Chapter 4

Uniqueness of $k^{th}$ derivative of differential-difference entire functions sharing a small function weakly with some weight

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The content in this chapter has been communicated.
4.1 Introduction

In this chapter, we continue to investigate the uniqueness problems of finite order transcendental entire functions whose certain non-linear differential-difference polynomial share a small function in a relaxed manner. The results obtained generalize the results of P. Sahoo [69], P. Sahoo and H. Karmakar [70].

Following are well-known definitions in the literature of weighted sharing.

In 2001, I. Lahiri [37] first introduced a gradation of sharing of values which is known as weighted sharing, which is a scaling between sharing IM and sharing CM.

Definition 4.1.1. Weighted sharing [37]

Let $k$ be a non-negative integer or infinity. For $\alpha \in \mathbb{C}$, $F_k(\alpha, f)$ denotes the set of all $\alpha$-points of $f(z)$ where an $\alpha$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. If $F_k(\alpha, f) = F_k(\alpha, g)$, then $f(z)$ and $g(z)$ share the value ‘$\alpha$’ with weight $k$.

Definition 4.1.2. [49]

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions sharing $\alpha$ “IM”, for $\alpha \in S(f) \cap S(g)$ where $S(f)$ and $S(g)$ denotes the set of all small functions of $f(z)$ and $g(z)$ respectively and $k$ be a positive integer or $\infty$.

(i) $N^{F}_k(r, \alpha; f, g \ \leq k)$ denotes the reduced counting function of those $\alpha$—points of $f(z)$ whose multiplicities are equal to the corresponding $\alpha$—points of $g(z)$, both of their multiplicities are not greater than $k$. 

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(ii) $\overline{N}^0(r, \alpha; f, g \setminus k > k)$ denotes the reduced counting function of those $\alpha-$points of $f(z)$ which are $\alpha-$points of $g(z)$, both of their multiplicities are not less than or equal to $k$.

Clearly $\overline{N}^0(r, \alpha; f, g \setminus k > k) = \overline{N}^0(r, \alpha; f, g \setminus k \geq k + 1)$.

Recently, S. H. Lin and W. C. Lin [49] introduced concept of weakly weighted sharing defined as follows:

**Definition 4.1.3. [49]**

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. For $\alpha \in S(f) \cap S(g)$, if $k$ be a positive integer or $\infty$ and

$$\overline{N}(r, \alpha; f \setminus k) - \overline{N}^E(r, \alpha; f, g \setminus k) = S(r, f),$$

$$\overline{N}(r, \alpha; g \setminus k) - \overline{N}^E(r, \alpha; f, g \setminus k) = S(r, g),$$

$$\overline{N}(r, \alpha; f \setminus k + 1) - \overline{N}^0(r, \alpha; f, g \setminus k + 1) = S(r, f),$$

$$\overline{N}(r, \alpha; g \setminus k + 1) - \overline{N}^0(r, \alpha; f, g \setminus k + 1) = S(r, g)$$

or if $k = 0$ and

$$\overline{N}(r, \alpha, f) - \overline{N}^0(r, \alpha; f, g) = S(r, f),$$

$$\overline{N}(r, \alpha, g) - \overline{N}^0(r, \alpha; f, g) = S(r, g),$$

then $f(z)$, $g(z)$ weakly share $\alpha$ with weight $k$ and is denoted by $(\alpha, k)$.

**Definition 4.1.4. [4]**

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and $\alpha \in \mathbb{C} \cup \{\infty\}$. 

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\( \overline{N}(r, \alpha; f \setminus = p; g \setminus = q) = S(r, f) \) is reduced counting function of common \( \alpha \)-points of \( f(z) \) and \( g(z) \) with multiplicities \( p \) and \( q \) respectively.

A. Banerjee and S. Mukherjee [4] introduced concept of relaxed weighted sharing defined as follows:

**Definition 4.1.5. [4]**

Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions share ‘\( \alpha \)’ “IM”. Let \( k \) be a positive integer or \( \infty \) and \( \alpha \in \mathbb{C} \cup \{ \infty \} \). If

\[
\sum_{p,q \leq k} \overline{N}(r, \alpha; f \setminus = p; g \setminus = q) = S(r, f),
\]

then \( f(z) \) and \( g(z) \) share ‘\( \alpha \)’ with weight \( k \) in a relaxed manner and is denoted by \((\alpha, k)^{\alpha}\).

### 4.2 Preliminaries

In 2015, P. Sahoo [69] proved the following theorems.

**Theorem 4.2.1. [69]**

Let \( f(z) \) and \( g(z) \) be two finite order transcendental entire functions and \( \alpha(z) (\neq 0, \infty) \) be a small function with respect to both \( f(z) \) and \( g(z) \). Suppose that \( \eta \) is a non-zero complex constant, \( n \) and \( m (\geq 2) \) are integers satisfying \( n + m \geq 10 \). If \( f^n(z)(f(z) - 1)^m f(z + \eta) \) and \( g^n(z)(g(z) - 1)^m g(z + \eta) \) share \((\alpha(z), 2)\), then either \( f(z) \equiv g(z) \) or \( f(z) \) and \( g(z) \) satisfy the algebraic equation \( P(f, g) = 0 \), where

\[
R(w_1, w_2) = w_1^n (w_1 - 1)^m w_1(z + \eta) - w_2^n (w_2 - 1)^m w_2(z + \eta) \quad (4.2.1)
\]
Theorem 4.2.2. [69]

Let $f(z)$ and $g(z)$ be two finite order transcendental entire functions and $\alpha(z)(\neq 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that $\eta$ is a non-zero complex constant, $n$ and $m(\geq 2)$ are integers satisfying $n + m \geq 13$. If $f^n(z)(f(z) - 1)^m f(z + \eta)$ and $g^n(z)(g(z) - 1)^m g(z + \eta)$ share $(\alpha(z), 2)^*$, then the conclusion of Theorem 4.2.1 holds.

Theorem 4.2.3. [69]

Let $f(z)$ and $g(z)$ be two finite order transcendental entire functions and $\alpha(z)(\neq 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that $\eta$ is a non-zero complex constant, $n$ and $m(\geq 2)$ are integers satisfying $n + m \geq 19$. If $\overline{F}_2(\alpha(z), f^n(z)(f(z) - 1)^m f(z + \eta)) = \overline{F}_2(\alpha(z), g^n(z)(g(z) - 1)^m g(z + \eta))$, then the conclusion of Theorem 4.2.1 holds.

In the same year 2015, P. Sahoo and H. Karmakar [70] obtained the following theorems.

Theorem 4.2.4. [70]

Let $f(z)$ and $g(z)$ be two finite order transcendental entire functions and $\alpha(z)(\neq 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that $\eta$ is a non-zero complex constant, $n(\geq 1)$ and $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq 3k + 2m + 8$ when $m \leq k + 1$ and $n \geq 6k - m + 13$ when $m > k + 1$. If $(f^n(z)(f(z) - 1)^m f(z + \eta))^{(k)}$ and $(g^n(z)(g(z) - 1)^m g(z + \eta))^{(k)}$ share $(\alpha(z), 2)^*$, then either $f(z) = g(z)$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f, g) = 0.$
where $R(f, g) = 0$ is given by (4.2.1).

**Theorem 4.2.5. [70]**

Let $f(z)$ and $g(z)$ be two finite order transcendental entire functions and $\alpha(z)(\neq 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that $\eta$ is a non-zero complex constant, $n(\geq 1)$ and $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq 5k + 4m + 12$ when $m \leq k + 1$ and $n \geq 10k - m + 19$ when $m > k + 1$. If

$$E_2\left(\alpha(z), (f^n(z)(f(z) - 1)^m f(z + \eta))^k\right) = E_2\left(\alpha(z), (g^n(z)(g(z) - 1)^m g(z + \eta))^k\right)$$

then the conclusion of Theorem 4.2.1 holds.

### 4.3 Main results

In this chapter, we extend above theorems by considering differential-difference functions of the form $\left(f^n(z)(f(z) - 1)^m \prod_{j=1}^d f(z + c_j)^{\nu_j}\right)^k$ as follows.

**Theorem 4.3.1.**

Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order, $\alpha(z)(\neq 0)$ be a small function with respect to $f(z)$ and $g(z)$, $c_j (j = 1, 2, \ldots, d)$ be distinct finite non-zero complex numbers and $n, m, d, k$ and $\nu_j (j = 1, 2, \ldots, d)$ are non-negative integers satisfying $n \geq 3k + 2m + 2\sigma + 6$ when $m \leq k + 1$ and $n \geq 6k - m + 2\sigma + 11$ when $m > k + 1$, where $\sigma = \sum_{j=1}^d \nu_j$. If $\left(f^n(z)(f(z) - 1)^m \prod_{j=1}^d f(z + c_j)^{\nu_j}\right)^k$ and $\left(g^n(z)(g(z) - 1)^m \prod_{j=1}^d g(z + c_j)^{\nu_j}\right)^k$ share $\alpha(z)$, then either $f(z) \equiv g(z)$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by

$$R(u_1, u_2) = u_1^n (u_1 - 1)^m \prod_{j=1}^d u_1 (z + c_j)^{\nu_j} - u_2^n (u_2 - 1)^m \prod_{j=1}^d u_2 (z + c_j)^{\nu_j}. \quad (4.3.1)$$
Theorem 4.3.2.

Let \( f(z) \) and \( g(z) \) be transcendental entire functions of finite order; \( \alpha(z)(\neq 0) \) be a small function with respect to \( f(z) \) and \( g(z) \), \( c_j (j = 1, 2, \ldots, d) \) be distinct finite non-zero complex numbers and \( n, m, d, k \) and \( r_j (j = 1, 2, \ldots, d) \) are non-negative integers satisfying \( n \geq 5k + 4m + 4\sigma + 8 \) when \( m \leq k + 1 \) and \( n \geq 10k - m + 4\sigma + 15 \) when \( m > k + 1 \), where \( \sigma = \sum_{j=1}^{d} r_j \). If

\[
E_{2n} \left( \alpha(z), \left( \frac{f''(z)f(z) - 1}{f(z)} \right)^{(k)} \right) = E_{2n} \left( \alpha(z), \left( \frac{g''(z)(g(z) - 1)}{g(z)} \right)^{(k)} \right)
\]

then the conclusion of Theorem 4.3.1 holds.

4.4 Lemmas

We need the following Lemmas to prove our results.

Lemma 4.4.1. [4]

Let \( F \) and \( G \) be non-constant meromorphic functions that share \((1, 2)^+\) and \( H = \left( \frac{F''}{F} - \frac{G''}{G} \right) - \left( \frac{F'''}{F} - \frac{G'''}{G} \right) \neq 0 \). Then

\[
T(r, F) \leq N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) + N_2(r, F) + N_2(r, G) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right)
\]

\[
- n \left( r, \frac{1}{F} \right) + S(r, F) + S(r, G)
\]

and the same inequality is true for \( T(r, G) \).

Lemma 4.4.2. [50]

Let \( F \) and \( G \) be non-constant entire functions and \( p \geq 2 \) be an integer. If

\[
F_{p}(1, F) = F_{p}(1, G) \text{ and } H \neq 0,
\]

then

\[
T(r, F) \leq N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) + 2N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) + S(r, F) + S(r, G)
\]

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and the same inequality is true for $T(r, G)$.

**Lemma 4.4.3.** [15]

Let $f(z)$ be entire function of finite order and

$$F = f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{v_j}$$

Then, $T(r, F) = (n + m + \sigma)T(r, f) + S(r, f)$.  $\sigma = \sum_{j=1}^{d} v_j$.

**Lemma 4.4.4.**

Let $f(z)$ and $g(z)$ be transcendental entire functions, $n(\geq 1)$, $m(\geq 1)$, $k(\geq 0)$ be integers, and let

$$F = \left( f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{v_j} \right)^{(k)}$$
$$G = \left( g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{v_j} \right)^{(k)}$$

If there exists non-zero constants $c_1$ and $c_2$ such that $\mathcal{N} \left( r, \frac{1}{F - c_1} \right) = \mathcal{N} \left( r, \frac{1}{G} \right)$ and $\mathcal{N} \left( r, \frac{1}{G - c_2} \right) = \mathcal{N} \left( r, \frac{1}{F} \right)$, then $n \leq 2k + m + \sigma + 2$ when $m \leq k + 1$ and $n \leq 4k - m + \sigma + 4$ when $m > k + 1$.

**Proof:** Let $F_1 = f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{v_j}$ and

$$G_1 = g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{v_j}.$$ 

By Lemma 4.4.3, we have

$$T(r, F_1) = (n + m + \sigma)T(r, f) + S(r, f) \quad (4.4.1)$$
$$T(r, G_1) = (n + m + \sigma)T(r, g) + S(r, g) \quad (4.4.2)$$

Since $F$ is an entire function and from Second fundamental theorem, we get

$$T(r, F) \leq \mathcal{N} \left( r, \frac{1}{F} \right) + \mathcal{N} \left( r, \frac{1}{F - c_1} \right) + S(r, F) \quad (4.4.3)$$
From Lemma 1.4.4, (4.4.1) and (4.4.3), we deduce

\[(n + m + \sigma)T(r, f) \leq N_{k+1} \left( r, \frac{1}{f_1} \right) + N_{k+1} \left( r, \frac{1}{f_1} \right) + S(r, f) + S(r, g) \]

i.e.,

\[(n + m + \sigma)T(r, f) \leq N_{k+1} \left( r, \frac{1}{f^m(z)(f(z) - 1) \prod_{j=1}^{d} f(z + c_j)^{v_j}} \right) + N_{k+1} \left( r, \frac{1}{g^n(z)(g(z) - 1) \prod_{j=1}^{d} g(z + c_j)^{v_j}} \right) + S(r, f) + S(r, g) \quad (4.4.4)\]

If \( m \leq k + 1 \), then (4.4.4) reduces to

\[(n + m + \sigma)T(r, f) \leq (k + 1)T(r, f) + mT(r, f) + T \left( r, \prod_{j=1}^{d} f(z + c_j)^{v_j} \right) + (k + 1)T(r, g) + mT(r, g) + T \left( r, \prod_{j=1}^{d} g(z + c_j)^{v_j} \right) + S(r, f) + S(r, g) \]

i.e.,

\[(n + m + \sigma)T(r, f) \leq (k + 1 + m + \sigma)(T(r, f) + T(r, g)) + S(r, f) + S(r, g) \quad (4.4.5)\]

Similarly, we get

\[(n + m + \sigma)T(r, g) \leq (k + 1 + m + \sigma)(T(r, f) + T(r, g)) + S(r, f) + S(r, g) \quad (4.4.6)\]

Combining (4.4.5) and (4.4.6), we obtain

\[(n + m + \sigma)(T(r, f) + T(r, g)) \leq 2(k + 1 + m + \sigma)(T(r, f) + T(r, g)) + S(r, f) + S(r, g) \]
Thus, we get $n \leq 2k+m+\sigma+2$ as $f(z)$ and $g(z)$ are transcendental entire functions.

If $m > k+1$, then (4.4.4) reduces to

$$(n + m + \sigma)T(r, f) \leq (k + 1)T(r, f) + (k + 1)T(r, g) + T\left(r, \prod_{j=1}^{d} f(z + c_j)^{v_j}\right)$$

$$+ (k + 1)T(r, g) + (k + 1)T(r, g) + T\left(r, \prod_{j=1}^{d} g(z + c_j)^{v_j}\right)$$

$$+ S(r, f) + S(r, g)$$

i.e.,

$$(n + m + \sigma)T(r, f) \leq (2k + \sigma + 2)(T(r, f) + T(r, g)) + S(r, f) + S(r, g) \quad (4.4.7)$$

Similarly, we get

$$(n + m + \sigma)T(r, g) \leq (2k + \sigma + 2)(T(r, f) + T(r, g)) + S(r, f) + S(r, g) \quad (4.4.8)$$

Combining (4.4.7) and (4.4.8), we obtain

$$(n + m + \sigma)(T(r, f) + T(r, g)) \leq 2(2k + \sigma + 2)(T(r, f) + T(r, g)) + S(r, f) + S(r, g) \quad (4.4.9)$$

Thus, we get $n \leq 4k-m+\sigma+4$ as $f(z)$ and $g(z)$ are transcendental entire functions.

### 4.5 Proof of the main results

**Proof of the Theorem 4.3.1:**

Let $F = \frac{F^{(k)}}{\alpha(z)}$ and $G = \frac{G^{(k)}}{\alpha(z)}$ where $F_1 = f^{\alpha}(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{v_j}$ and $G_1 = g^{\alpha}(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{v_j}$. Then $F$ and $G$ are transcendental meromorphic
functions that share $(1.2)^n$ except the zeros and poles of $a(z)$.

Let $H \neq 0$. Then, using Lemma 4.4.1, we have

$$T(r, F) \leq N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) + N_1 \left( r, \frac{1}{F} \right) + S(r, F) + S(r, G) \quad (4.5.1)$$

From Lemma 1.4.4, $(4.5.1)$ reduces to

$$T(r, F_1) \leq N_{k+2} \left( r, \frac{1}{F_1} \right) + N_{k+2} \left( r, \frac{1}{G_1} \right) + N_{k+1} \left( r, \frac{1}{F_1} \right) + S(r, f) + S(r, g) \quad (4.5.2)$$

If $m \leq k + 1$, using Lemma 1.4.4, $(4.5.2)$ can be reduced to

$$(n+m+\sigma)T(r, f) \leq (k+2)T(r, f) + mT(r, f) + \sigma T(r, f) + (k+2)T(r, g)$$

$$+ mT(r, g) + \sigma T(r, g) + (k+1)T(r, f) + mT(r, f) + \sigma T(r, f)$$

$$+ S(r, f) + S(r, g)$$

i.e.,

$$(n+m+\sigma)T(r, f) \leq (2k+2m+2\sigma+3)T(r, f) + (k+m+\sigma+2)T(r, g) + S(r, f) + S(r, g) \quad (4.5.3)$$

Similarly, we get

$$T(r, g) \leq (2k+2m+2\sigma+3)T(r, g) + (k+m+\sigma+2)T(r, f) + S(r, f) + S(r, g) \quad (4.5.4)$$

Combining $(4.5.3)$ and $(4.5.4)$, we have

$$(n+m+\sigma)[T(r, f) + T(r, g)] \leq (3k+3m+3\sigma+5)[T(r, f) + T(r, g)] + S(r, f) + S(r, g)$$

which is contradiction to $n \geq 3k+2m+2\sigma+6$ as $f(z)$ and $g(z)$ are transcendental entire functions.
If \( m > k + 1 \), using Lemma 1.4.4, (4.5.2) can be reduced to

\[
(n + m + \sigma)T(r, f) \leq (k + 2)T(r, f) + (k + 2)T(r, f) + \sigma T(r, f) + (k + 2)T(r, g)
\]

\[
+ (k + 2)T(r, g) + \sigma T(r, g) + (k + 1)T(r, f) + (k + 1)T(r, f)
\]

\[
+ \sigma T(r, f) + S(r, f) + S(r, g)
\]

i.e.,

\[
(n + m + \sigma)T(r, f) \leq (4k + 2\sigma + 6)T(r, f) + (2k + \sigma + 4)T(r, g) + S(r, f) + S(r, g) \quad (4.5.5)
\]

similarly, we get

\[
(n + m + \sigma)T(r, g) \leq (4k + 2\sigma + 6)T(r, g) + (2k + \sigma + 4)T(r, f) + S(r, f) + S(r, g) \quad (4.5.6)
\]

Combining (4.5.5) and (4.5.6), we get

\[
(n + m + \sigma)[T(r, f) + T(r, g)] \leq (6k + 3\sigma + 10)[T(r, f) + T(r, g)] + S(r, f) + S(r, g)
\]

which is contradiction to \( n \geq 6k - m + 2\sigma + 11 \) as \( f(z) \) and \( g(z) \) are transcendental entire functions.

Thus \( II \equiv 0 \) that is \( \left( \frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right) = 0 \) which implies \( \left( \frac{F''}{F'} - \frac{2F'}{F - 1} \right) = \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right) \).

Integrating both sides twice, we get

\[
\frac{1}{F - 1} = \frac{A}{G - 1} + B \quad (4.5.7)
\]

where \( A(\neq 0) \) and \( B \) are constants.

From (4.5.7), it is clear that \( f, g \) share 1 CM and hence they share \((1,2)\).

We now discuss the following cases:
Case 1: Let \( B \neq 0 \) and \( A = B \). Then from (4.5.7), we get
\[
\frac{1}{F - 1} = \frac{BG}{G - 1}
\]
(4.5.8)

If \( B = -1 \), then from (4.5.8), we have \( FG = 1 \), that is
\[
\left( f^n(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{v_j} \right)^{(k)} \left( g^n(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{v_j} \right)^{(k)} = \alpha^2(z)
\]
(4.5.9)

From (4.5.9), we have
\[
N\left( r, \frac{1}{F} \right) = S(r, f) \quad \text{and} \quad N\left( r, \frac{1}{F^{-1}} \right) = S(r, f),
\]
using these, we obtain
\[
\delta(0, f) = 1 - \lim_{r \to \infty} \frac{N\left( r, \frac{1}{F} \right)}{T(r, f)} = 1;
\]
\[
\delta(1, f) = 1 - \lim_{r \to \infty} \frac{N\left( r, \frac{1}{F^{-1}} \right)}{T(r, f)} = 1 \quad \text{and}
\]
\[
\delta(\infty, f) = 1 - \lim_{r \to \infty} \frac{N(r, f)}{T(r, f)} = 1
\]

Thus, \( \delta(0, f) + \delta(1, f) + \delta(\infty, f) = 3 \) which is not possible.

If \( B \neq -1 \), from (4.5.8), we can deduce that \( \frac{1}{F} = \frac{BG}{(B+1)G-1} \).

Thus, \( N\left( r, \frac{1}{\frac{B+1}{B+1}} \right) = N\left( r, \frac{1}{F} \right) \).

From Second Fundamental Theorem, we have
\[
T(r, G) \leq \mathcal{N} \left( r, \frac{1}{G} \right) + \mathcal{N} \left( r, \frac{1}{G - \frac{1}{B+1}} \right) + S(r, G)
\]
\[
\leq \mathcal{N} \left( r, \frac{1}{G} \right) + \mathcal{N} \left( r, \frac{1}{F} \right) + S(r, g)
\]
i.e.,
\[
T(r, G) \leq N_1 \left( r, \frac{1}{G^{(k)}} \right) + N_1 \left( r, \frac{1}{F^{(k)}} \right) + S(r, f) + S(r, g)
\]
(4.5.10)
Using Lemma 1.4.4 and (4.4.1), (4.5.10) can be deduced to

\[
(n + m + \sigma)T(r, g) \leq N_{k+1} \left( r, \frac{1}{f^m(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{\nu_j}} \right) \\
+ N_{k+1} \left( r, \frac{1}{g^n(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{\nu_j}} \right) \\
+ S(r, f) + S(r, g) \tag{4.5.11}
\]

If \( m \leq k + 1 \), from (4.5.11), we deduce

\[
(n + m + \sigma)T(r, g) \leq (k + m + \sigma + 1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g) \tag{4.5.12}
\]

Similarly, we get

\[
(n + m + \sigma)T(r, f) \leq (k + m + \sigma + 1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g) \tag{4.5.13}
\]

Combining (4.5.12) and (4.5.13), we get

\[
(n + m + \sigma)(T(r, f) + T(r, g)) \leq 2(k + m + \sigma + 1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g)
\]

which is contradiction to \( n \geq 3k + 2m + 2\sigma + 6 \) as \( f(z) \) and \( g(z) \) are transcendental entire functions.

If \( m > k + 1 \), from (4.5.11), we deduce

\[
(n + m + \sigma)T(r, g) \leq (2k + \sigma + 2)(T(r, f) + T(r, g)) + S(r, f) + S(r, g) \tag{4.5.14}
\]

Similarly, we get

\[
(n + m + \sigma)T(r, f) \leq (2k + \sigma + 2)(T(r, f) + T(r, g)) + S(r, f) + S(r, g) \tag{4.5.15}
\]

Combining (4.5.14) and (4.5.15), we get

\[
(n + m + \sigma)(T(r, f) + T(r, g)) \leq 2(2k + \sigma + 2)(T(r, f) + T(r, g)) + S(r, f) + S(r, g)
\]
which is contradiction to \( n \geq 6k - m + 2\sigma + 11 \) as \( f(z) \) and \( g(z) \) are transcendental entire functions.

**Case 2:** Let \( B \neq 0 \) and \( A \neq B \). Then from (4.5.7), we get

\[
F = \frac{(1 + B)G - (B - A + 1)}{BG + (A - B)}
\]

Thus, we have \( \overline{N} \left( r, \frac{1}{G - \frac{(B - A + 1)}{A}} \right) = \overline{N} \left( r, \frac{1}{F} \right) \).

Proceeding in a manner similar to Case 1, we get a contradiction.

**Case 3:** Let \( B = 0 \) and \( A \neq 0 \). Then, from (4.5.7), we get

\[
F = \frac{G - 1 + A}{A} \quad \text{and} \quad G = AF - (A - 1). \quad \text{If} \quad A \neq 1, \quad \text{it follows that}
\]

\[
\overline{N} \left( r, \frac{1}{F} \right) = \overline{N} \left( r, \frac{1}{G} \right) \quad \text{and} \quad \overline{N} \left( r, \frac{1}{G - (A - 1)} \right) = \overline{N} \left( r, \frac{1}{F} \right).
\]

Applying Lemma 4.4.4, we arrive at a contradiction.

Therefore \( A = 1 \) which implies \( F = G \), that is

\[
\left( f^n (f(z) - 1)^m \prod_{j=1}^{d} f(z + cj)^{v_j} \right)^{(k)} = \left( g^n (g(z) - 1)^m \prod_{j=1}^{d} g(z + cj)^{v_j} \right)^{(k)}
\]

Integrating once, we obtain

\[
\left( f^n (f(z) - 1)^m \prod_{j=1}^{d} f(z + cj)^{v_j} \right)^{(k-1)} = \left( g^n (g(z) - 1)^m \prod_{j=1}^{d} g(z + cj)^{v_j} \right)^{(k-1)} + c_{k-1}
\]

where \( c_{k-1} \) is a constant. If \( c_{k-1} \neq 0 \), by Lemma 4.4.4, it follows that \( n \leq 2\bar{k} + m + \sigma \)

when \( m \leq k + 1 \) and \( n \leq 4\bar{k} - m + \sigma \) when \( m > k + 1 \), a contradiction to the hypothesis. Hence \( c_{k-1} = 0 \). Repeating the process \( \bar{k} \) times, we deduce that

\[
f^n (f(z) - 1)^m \prod_{j=1}^{d} f(z + cj)^{v_j} = g^n (g(z) - 1)^m \prod_{j=1}^{d} g(z + cj)^{v_j} \quad (4.5.16)
\]
Let \( h = \frac{f}{g} \).

If \( h \) is a constant, let \( h(z) = t \), then substituting \( f = gt \) in (4.5.16), we get

\[
\ell^n g^n(z)(tg(z) - 1)^m \prod_{j=1}^{d} \ell^m g(z + c_j)^{v_j} = g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} (g + c_j)^{v_j}
\]

implies

\[
\ell^{n+\sigma}(tg(z) - 1)^m = (g(z) - 1)^m
\]

i.e.,

\[
\ell^{n+\sigma}(\ell^m g^m(z) - mC_1 \ell^{m-1} g^{m-1}(z) - \ldots - 1) = (g^m(z) - mC_1 g^{m-1}(z) - \ldots - 1)
\]

which implies \( \ell^{n+\sigma+m} = \ell^{n+\sigma+m-1} = \ldots = \ell^{n+\sigma} = 1 \), that is \( t = 1 \). Thus \( f(z) \equiv g(z) \).

If \( h(z) \) is not a constant, then \( f(z) \) and \( g(z) \) satisfy the algebraic equation \( R(f, g) = 0 \), where \( R(f, g) \) is given by

\[
R(u_1, u_2) = u_1^n (u_1 - 1)^m \prod_{j=1}^{d} (u_1 + c_j)^{v_j} - u_2^n (u_2 - 1)^m \prod_{j=1}^{d} (u_2 + c_j)^{v_j}.
\]

Hence the proof of Theorem 4.3.1.

**Proof of the Theorem 4.3.2:**

Let \( F, G, F_1, G_1 \) be as in Theorem 4.3.1. Then \( F \) and \( G \) are transcendental meromorphic functions such that \( \overline{T}_{2d}(1, F) = \overline{T}_{2d}(1, G) \) except for the zeros and poles of \( a_0(z) \).

Let \( H \neq 0 \). Then, from Lemma 1.4.4, Lemma 4.4.2 and from (4.4.1), we deduce

\[
(n + m + \sigma)T(r, f) \leq N_{k+2} \left( r, \frac{1}{F_1} \right) + N_{k+2} \left( r, \frac{1}{G_1} \right) + 2N_{k+1} \left( r, \frac{1}{F_1} \right)
\]

\[
+ N_{k+1} \left( r, \frac{1}{G_1} \right) + S(r, f) + S(r, g)
\]

(4.5.17)
If \( m \leq k + 1 \), then from (4.5.17), we deduce that

\[
(n + m + \sigma)T(r, f) \leq (3k + 3m + 3\sigma + 4)T(r, f) + (2k + 2m + 2\sigma + 3)T(r, g) + S(r, f) + S(r, g)
\]

Similarly, we get

\[
(n + m + \sigma)T(r, g) \leq (3k + 3m + 3\sigma + 4)T(r, g) + (2k + 2m + 2\sigma + 3)T(r, f) + S(r, f) + S(r, g)
\]

Combining (4.5.18) and (4.5.19), we obtain

\[
(n + m + \sigma)(T(r, f) + T(r, g)) \leq (5k + 5m + 5\sigma + 7)(T(r, f) + T(r, g)) + S(r, f) + S(r, g)
\]

which is contradiction to \( n \geq 5k + 4m + 4\sigma + 8 \) as \( f(z) \) and \( g(z) \) are transcendental entire functions.

If \( m > k + 1 \), then from (4.5.17), we deduce

\[
(n + m + \sigma)T(r, f) \leq (6k + 3\sigma + 8)T(r, f) + (4k + 2\sigma + 6)T(r, g) + S(r, f) + S(r, g)
\]

Similarly, we get

\[
(n + m + \sigma)T(r, g) \leq (6k + 3\sigma + 8)T(r, g) + (4k + 2\sigma + 6)T(r, f) + S(r, f) + S(r, g)
\]

Combining (4.5.20) and (4.5.21), we obtain

\[
(n + m + \sigma)(T(r, f) + T(r, g)) \leq (10k + 5\sigma + 14)(T(r, f) + T(r, g)) + S(r, f) + S(r, g)
\]

which is contradiction to \( n \geq 10k - m + 5\sigma + 15 \) as \( f(z) \) and \( g(z) \) are transcendental entire functions.
Thus $H \equiv 0$ and the rest of the theorem follows as in the proof of the Theorem 4.3.1.

Hence the proof of Theorem 4.3.2.

4.6 Conclusions

1. If $k = 0$ and $\sigma = 1$ in Theorem 4.3.1, then Theorem 4.3.1 reduces to Theorem 4.2.2.

2. If $\sigma = 1$ in Theorem 4.3.1, then Theorem 4.3.1 reduces to Theorem 4.2.4.

3. If $k = 0$ and $\sigma = 1$ in Theorem 4.3.2, then Theorem 4.3.2 reduces to Theorem 4.2.3.

4. If $\sigma = 1$ in Theorem 4.3.2, then Theorem 4.3.2 reduces to Theorem 4.2.5.