delta_{\epsilon_i} f(z)) = [L(z, \Delta_{\epsilon_i} b(z)) - a(z)] e^{h(z)} + L(z, \Delta_{\epsilon_i} b(z)). \]

**Proof of the Theorem 5.3.2:**

Since, by hypothesis, \( L(z, \Delta_{\epsilon_i} f(z)) \) and \( L(z, \Delta_{\epsilon_i} g(z)) \) share a CM, we have

\[ \frac{L(z, \Delta_{\epsilon_i} f(z)) - a}{L(z, \Delta_{\epsilon_i} g(z)) - a} = e^{h(z)} \]

where \( h(z) \) is a polynomial \hspace{1cm} (5.4.17)

From (5.4.17), we get

\[ T(r, e^{h(z)}) = T\left(r, \frac{L(z, \Delta_{\epsilon_i} f(z)) - a}{L(z, \Delta_{\epsilon_i} g(z)) - a}\right) \]

\[ \leq T(r, L(z, \Delta_{\epsilon_i} f(z))) + T(r, L(z, \Delta_{\epsilon_i} g(z))) + O(1) \]

\[ \leq T\left(r, \sum_{i=1}^{k} a_i \Delta_{\epsilon_i} f(z)\right) + T\left(r, \sum_{i=1}^{k} a_i \Delta_{\epsilon_i} g(z)\right) + O(1) \]

\[ \leq \sum_{i=1}^{k} T(r, a_i \Delta_{\epsilon_i} f(z)) + \sum_{i=1}^{k} T(r, a_i \Delta_{\epsilon_i} g(z)) + O(1) \]

i.e.,

\[ T(r, e^{h(z)}) \leq 2kT(r, f(z)) + 2kT(r, g(z)) + S(r, f) + S(r, g) \]

(5.4.18)

From (5.4.18), we observe that \( \rho(e^{h(z)}) \leq \max\{\rho(f), \rho(g)\} \). Since \( \rho(e^{h(z)}) = \deg h(z) \) is an integer it can be deduced that \( \deg h(z) \leq \max\{\rho(f), \rho(g)\} \).
Suppose that \( h(z) \) is not a constant. Differentiating (5.4.17), we get

\[
\left( L(z, \Delta_{c_i} f(z)) \right)' - \left( L(z, \Delta_{c_i} f(z)) \right)' \cdot e^{h(z)} - h'(z) \left( L(z, \Delta_{c_i} g(z)) - a \right) e^{h(z)} \equiv 0
\]

(5.4.19)

Let \( m = \max\{\rho(f), \rho(g), 1\} \). Then \( m \geq \max\{\rho(f), \rho(g)\} \geq \deg h(z) \).

By hypothesis, \( L(z, \Delta_{c_i} f(z)) - b \) and \( L(z, \Delta_{c_i} g(z)) - b \) have at least \( m \) distinct common zeros of multiplicity \( \geq 2 \).

Suppose that \( z_j (j = 1, 2, \cdots, m) \) satisfies

\[
L(z_j, \Delta_{c_i} f(z)) = L(z_j, \Delta_{c_i} g(z)) = b
\]

i.e.,

\[
L(z_j, \Delta_{c_i} f)' = L(z_j, \Delta_{c_i} g)' = 0
\]

(5.4.20)

From (5.4.19) and (5.4.20), we deduce \( h'(z_j) = 0, \; j = 1, 2, \cdots, m \). It implies \( h(z) \) is a polynomial of \( \deg h(z) \geq m + 1 \), which is a contradiction, since \( m \geq \deg h(z) \).

Now, suppose \( h(z) \) is a constant. Then \( e^{h(z)} \) is also a constant say \( l \).

From (5.4.17), we get

\[
\frac{L(z, \Delta_{c_i} f(z)) - a}{L(z, \Delta_{c_i} g(z)) - a} = l
\]

(5.4.21)

By hypothesis of Theorem 5.3.2 it is clear that \( L(z, \Delta_{c_i} f(z)) - b \) and \( L(z, \Delta_{c_i} g(z)) - b \) have at least a common zero \( z_0 \) such that \( L(z_0, \Delta_{c_i} f(z_0)) = L(z_0, \Delta_{c_i} g(z_0)) = b \).

From (5.4.21), we get

\[
\frac{L(z, \Delta_{c_i} f(z_0)) - a}{L(z, \Delta_{c_i} g(z_0)) - a} = \frac{b - a}{b - a} = l
\]

i.e.,

\[
l = 1
\]

116
Hence, from (5.4.21), we obtain \( I_z(z, \Delta_w f(z)) = I_z(z, \Delta_w g(z)) \).

**Proof of the Theorem 5.3.3:**

Let \( w' = e^{2z} \).

By Second fundamental theorem, we have

\[
m \left( r, \frac{1}{F(z) - 1} \right) = \sum_{p=0}^{n-1} m \left( r, \frac{1}{f(z) - w^p} \right) \leq 2T(r, f) - m(r, f) - \left( 2N(r, f) - N \left( r, f' \right) + N \left( r, \frac{1}{f} \right) \right) + S(r, f) \leq T(r, f) + \overline{N}(r, f) + S(r, f)
\]

i.e.,

\[
m \left( r, \frac{1}{F(z) - 1} \right) \leq 2T(r, f) + S(r, f) \quad (5.4.22)
\]

Since \( F(z) \) and \( L(z, F) \) share 1, \( \infty \) CM, we have

\[
\frac{F(z) - 1}{L(z, F) - 1} = e^{h(z)}, \text{ where } h(z) \text{ is a polynomial} \quad (5.4.23)
\]

By First Fundamental theorem, we have

\[
T(r, e^h) = T(r, e^{-h}) + O(1) = m(r, e^{-h(z)}) + N \left( r, e^{-h(z)} \right) + O(1) \quad (5.4.24)
\]

\[
= m \left( r, e^{-h(z)} \right) + O(1)
\]

From (5.4.23) and (5.4.24), we obtain

\[
T \left( r, e^{h(z)} \right) = m \left( r, \frac{L(z, F) - 1}{F(z) - 1} \right) + O(1)
\]

\[
\leq m \left( r, \frac{L(z, F)}{F(z) - 1} \right) + m \left( r, \frac{1}{F(z) - 1} \right) + O(1)
\]
i.e.,
\[ T \left( r, e^{h(z)} \right) \leq m \left( r, \frac{1}{F(z) - 1} \right) + S(r, F) + O(1) \] (5.4.25)

From (5.4.22) and (5.4.25), we get
\[ T \left( r, e^{h(z)} \right) \leq 2T(r, f) + S(r, f) \] (5.4.26)

Now, (5.4.23), can be written as
\[ F + e^{h(z)} - e^{h(z)}L(z, F) = 1 \]

Let \( F_1 = F, \ F_2 = e^{h(z)}, \ F_3 = -e^{h(z)}L(z, F) \) such that \( F_1 + F_2 + F_3 = 1 \) and \( T(r) = \max_{1 \leq j \leq 3} \{ T(r, F_j) \}, \ S(r) = o(T(r)). \)

Consider,
\[ \overline{N}(r, F_1) = \overline{N}(r, F) = \overline{N}(r, f) + S(r, f) \]

\[ \overline{N} \left( r, \frac{1}{F_1} \right) = \overline{N} \left( r, \frac{1}{f} \right) \] (5.4.27)

\[ \overline{N}(r, F_2) = \overline{N} \left( r, \frac{1}{F_2} \right) = 0 \]

\[ \overline{N}(r, F_3) = \overline{N}(r, L(z, F)) \]

\[ = \overline{N} \left( r, a_1 F(z + c_1) + a_2 F(z + c_2) + \cdots + a_k F(z + c_k) \right) \]

\[ \leq k\overline{N}(r, f) + S(r, f) \]

i.e.,
\[ \overline{N}(r, F_3) \leq k\overline{N}(r, f) + S(r, f) \] (5.4.28)

and
\[ \overline{N} \left( r, \frac{1}{F_3} \right) = \overline{N} \left( r, \frac{1}{L(z, F)} \right) \leq k\overline{N} \left( r, \frac{1}{f} \right) + S(r, f) \] (5.4.29)
From (5.4.27)-(5.4.29), we obtain

\[
3 \sum_{j=1}^{3} N(r, F_j) + 3 \sum_{j=1}^{3} N_2 \left( r, \frac{1}{F_j} \right) = N(r, F_1) + N(r, F_3) + N_2 \left( r, \frac{1}{F_1} \right) \\
+ N \left( r, \frac{1}{F_3} \right) \\
\leq N(r, f) + kT(r, f) + 2N \left( r, \frac{1}{f^n} \right) \\
+ 2N \left( r, \frac{1}{L(z, f)} \right) + S(r, f) \\
\leq N(r, f) + kT(r, f) + 2N \left( r, \frac{1}{f} \right) \\
+ 2N \left( r, \frac{1}{L(z, f)} \right) + S(r, f) \\
\leq T(r, f) + kT(r, f) + 2T(r, f) \\
+ 2kT(r, f) + S(r, f) \\
\leq (3k + 3)T(r, f) + S(r, f) \\
\leq \frac{(3k + 3)}{n} T(r) + S(r)
\]

i.e.,

\[
3 \sum_{j=1}^{3} N(r, F_j) + 3 \sum_{j=1}^{3} N_2 \left( r, \frac{1}{F_j} \right) \leq \frac{(3k + 3)}{n} T(r) + S(r)
\]

Noting that \( n \geq 3k + 4 \), from Lemma 1.4.14 and since \( F_1 = f^n \) is not constant, we get either \( F_2 \equiv 1 \) or \( F_3 \equiv 1 \).

If \( F_3 \equiv 1 \) implies

\[
-e^{h(z)} L(z, F) = 1 \\
L(z, F) = \frac{-1}{e^{h(z)}}
\]
which implies

\[ T(r, L(z, F)) \leq T\left(r, e^{h(z)}\right) + O(1) \]

From Lemma 1.4.2 and (5.4.26), we get

\[ nT(r, f) + S(r, f) \leq 2T(r, f) + S(r, f) \]

which is a contradiction to \( n \geq 3k + 4 \).

If \( F_2 \equiv 1 \) implies \( e^{h(z)} \equiv 1 \), from (5.4.23), we get \( F(z) \equiv L(z, F) \).

**Proof of the Theorem 5.3.4:**

In this Theorem, \( f(z) \) is an entire function. Taking \( N(r, f) = N(r, L(z, F)) = 0 \), in the Theorem 5.3.3, we obtain the conclusion of Theorem 5.3.4.

### 5.5 Conclusions

1. If \( i = 1 \) and \( a_1 = 1 \) in Theorem 5.3.1, then Theorem 5.3.1 reduces to Theorem 5.2.2.

2. If \( i = 1 \) and \( a_1 = 1 \) in Theorem 5.3.2, then Theorem 5.3.2 reduces to Theorem 5.2.3.