Chapter 5

An ideal-based $k$-zero-divisor hypergraph of a commutative ring

In this chapter, we introduce a new graph called ideal-based $k$-zero-divisor hypergraph $H^k_I(R)$ of a commutative ring $R$. In 2009, Redmond introduced the ideal-based zero-divisor graph [51] and Eslahchi et. al. generalized the zero-divisor graph to $k$-zero-divisor hypergraph [38]. In this notion, we extend the concept $k$-zero-divisor hypergraph to ideal-based $k$-zero-divisor hypergraph. At the first instance, we obtain certain fundamental properties of $H^k_I(R)$. Also we investigate the relation between $H^k_I(R)$ and $H_k(R/I)$. Finally, we obtain some conditions for $H^k_I(R)$ is planar and toroidal. The contents of this chapter has been communicated as a paper to “Discrete Mathematics” [57].
5.1 Ideal-based $k$-zero-divisor

In this section, we define and study some properties of ideal-based $k$-zero-divisor of a commutative ring with respect to an ideal.

**Definition 5.1.1.** Let $R$ be a commutative ring and $I$ be an ideal of $R$ and $k \geq 2$ a fixed integer. A non-zero non-unit element $x_1$ in $R$ is said to be an *ideal-based $k$-zero-divisor* if there exist $k - 1$ distinct non-zero non-unit elements $x_2, x_3, \ldots, x_k$ in $R$ different from $x_1$ such that $\prod_{i=1}^{k} x_i \in I$ and the product of any proper subset of $\{x_1, x_2, \ldots, x_k\}$ is not in $I$.

By $Z_I(R,k)$, we denote the set of all *ideal-based $k$-zero-divisors* of $R$. It is clear that every element of the set $\{x_2, x_3, \ldots, x_k\}$ is an ideal-based $k$-zero divisor in $R$.

**Remark 5.1.2.** Clearly an ideal-based 2-zero-divisor is in the vertex set of $\Gamma_I(R)$, but the converse is not true in general. For example, if $R = \mathbb{Z}_4$ and $I = (0)$, then 2 is in the vertex set of $\Gamma_I(R)$ but not in $Z_I(R,2)$.

**Example 5.1.3.** Let $R = \mathbb{Z}_{24}$ and $I = (8)$. Then 2 is an ideal-based 3-zero divisor, since $2 \cdot 6 \cdot 10 \in I$ and the product of any proper subset of $\{2, 6, 10\}$ is not in $I$. 
Remark 5.1.4. Let $R$ be a commutative ring and $I$ be a proper ideal of $R$. Then:

(1) If $I = (0)$, then $Z_I(R, k) = Z(R, k)$.

(2) If $I$ is prime, then $Z_I(R, k) = \emptyset$.

Proposition 5.1.5. For $k \geq 2$, $Z_I(R, k + 1) \subseteq Z_I(R, k)$.

Proof. Let $x \in Z_I(R, k+1)$. Then there exist $k$ elements $x_1, x_2, \ldots, x_k \in R - I$ such that $xx_1x_2 \cdots x_k \in I$ and $xx_1x_2 \cdots x_{i-1}x_{i+1}x_k \notin I$ and $x_1x_2 \cdots x_k \notin I$ for any $i = 1, 2, \ldots, k$. Now replace some $x_i$ by $x_sx_t$, $s \neq t, 1 \leq s, t \leq k$, then $x \in Z_I(R, k)$. \qed

Theorem 5.1.6. Let $R$ be a commutative ring and $I$ be a non-zero proper ideal of $R$. Then $I$ is a $k$-prime ideal of $R$ if and only if $Z_I(R, k) = \emptyset$.

Proof. Since $I$ is a $k$-prime ideal of $R$, for any set $\{x_1, x_2, \ldots, x_k\}$, the product of a proper subset of $\{x_1, x_2, \ldots, x_k\}$ is in $I$. Hence $Z_I(R, k) = \emptyset$.

Conversely, suppose $Z_I(R, k) = \emptyset$. If $x_1 \in R - I$, and $\prod_{i=1}^{k} x_i \in I$ for some $x_2, \ldots, x_k \in R - I$, we have there is a product of a proper subset of $\{x_1, x_2, \ldots, x_k\}$ is in $I$. Hence $I$ is a $k$-prime ideal of $R$. \qed
5.2 An ideal-based $k$-uniform hypergraph

In this section, we define and study some properties of the ideal-based $k$-zero-divisor hypergraph $\mathcal{H}_I^k(R)$.

**Definition 5.2.1.** Let $R$ be a commutative ring and $I$ be an ideal of $R$ and $k \geq 2$ a fixed integer. An *ideal-based $k$-uniform hypergraph* $\mathcal{H}_I^k(R)$ of $R$ with vertex set $Z_I(R,k)$ and for distinct elements $x_1, x_2, \ldots, x_k$ in $Z_I(R,k)$, the set $\{x_1, x_2, \ldots, x_k\}$ is an edge in $\mathcal{H}_I^k(R)$ if and only if $\prod_{i=1}^{k} x_i \in I$ and the product of elements of $(k-1)$-subset of $\{x_1, x_2 \ldots x_k\}$ is not in $I$.

**Example 5.2.2.** Let $R = \mathbb{Z}_{16}$ and $I = (8)$. Then $Z_I(R,3) = \{2, 6, 10, 14\}$ and $\mathcal{H}_I^3(R)$ is given in Figure 5.1.

![Figure 5.1: $\mathcal{H}_I^3(\mathbb{Z}_{16})$ where $I = (8)$](image_url)

**Theorem 5.2.3.** Let $R$ be a commutative ring and $I$ be a proper ideal of $R$. Then the following conditions hold.

(i) If $I = (0)$, then $\mathcal{H}_I^k(R) = \mathcal{H}_k(R)$ for any $k \geq 3$.

(ii) If $I$ is a prime ideal of $R$, then $\mathcal{H}_I^k(R)$ is empty for any $k \geq 3$.

(iii) If $I$ is a $k$-prime ideal of $R$, then $\mathcal{H}_I^k(R)$ is empty for any $k \geq 3$. 

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(iv) Let $R = R_1 \times R_2 \times \cdots \times R_n$, where each $(R_i, m_i)$ is a local ring. If $I = \prod_{i=1}^{s-1} R_i \times m_s \times \prod_{j=s+1}^{t-1} R_j \times m_t \times \prod_{l=t+1}^{n} R_l$ for $s \neq t$, then for $k \geq 3$, $\mathcal{H}_I^k(R)$ is empty.

**Proof.** (i) - (iii) follows from Remark 5.1.4 and Theorem 5.1.6.

(iv) Suppose $\mathcal{H}_I^k(R)$ is not an empty hypergraph. Then there exist $x_1, \ldots, x_k \in R - I$ such that $\prod_{i=1}^{k} x_i \in I$ and product of any $k-1$ proper subset of $\{x_1, \ldots, x_k\}$ is not in $I$, where $x_i = (x_{i_1}, \ldots, x_{i_s}, \ldots, x_{i_t}, \ldots, x_{i_n})$, for $1 \leq i \leq k$. Since $m_s$ and $m_t$ are prime, $\prod_{i=1}^{k} x_{i_s} \in m_s$, $\prod_{i=1}^{k} x_{i_t} \in m_t$ implies that $x_{j_s} \in m_s$ and $x_{l_t} \in m_t$ for some $1 \leq j, l \leq k$. It is easy to see that a product of a proper 2-subset of $\{x_1, \ldots, x_k\}$ is in $I$, a contradiction.

\[\square\]

**Theorem 5.2.4.** Let $R = F_1 \times \cdots \times F_n$ be a commutative ring, where each $F_i$ is a field. For $3 \leq k \leq n$, if $I = \prod_{m<k} (0) \times \prod_{n-m} F_i$, then $\mathcal{H}_I^k(R)$ is empty.

**Proof.** Let $\Lambda = \{j : F_j = (0) \text{ in } I\}$. Then $|\Lambda| = m < k$. Suppose $\mathcal{H}_I^k(R)$ is non-empty. Then there exist $x_1, x_2, \ldots, x_k \in R - I$, where $x_i = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$, $1 \leq i \leq k$, such that $\prod_{i=1}^{k} x_i = y \in I$ and $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_k \notin I$ for $1 \leq i \leq k$. Since $y = (y_{i_1}, y_{i_2}, \ldots, y_{i_n}) \in I$ and $I = \prod_{m<k} (0) \times \prod_{n-m} F_i$, $y_{jl} = 0$ for all $l \in \Lambda$. It is easy to see that the product of a $m$-subset of $\{x_1, x_2, \ldots, x_k\}$ is in $I$, which is a contradiction. \[\square\]
Remark 5.2.5. Let \( R = F_1 \times \cdots \times F_n \) be a commutative ring, where each \( F_i \) is a field. For \( 3 \leq k \leq n \), if \( I = \prod_{3 \leq k \leq m \leq n} (0) \times \prod_{n-m} F_i \), then \( Z_I(R, k) \neq \emptyset \) and \( \mathcal{H}^k_I(R) \) has at least one edge. Further, if \( k = n \) then \( \mathcal{H}^k_I(R) = \mathcal{H}_k(R) \).

The next theorem shows that \( \mathcal{H}^k_I(R) \) is non-linear for any \( k \geq 3 \), where \( R \) is the product of finite fields.

Theorem 5.2.6. Let \( R = R_1 \times \cdots \times R_n \), where \( R_i \) is a local ring for each \( i \), \( 1 \leq i \leq n \) and \( I = \prod_{m \geq k \geq 3} I_i \times \prod_{n-m} R_j \) be a proper ideal of \( R \) where \( I_i \subset R_i \). Then:

(i) For \( 3 \leq k < n \), \( \mathcal{H}^k_I(R) \) is non-linear.

(ii) For \( k = n \), if \( |R_i| \geq 3 \) for some \( i \), then \( \mathcal{H}^k_I(R) \) is non-linear.

Proof. Without loss of generality assume that, \( I = \prod_{i=1}^m I_i \times \prod_{j=m+1}^n R_j \).

(i) For \( 3 \leq k \leq m < n \), let \( x_1 = (1, 1, \ldots, 1, 0, 0, \ldots, 0) \), \( x_2 = (1, 1, \ldots, 1, 0, 1, 0, \ldots, 0) \), \( \ldots \), \( x_k = (0, 1, \ldots, 1, 1, 0, \ldots, 0) \), then \( e_1 = \{x_1, x_2, \ldots, x_k\} \) is an edge in \( \mathcal{H}^k_I(R) \).

For \( m+1 \leq j \leq n \), define \( \lambda_l = \begin{cases} 1 & \text{if } l = j \\ 0 & \text{otherwise} \end{cases} \). Let \( y_i = x_i + \lambda_{m+1} \). Then clearly \( e_2 = \{y_1, x_2, \ldots, x_k\} \) is an edge in \( \mathcal{H}^k_I(R) \) and \( |e_1 \cap e_2| > 1 \). Thus \( \mathcal{H}^k_I(R) \) is non-linear.

(ii) Without loss of generality, assume that \( |R_1| \geq 3 \). Now let \( x_1 = (1, 1, \ldots, 1, 0) \), \( x_2 = (1, \ldots, 1, 1, 0) \), \( \ldots \), \( x_k = (0, 1, \ldots, 1) \), \( y_1 = (a, 1, \ldots, 1) \), \( y_2 = (a, 1, \ldots, 1, 1) \), \( \ldots \), \( y_{m+1} = (a, 1, \ldots, 1, 1, 0) \), then \( e_1 = \{x_1, x_2, \ldots, x_k\} \) is an edge in \( \mathcal{H}^k_I(R) \) and \( |e_1 \cap e_2| > 1 \). Thus \( \mathcal{H}^k_I(R) \) is non-linear.
1, 0), where \( a \in R_1^* \). Then \( e_1 = \{x_1, x_2, \ldots, x_k\} \) and \( e_2 = \{y_1, x_2, \ldots, y_k\} \) \( \in E(H^k_i(R)) \) and \( |e_1 \cap e_2| > 1 \), and so \( H^k_i(R) \) is non-linear. \( \square \)

In Theorem 5.2.6 (ii), if \( |R_i| = 2 \) for all \( i \), then \( H^k_i(R) = H_k(R) \) and \( |E(H^k_i(R))| = 1 \). Hence we cannot discuss about linearity.

**Theorem 5.2.7.** Let \( R = R_1 \times \cdots \times R_n \) be a commutative ring, where \( n \geq 2 \), \( (R_i, m_i) \) is a local ring with \( m_i \neq \{0\} \) and \( I = R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_{j-1} \times I_j \times R_{j+1} \times \cdots \times R_n \) be an ideal of \( R \) such that \( \Gamma_{I_i}(R_i) \) and \( \Gamma_{I_j}(R_j) \) are non-empty. Then \( H^3_i(R) \) is non-linear.

**Proof.** Since \( \Gamma_{I_i}(R_i) \) and \( \Gamma_{I_j}(R_j) \) are non-empty, there exist \( x, y \in R_i - I_i \) and \( a, b \in R_j - I_j \) such that \( xy \in I_i \) and \( ab \in I_j \).

Let \( a_1 = (0, \ldots, 0, x, 0, \ldots, 0, 1, 0, \ldots, 0) \),

\( a_2 = (0, \ldots, 0, y, 0, \ldots, 0, a, 0, \ldots, 0) \),

\( a_3 = (0, \ldots, 0, 1, 0, \ldots, 0, b, 0, \ldots, 0) \),

\( a_4 = (0, \ldots, 0, u, 0, \ldots, 0, b, 0, \ldots, 0) \).

Then \( e_1 = \{a_1, a_2, a_3\} \) and \( e_2 = \{a_1, a_2, a_4\} \in E(H^3_i(R)) \) and \( |e_1 \cap e_2| > 1 \) and hence \( H^3_i(R) \) is non-linear. \( \square \)

### 5.3 Connectedness of \( H^3_i(R) \)

In this section, we show that \( H^3_i(R) \) is connected with diameter at most 4 provided \( x^2 \notin I \) for all ideal-based 3-zero-divisor hypergraph.
Also we find the chromatic number of $\mathcal{H}_I^h(R)$, when $R$ is a product of finite fields.

**Theorem 5.3.1.** Let $\mathcal{H}_I^3(R)$ be the ideal-based 3-zero-divisor hypergraph of a ring $R$ such that $x^2 \notin I$ for every $x \in Z_I(R,3)$. Then $\mathcal{H}_I^3(R)$ is connected and $\text{diam}(\mathcal{H}_I^3(R)) \leq 4$.

**Proof.** It is enough to show that for each $e_1 = \{x_1, x_2, x_3\}$ and $e_2 = \{y_1, y_2, y_3\}$ of $\mathcal{H}_I^3(R)$, there exist edges $e_3$ and $e_4$ which satisfy one of the following conditions:

$$e_3 \cap e_1 \neq \emptyset, \quad e_3 \cap e_2 \neq \emptyset$$

(5.1)

or

$$e_3 \cap e_1 \neq \emptyset, \quad e_4 \cap e_2 \neq \emptyset, \quad e_4 \cap e_3 \neq \emptyset$$

(5.2)

Consequently, we can always assume that $x_i \neq y_j$ and $x_i \neq -y_j$ for $i, j \in \{1, 2, 3\}$. Let $G$ be a bipartite graph with vertex set $V(G) = e_1 \cup e_2$ and $x_i y_j \in E(G)$ if and only if $x_i y_j \in I$. The rest of our proof depends on the number of edges of $G$.

**Case 1.** Suppose $|E(G)| \leq 2$. Then $G$ has two isolated vertices, one in $e_1$ and other in $e_2$. Without loss of generality, assume that $\deg(x_3) = \deg(y_3) = 0$ in $G$. If there exists an element $z \in \{x_1, x_2, y_1, y_2\}$ such that $x_3 y_3 z \in I$, then $e_3 = \{x_3, y_3, z\}$ satisfies (5.1). Suppose that this is not the case. If $x_3 y_3 \notin \{x_1, x_2, y_1, y_2\}$, then $e_3 = \{x_1, x_2, x_3 y_3\}$ and $e_4 = \{y_1, y_2, x_3 y_3\}$ satisfy (5.2). If not, without loss of generality assume that $x_3 y_3 = x_1$, then $e_3 = \{x_1, y_1, y_2\}$ satisfies (5.1).
Case 2. Suppose $|E(G)| = 3$. In this case, we have four different subcases.

Case 2a. Assume that the degree of each vertex of $G$ is one and $E(G) = \{x_1y_1, x_2y_2, x_3y_3\}$.

Consider the set $\{x_1 + y_2, x_2, y_1\}$. If $x_1 + y_2 = x_2$, then $x_2y_1y_3 = (x_1 + y_2)y_1y_3 \in I$, and $x_2y_1, x_2y_3 \notin I$ and so $e_3 = \{x_2, y_1, y_3\}$ satisfies (5.1). If $x_1 + y_2 = y_1$, then $y_1x_2x_3 = (x_1 + y_2)x_2x_3 \in I$ and $y_1x_2, y_1x_3 \notin I$, therefore $e_3 = \{y_1, x_2, x_3\}$ satisfies (5.1). Otherwise $e_3 = \{x_1 + y_2, x_2, y_1\}$ satisfies (5.1).

Case 2b. Assume that the degree of exactly one of the vertices of $G$ is two. Assume that $E(G) = \{x_1y_1, x_1y_2, x_3y_3\}$.

Consider the set $\{x_3, x_2y_1, x_1 + y_3\}$. If $x_3 = x_2y_1$, then $x_1x_3 = x_1x_2y_1 \in I$ is a contradiction. If $x_3 = x_1 + y_3$, then $x_3y_1y_2 = (x_1 + y_3)y_1y_2 \in I$ and $x_3y_1, x_3y_2 \notin I$, therefore $e_3 = \{x_3, y_1, y_2\}$ satisfies (5.1). If $x_2y_1 = x_1 + y_3$, then $x_2y_1y_2y_1 = (x_1 + y_3)y_2y_1 \in I$. In this case, if $x_2 = y_1y_2$, then $x_1x_2 \in I$ and $y_1 = y_1y_2$, then $y_1y_3 \in I$, which in both cases we have a contradiction. Hence $e_3 = \{x_2, y_1y_2, y_1\}$ is an edge which satisfies (5.1). If the above conditions does not hold, then $e_3 = \{x_3, x_2y_1, x_1 + y_3\}$ is an edge. Now consider the set $\{x_2y_1, y_2, y_3\}$. If $x_2y_1 = y_2$, then $x_2y_1y_3y_1 = y_1y_2y_3 \in I$. In this case if $x_2 = y_1y_3$, then $x_2x_3 = y_1y_3x_3 \in I$, a contradiction. If $y_1 = y_1y_3$, then $y_1y_2 \in I$, again a contradiction. Hence $e_3 = \{x_2, y_1y_3, y_1\}$ satisfies (5.1). In other case the edges $e_3 = \{x_3, x_2y_1, x_1 + y_3\}$ and $e_4 = \{x_2y_1, y_2, y_3\}$ satisfy (5.2).

Case 2c. Let the degree of two vertices of $G$ be two. Without loss of
generality, assume that $E(G) = \{x_1y_1, x_1y_2, x_2y_2\}$.

In this case $\text{deg}(x_3) = \text{deg}(y_3) = 0$, and the proof follows from Case 1.

**Case 2d.** Assume that the degree of one vertex of $G$ is three. Suppose $E(G) = \{x_1y_1, x_1y_2, x_1y_3\}$.

Suppose $x_1^2x_2 \notin I$. Consider the set $\{x_1x_2 - y_1, x_1, x_3\}$. If $x_1 = x_1x_2 - y_1$, then $y_1y_2 = (x_1x_2 - x_1)y_2 \in I$, is a contradiction. If $x_3 = x_1x_2 - y_1$, then $x_3y_2y_3 = (x_1x_2 - y_1)y_2y_3 \in I$ and hence $e_3 = \{x_3, y_2, y_3\}$ is an edge satisfies (5.1). Otherwise $e_3 = \{x_1x_2 - y_1, x_1, x_3\}$ is an edge. Similarly, consider the set $\{x_1x_2 - y_1, y_2, y_3\}$. If $y_2 = x_1x_2 - y_1$, then $x_1^2x_2 = x_1(y_1 + y_2) \in I$, a contradiction. If $y_3 = x_1x_2 - y_1$, then $x_1^2x_2 = x_1(y_1 + y_3) \in I$, a contradiction. Therefore $e_4 = \{x_1x_2 - y_1, y_2, y_3\}$ is an edge with $e_3$ and $e_4$ satisfy (5.2). Suppose $x_1^2x_2 \in I$.

Consider the set $\{x_1 - y_1, x_1, x_2\}$. If $x_2 = x_1 - y_1$, then $x_2y_2y_3 = (x_1 - y_1)y_2y_3 \in I$, and hence $e_3 = \{x_2, y_2, y_3\}$ satisfies (5.1). In other case $e_3 = \{x_1 - y_1, x_1, x_2\}$ is an edge. Similarly we consider the set $\{x_1 - y_1, y_2, y_3\}$. If $x_1 - y_1 = y_2$ or $y_3$, then $x_1^2 \in I$, a contradiction. Therefore $e_4 = \{x_1 - y_1, y_2, y_3\}$ is an edge with $e_3$ and $e_4$ satisfy (5.2).

**Case 3.** Suppose $|E(G)| = 4$. In this case, we have the following four different non-isomorphic subcases.

**Case 3a.** Assume that the degree of one vertex of $G$ is three. Assume that $E(G) = \{x_1y_1, x_1y_2, x_1y_3, x_2y_2\}$.

Consider the set $\{x_2, y_1, x_1 + y_2\}$. If $x_2 = x_1 + y_2$, then $y_2^2 \in I$, is a contradiction. If $y_1 = x_1 + y_2$, then $x_1^2 \in I$, which is a contradiction.
Therefore $e_3 = \{x_2, y_1, x_1 + y_2\}$ is an edge satisfies (5.1).

**Case 3b.** Assume that the degree of two vertices of $G$ is two. In this case, there are two different non-isomorphic cases.

For the first case, Without loss of generality, we can assume that $E(G) = \{x_1y_1, x_1y_2, x_2y_2, x_3y_3\}$.

Consider the set $\{x_2, y_1, x_1 + y_2\}$. If $x_2 = x_1 + y_2$, then $y_2^2 \in I$, which is a contradiction. If $y_1 = x_1 + y_2$, then $x_1^2 \in I$, a contradiction. Therefore $e_3 = \{x_2, y_1, x_1 + y_2\}$ is an edge satisfies (5.1).

For the second case, we assume that $E(G) = \{x_1y_1, x_1y_2, x_2y_3, x_3y_3\}$.

Consider the set $\{x_1 + y_3, y_2, x_3\}$. If $y_2 = x_1 + y_3$, then $x_2y_1y_2 = x_2(x_1 + y_3)y_2 \in I$, and $x_2y_1, x_2y_2 \notin I$ and hence $e_3 = \{x_2, y_1, y_2\}$ is an edge which satisfies (5.1). If $x_3 = x_1 + y_3$, then $x_2x_3y_1 = x_2(x_1 + y_3)y_1 \in I$, and $x_2y_1, x_3y_1 \notin I$ and hence $e_3 = \{x_2, x_3, y_1\}$ is an edge which satisfies (5.1). Otherwise $e_3 = \{x_1 + y_3, y_2, x_3\}$ is an edge that satisfies (5.1).

**Case 3c.** Let the degree of three vertices of $G$ be two. Without loss of generality assume that $E(G) = \{x_1y_1, x_1y_2, x_2y_2, x_2y_3\}$.

Consider the set $\{x_1, y_3, x_2 + y_2\}$. If $x_1 = x_2 + y_2$, then $y_2^2 \in I$, a contradiction. If $y_3 = x_2 + y_2$, then $x_2^2 \in I$, a contradiction. Therefore $e_3 = \{x_1, y_3, x_2 + y_2\}$ is an edge that satisfies (5.1).

**Case 3d.** Let the degree of four vertices of $G$ be two. Without loss of generality, suppose that $E(G) = \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$.

In this case $\deg(x_3) = \deg(y_3) = 0$, and the proof follows from Case 1.
Case 4. Suppose $|E(G)| = 4$. In this case, we have the following five different cases.

Case 4a. Assume that $E(G) = \{x_1y_1, x_1y_2, x_1y_3, x_2y_2, x_3y_2\}$.

Consider the set $\{x_1 + y_2, x_2, y_1\}$. If $x_2 = x_1 + y_2$, then $y_2^2 = y_2(x_2 - x_1) \in I$, which is a contradiction. If $y_1 = x_1 + y_2$, then $x_1^2 = x_1(y_1 - y_2) \in I$, a contradiction. Therefore $e_3 = \{x_1 + y_2, x_2, y_1\}$ is an edge that satisfies (5.1).

Case 4b. Assume that $E(G) = \{x_1y_1, x_1y_2, x_2y_1, x_2y_2, x_3y_3\}$.

Consider the set $\{x_3 + y_1, x_1, y_3\}$. If $x_1 = x_3 + y_1$, then $x_1x_2y_3 = (x_3 + y_1)x_2y_3$, and therefore the edge $e_3 = \{x_1, x_2, y_3\}$ satisfies (5.1). If $y_3 = x_3 + y_1$, then $x_1x_2y_3 = x_1x_2(x_3 + y_1) \in I$, and hence the edge $e_3 = \{x_1, x_2, y_3\}$ satisfies (5.1). Otherwise $e_3 = \{x_1 + y_2, x_2, y_1\}$ is an edge that satisfies (5.1).

Case 4c. Assume that $E(G) = \{x_1y_1, x_1y_2, x_2y_2, x_2y_3, x_3y_1\}$.

Consider the set $\{x_1 + y_2, x_2, y_1\}$. If $x_2 = x_1 + y_2$, then $y_2^2 = y_2(x_2 - x_1) \in I$, a contradiction. If $y_1 = x_1 + y_2$, then $x_1^2 = x_1(y_1 - y_2) \in I$, which is a contradiction. Therefore $e_3 = \{x_1 + y_2, x_2, y_1\}$ is an edge that satisfies (5.1).

Case 4d. Assume that $E(G) = \{x_1y_2, x_2y_1, x_2y_2, x_2y_3, x_3y_2\}$.

Consider the set $\{x_2 + y_2, x_3, y_3\}$. If $x_3 = x_2 + y_2$, then $y_2^2 = y_2(x_3 - x_2) \in I$, which is a contradiction. If $y_3 = x_2 + y_2$, then $x_2^2 = x_2(y_3 - y_2) \in I$, a contradiction. Therefore $e_3 = \{x_2 + y_2, x_3, y_3\}$ is an edge that satisfies (5.1).

Case 4e. Assume that $E(G) = \{x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2\}$.
Consider the set \( \{x_3y_3, x_2, x_1 + y_2\} \). If \( x_2 = x_3y_3 \), then \( x_1x_2 = x_3y_3x_1 \in I \), a contradiction. If \( x_3y_3 = x_1 + y_2 \), then \( x_1^2 = x_1(x_3y_3 - y_2) \in I \), a contradiction. If \( x_2 = x_1 + y_2 \), then \( y_1y_2 = y_1(x_2 - x_1) \in I \), a contradiction. Therefore \( e_3 = \{x_3y_3, x_2, x_1 + y_2\} \) is an edge. Similarly consider the set \( \{x_3y_3, y_1, y_2\} \). If \( y_1 = x_3y_3 \), the \( x_2x_3y_3 = x_2y_1 \in I \), and hence the edge \( e_3 = \{x_2, x_3, y_3\} \) satisfies (5.1). If \( y_2 = x_3y_3 \), the \( x_2x_3y_3 = x_2y_2 \in I \), and hence the edge \( e_3 = \{x_2, x_3, y_3\} \) satisfies (5.1). Otherwise \( e_4 = \{x_3y_3, y_1, y_2\} \) is an edge with \( e_3 \) and \( e_4 \) satisfy (5.2).

**Case 5.** Suppose \( |E(G)| = 6 \). We have the following subcases.

**Case 5a.** Without loss of generality, assume that \( E(G) = \{x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3\} \).

Consider the set \( \{x_1 + y_1, x_2, x_3\} \). If \( x_2 = x_1 + y_1 \), then \( y_1^2 = y_1(x_2 - x_1) \in I \), a contradiction. If \( x_3 = x_1 + y_1 \), then \( x_3y_2y_3 = (x_1 + y_1)y_2y_3 \in I \), and so \( e_3 = \{x_3, y_2, y_3\} \) is an edge which satisfies (5.1). In other case \( e_3 = \{x_1 + y_1, x_2, x_3\} \) is an edge. Similarly, consider the set \( \{x_1 + y_1, y_2, y_3\} \). If \( x_1 + y_1 = y_2 \) or \( y_3 \), then \( x_1^2 \in I \), a contradiction. Therefore the edges \( e_3 = \{x_1 + y_1, x_2, x_3\} \) and \( e_4 = \{x_1 + y_1, y_2, y_3\} \) satisfy (5.2).

**Case 5b.** Without loss of generality, assume that \( E(G) = \{x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_3y_1\} \).

Consider the set \( \{x_1 + y_1, x_2, x_3\} \). If \( x_1 + y_1 = x_2 \), then \( y_1y_2 = (x_2 - x_1)y_2 \in I \), a contradiction. If \( x_1 + y_1 = x_3 \), then \( y_1^2 = y_1(x_3 - x_1) \in I \), a contradiction. Therefore \( e_3 = \{x_1 + y_1, x_2, x_3\} \) is an edge. Similarly consider the set \( \{x_1 + y_1, y_2, y_3\} \). If \( x_1 + y_1 = y_2 \), then \( x_1^2 = (y_2 - y_1)x_1 \in I \), a contradiction.
I, a contradiction. If \( x_1 + y_1 = y_3 \), then \( x_1^2 = x_1(y_3 - y_1) \in I \), which is a contradiction. Therefore the edges \( e_3 = \{x_1 + y_1, x_2, x_3\} \) and \( e_4 = \{x_1 + y_1, y_2, y_3\} \) satisfy (5.2).

**Case 5c.** Without loss of generality, assume that \( E(G) = \{x_1y_1, x_1y_2, x_2y_1, x_2y_2, x_2y_3, x_3y_3\} \).

Consider the set \( \{x_2 + y_3, x_3, y_1\} \). If \( x_2 + y_3 = x_3 \), then \( y_3^2 = (x_3 - x_2)y_3 \in I \), which is a contradiction. If \( x_2 + y_3 = y_1 \), then \( x_2^2 = x_2(y_1 - y_3) \in I \), a contradiction. Therefore \( e_3 = \{x_1 + y_1, x_2, x_3\} \) is an edge which satisfies (5.1).

**Case 5d.** Without loss of generality, assume that \( E(G) = \{x_1y_1, x_1y_2, x_2y_2, x_2y_3, x_3y_1, x_3y_3\} \).

Consider the set \( \{x_3 + y_3, x_1, y_3\} \). If \( x_3 + y_1 = x_1 \), then \( y_1^2 = (x_1 - x_3)y_1 \in I \), which is a contradiction. If \( x_3 + y_1 = y_3 \), then \( x_3^2 = x_3(y_3 - y_1) \in I \), a contradiction. Therefore the edge \( e_3 = \{x_3 + y_1, x_1, y_3\} \) satisfies (5.1).

**Case 6.** Suppose that \( 7 \leq |E(G)| \leq 9 \). In this case, there always exist two vertices with degree three, one from \( e_1 \) and another one from \( e_2 \). Let \( \text{deg}(x_1) = \text{deg}(y_1) = 3 \). Consider the set \( \{x_1 + y_1, x_2, x_3\} \) and \( \{x_1 + y_1, y_2, y_3\} \). If \( x_1 + y_1 = x_2 \) or \( x_3 \), then \( y_1^2 \in I \) and if \( x_1 + y_1 = y_2 \) or \( y_3 \), then \( x_1^2 \in I \), which is a contradiction. Therefore \( e_3 = \{x_1 + y_1, x_2, x_3\} \) and \( e_4 = \{x_1 + y_1, y_2, y_3\} \) are two edges that satisfy (5.2). \( \square \)

**Theorem 5.3.2.** Let \( R = F_1 \times \cdots \times F_n \) be a finite commutative ring where each \( F_i \) is a field and \( I = \prod_{m \geq k \geq 3} (0) \times \prod_{n-m} F_i \) be an ideal of \( R \).
Then

(i) If \( m = k \), then \( \chi(\mathcal{H}_I^k(R)) = 2 \).

(ii) If \( m = k + t \), then \( \chi(\mathcal{H}_I^k(R)) \leq 2 + t \) for all \( t \geq 0 \).

**Proof.** (i) Let \( m = k \). Let \( \Lambda = \{ i \mid I_i = (0) \text{ in } I \} \). Then

\[
Z_I(R, k) = \{ (x_1, x_2, \ldots, x_n) : \text{exactly one of the } x_i \text{ is zero for } i \in \Lambda \}
\]

Consider \( c : Z_I(R, k) \to \{ 1, 2 \} \) given by,

\[
c(x) = \begin{cases} 
1 & \text{if } x_i = 0 \text{ for } i = \min \Lambda \\
2 & \text{otherwise}
\end{cases}
\]

It is easy to see that \( c \) is a 2-coloring of \( \mathcal{H}_I^k(R) \), and since \( \mathcal{H}_I^k(R) \) has at least one edge, \( \chi(\mathcal{H}_I^k(R)) = 2 \).

(ii) Let \( m = k + t \), \( t \geq 0 \) a fixed integer. The proof is by induction on \( t \). From (i), if \( t = 0 \), then the result is true. Now, assume that \( t \geq 1 \) and the result is true for \( k + t \). Let \( c : Z_I(R, k) \to \{ 1, 2, \ldots, t + 2 \} \) be a \( t + 2 \)-coloring of \( H_I^k(R) \) where \( I = \prod_{m=k+t}^{m=k+t+1} (0) \times \prod_{n-m} R_i \) and let \( \Omega = \{ j \mid I_j = (0) \text{ in } I \} \).

Now let \( I = \prod_{m=k+t+1}^{m=k+t+1} (0) \times \prod_{n-m} R_i \) and \( \Omega' = \Omega \cup \{ i+1 \} \), where \( i \in \Omega \) and \( i + 1 \notin \Omega \). Consider the function \( c : Z_I(R, k) \to \{ 1, 2, \ldots, t + 2, t + 3 \} \) given by,

\[
c'(x) = \begin{cases} 
c(x) & \text{if } x_{i+1} = 0 \\
t + 3 & \text{otherwise}
\end{cases}
\]
From this, it is easy to show that $c'$ is a $t + 3$-coloring of $\mathcal{H}^k_i(R)$ where $I = \prod_{m=k+t+1}^{m} (0) \times \prod_{n-m} R_i$. \hfill \Box

5.4 Girth of $\mathcal{H}^k_i(R)$

In this section, we only focus on the girth of $\mathcal{H}^3_i(R)$. Also we find the girth of $\mathcal{H}^k_i(R), k \geq 3$, where $R$ is the product of finite fields.

**Theorem 5.4.1.** Let $R = R_1 \times \cdots \times R_n$ be a commutative ring, where $n \geq 2$, each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and $I = R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_{j-1} \times I_j \times R_{j+1} \times \cdots \times R_n$ be an ideal of $R$ such that $\Gamma_{I_i}(R_i)$ and $\Gamma_{I_j}(R_j)$ are non-empty. Then $\text{gr}(\mathcal{H}^3_i(R)) = 3$.

**Proof.** Since $\Gamma_{I_i}(R_i)$ and $\Gamma_{I_j}(R_j)$ are non-empty, there exist $x, y \in R_i - I_i$ and $a, b \in R_j - I_j$ such that $xy \in I_i$ and $ab \in I_j$.

Let $a_1 = (0, \ldots, 0, x, 0, \ldots, 0, 1, 0, \ldots, 0)$,

$a_2 = (0, \ldots, 0, y, 0, \ldots, 0, a, 0, \ldots, 0)$,

$a_3 = (0, \ldots, 0, 1, 0, \ldots, 0, b, 0, \ldots, 0)$,

$a_4 = (0, \ldots, 0, u, 0, \ldots, 0, b, 0, \ldots, 0)$,

$a_5 = (0, \ldots, 0, x, 0, \ldots, 0, v, 0, \ldots, 0)$, where $u \in R_i^\times$ and $v \in R_j^\times$.

Then $e_1 = \{a_1, a_2, a_3\}$, $e_2 = \{a_1, a_2, a_4\}$ and $e_3 = \{a_2, a_4, a_5\} \in$
Then Theorem 5.4.3.

Proof. Let \( R = R_1 \times \cdots \times R_n \) be a commutative ring, where \( n \geq 2 \), each \((R_i, \mathfrak{m}_i)\) is a local ring with \( \mathfrak{m}_i \neq \{0\} \). Let \( I = R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_n \) be an ideal of \( R \) such that \( \mathfrak{m}_i \) is principal and \( \mathfrak{m}_i^t = (0) \) for \( l \geq 3 \) and \( I_i = \mathfrak{m}_i^t, 3 \leq t \leq l \). Then \( \text{gr}(\mathcal{H}_I^3(R)) = 3 \).

\textbf{Proof.} Since \( R_j \) is local with \( \mathfrak{m}_j \neq \{0\}, |R_j| \geq 4 \) for all \( j \). Since \( I_i = \mathfrak{m}_i^t, 3 \leq t \leq l \), there exists \( x \in \mathfrak{m}_i \) such that \( x^t \in I_i \) and \( x^{t-1} \notin I_i \).

Suppose \( t > 3 \). Let \( a_1 = (0, \ldots, 0, x^\alpha, 0, \ldots, 0), a_2 = (a, \ldots, 0, x^\beta, 0, \ldots, 0), a_3 = (b, \ldots, 0, x^\gamma, 0, \ldots, 0), a_4 = (c, \ldots, 0, x^\alpha, 0, \ldots, 0), a_5 = (c, \ldots, 0, x^\beta, 0, \ldots, 0) \) such that \( x^\alpha x^\beta x^\gamma \in I_i \), and \( x^\alpha x^\beta, x^\alpha x^\gamma, x^\beta x^\gamma \notin I_i \), where \( a, b, c \in R_i^* \). Let \( e_1 = \{a_2, a_3, a_4\}, e_2 = \{a_1, a_2, a_3\}, e_3 = \{a_1, a_3, a_5\} \). Then \( e_1 - e_2 - e_3 - e_1 \) is a cycle of length 3 and hence \( \text{gr}(\mathcal{H}_I^3(R)) = 3 \).

Suppose \( t = 3 \). Then there exists \( x \in \mathfrak{m}_i \) such that \( x^3 \in I_i \) and \( x^2 \notin I_i \). Let \( a_1 = (0, \ldots, 0, x, 0, \ldots, 0), a_2 = (a, \ldots, 0, x, 0, \ldots, 0), a_3 = (b, \ldots, 0, x, 0, \ldots, 0), a_4 = (c, \ldots, 0, x, \ldots, 0) \) \( \in \mathbb{Z}_I(R, k) \), where \( a, b, c \in R_i^* \). Let \( e_1 = \{a_1, a_2, a_4\}, e_2 = \{a_1, a_3, a_4\}, e_3 = \{a_1, a_2, a_3\} \). Then \( e_1 - e_2 - e_3 - e_1 \) is a cycle of length 3 and hence \( \text{gr}(\mathcal{H}_I^3(R)) = 3 \).

\textbf{Theorem 5.4.3.} Let \( R = F_1 \times F_2 \times \cdots \times F_n \) be a finite commutative ring, where each \( F_i \) is a field and \( I = \prod_{m \geq k \geq 3} (0) \times \prod_{n \geq m} F_i \). Then
\[ gr(H^k_I(R)) = 3. \]

**Proof.** Without loss of generality, assume that

\[ I = \left(0\right) \times \cdots \times \left(0\right) \times F_{m+1} \times \cdots \times F_n. \]

Let \( x_1 = (1, 1, \ldots, 1, 0, \ldots, 0) \), \( x_2 = (1, 1, \ldots, 1, 0, 1, \ldots, 0) \),
\[ \ldots, x_k = (0, 1, \ldots, 1, 1, 0, \ldots, 0) \in Z_I(R, k). \]

For \( m + 1 \leq j \leq n \), define \( \lambda_j \) such that 1 on its \( j^{th} \) place and 0 otherwise. Let \( y_i = x_i + \lambda_{m+1} \), \( 1 \leq i \leq k \). Then \( e_1 = \{y_1, x_2, \ldots, x_k\} \), \( e_2 = \{y_1, y_2, \ldots, y_k\} \), \( e_3 = \{x_1, x_2, y_3, x_4, \ldots, x_k\} \in E(H^k_I(R)) \) and \( e_1 - e_2 - e_3 - e_1 \) is a cycle in \( H^k_I(R) \) and so \( gr(H^k_I(R)) = 3. \) \( \square \)

**Corollary 5.4.4.** Let \( R = R_1 \times R_2 \times \cdots \times R_n \) be a commutative ring, where \( n \geq 2 \), each \( (R_i, m_i) \) is a local ring with \( m_i \neq \{0\} \) and \( I = \prod_{m \geq k \geq 2} I_j \times \prod_{n-m} R_i \) be a proper ideal of \( R \) and \( I_j \) is an ideal of \( R_j \). Then \( gr(H^k_I(R)) = 3 \) for \( 3 \leq k \leq n. \)

**Proof.** For \( 3 \leq k < n \), the proof follows from Theorem 5.4.3. For \( m = k = n \), let \( x_1 = (1, 1, \ldots, 1, 0) \), \( x_2 = (1, 1, \ldots, 1, 0, 1) \) \( \ldots x_k = (0, 1, \ldots, 1, 1), y_1 = (a, 1, \ldots, 1, 0), y_2 = (1, b, \ldots, 1, 0), \) where \( a \in R_1^* \) and \( b \in R_2^* \). Then \( e_1 = \{x_1, x_2, \ldots, x_k\} \), \( e_2 = \{y_1, x_2, \ldots, x_k\} \), \( e_3 = \{x_1, y_2, x_3, \ldots, x_k\} \in E(H^k_I(R)) \) and \( e_1 - e_2 - e_3 - e_1 \) is a cycle of length 3 and so \( gr(H^k_I(R)) = 3. \) \( \square \)
Theorem 5.4.5. Let $R$ be a commutative ring and $I$ be a non-zero proper ideal of $R$ such that $x^2 \notin I$ for all $x \in Z_1(R,k)$. If $\mathcal{H}_k(R/I)$ is non-empty, then $\mathcal{H}^k_{I}(R)$ is non-empty and $gr(\mathcal{H}^k_{I}(R)) = 3$ for $3 \leq k \leq n$.

Proof. Since $\mathcal{H}_k(R/I)$ is non-empty, there exist $x_1+I, x_2+I, \ldots, x_k+I \in V(\mathcal{H}_k(R/I))$ such that $\{(x_1+I), (x_2+I), \ldots, (x_k+I)\}$ is an edge in $\mathcal{H}_k(R/I)$. It is clear that $\{x_1, x_2, \ldots, x_k\} \in E(\mathcal{H}^k_{I}(R))$. Now consider the edges $e_1 = \{x_1, x_2+a, x_3, \ldots, x_k\}$, $e_2 = \{x_1+a, x_2+a, \ldots, x_k+a\}$, $e_3 = \{x_1, x_2, x_3+a, x_4, \ldots, x_k\}$, where $a \in I^*$. Then $e_1 - e_2 - e_3 - e_1$ is a cycle of $\mathcal{H}^k_{I}(R)$ and hence $gr(\mathcal{H}^k_{I}(R)) = 3$. \hfill \Box

5.5 Relation between $\mathcal{H}^k_{I}(R)$ and $\mathcal{H}_k(R/I)$

In this section, we investigate the relation between $\mathcal{H}^k_{I}(R)$ and $\mathcal{H}_k(R/I)$. The following theorem describes the adjacency relations of $\mathcal{H}^k_{I}(R)$ and $\mathcal{H}_k(R/I)$.

Theorem 5.5.1. Let $I$ be a non-zero proper ideal of a commutative ring $R$ and $x_1, x_2, \ldots, x_k \in R - I$.

(i) If $\{x_1+I, x_2+I, \ldots, x_k+I\}$ is an edge in $\mathcal{H}_k(R/I)$, then $\{x_1, x_2, \ldots, x_k\}$ is an edge in $\mathcal{H}^k_{I}(R)$.

(ii) If $x_1, x_2, \ldots, x_k$ is an edge in $\mathcal{H}^k_{I}(R)$ and $x_i + I \neq x_j + I$ for all $1 \leq i, j \leq k$, then $\{x_1 + I, x_2 + I, \ldots, x_k + I\}$ is an edge in $\mathcal{H}_k(R/I)$. 134
Proof. (i) Since \( \{x_1 + I, x_2 + I, \ldots, x_k + I\} \) is an edge in \( \mathcal{H}_k(R/I) \), \( (x_1 + I)(x_2 + I) \cdots (x_k + I) = 0 + I \) and \( (x_1 + I) \cdots (x_{i-1} + I)(x_{i+1} + I) \cdots (x_k + I) \neq 0 + I \). From this \( x_1 x_2 \cdots x_k \in I \) and \( x_1 \cdots x_{i-1} x_{i+1} \cdots x_k \notin I \), which completes the proof.

(ii) Since \( x_i + I \neq x_j + I \) and \( x_1, x_2, \ldots, x_k \) is an edge in \( \mathcal{H}_k^I(R) \), then \( (x_1 + I)(x_2 + I) \cdots (x_k + I) = 0 + I \) and \( (x_1 + I) \cdots (x_{i-1} + I)(x_{i+1} + I) \cdots (x_k + I) \neq 0 + I \), for \( 1 \leq i \leq k \), which completes the proof. □

Theorem 5.5.2. Let \( R \) be a commutative ring and \( I \) be a non-zero proper ideal of \( R \). For \( k \geq 3 \), if \( \mathcal{H}_k^R(R) \) is complete and \( \mathcal{H}_k(R/I) \) is non-empty, then \( \mathcal{H}_k(R/I) \) is complete.

Proof. The proof is clear. □

Remark 5.5.3. The converse of Theorem 5.5.2 is not true. For example, \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( I = (0) \times (0) \times (0) \times \mathbb{Z}_2 \). Here \( \mathcal{H}_3^R(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( \mathcal{H}_3^R(R) \) has only one edge, which is complete. But in \( \mathcal{H}_3^R(R) \), \( \{(1, 1, 0, 0), (1, 0, 1, 0), (1, 1, 0, 1)\} \) is not an edge.

The following example shows the interplay between \( \mathcal{H}_3^R(R) \) and \( \mathcal{H}_3(R) \).

Example 5.5.4. (i) Let \( R = \mathbb{Z}_{16} \) and \( I = (8) \). Then \( Z_I(R, 3) = \{2, 6, 10, 14\} \) and \( Z(R, 3) = \{2, 4, 6, 10, 12, 14\} \). Clearly \( Z_I(R, 3) \subset Z(R, 3) \). But \( \mathcal{H}_3^I(R) \) is not a subgraph of \( \mathcal{H}_3(R) \), since \( \{2, 6, 10\} \) is an edge in \( \mathcal{H}_3^I(R) \) but not in \( \mathcal{H}_3(R) \).
(ii) Let $R = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $I = (0) \times (0) \times \mathbb{Z}_2$. In this case, $Z_I(R, 3) = \{(2, 1, 0), (2, 1, 1), (1, 0, 1), (3, 0, 1)\} \subseteq Z(R, 3) = \{(1, 1, 0), (2, 1, 0), (3, 1, 0), (2, 1, 1), (1, 0, 1), (3, 0, 1), (0, 1, 1), (2, 0, 1)\}$ and $H^3_I(R)$ is a subgraph of $H_3(R)$. Also $R = \frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $I = (0) \times (0) \times \mathbb{Z}_2$. Also $H^3_J(R)$ is not a subgraph of $H_3(R)$ for $J = (2) \times (0) \times (0)$, since $\{(2, 1, 1), (1, 0, 1), (1, 1, 0)\}$ is an edge in $H^3_J(R)$ but not in $H_3(R)$.

### 5.6 Planarity of $H^3_I(R)$

In this section, we find some necessary conditions for a finite ring $R$ and a non-zero ideal $I$ of $R$ to have $H^3_I(R)$ is planar.

**Example 5.6.1.** Let $R = \mathbb{Z}_{16}$ and $I = (8)$. Then $Z_I(R, 3) = \{2, 6, 10, 14\}$, $E(H_{(8)}(\mathbb{Z}_{16})) = \{e_1, e_2, e_3, e_4\}$, where $e_1 = \{2, 6, 10\}$, $e_2 = \{2, 6, 14\}$, $e_3 = \{2, 10, 14\}$, $e_4 = \{6, 10, 14\}$. The incidence graph $I(H^3_{(8)}(\mathbb{Z}_{16}))$ is shown in the following figure.

![Figure 5.2: $I(H^3_{(8)}(\mathbb{Z}_{16}))$](image-url)
Remark 5.6.2. It is obvious that $\gamma(\mathcal{H}_k(R)) \leq \gamma(\mathcal{H}_k^1(R))$ for any $k \geq 3$ and $I \neq (0)$.

Theorem 5.6.3. Let $R$ be a commutative ring and $I$ be a non-zero ideal of $R$. If $\mathcal{H}_3(R)$ has exactly one edge and $|I| = 2$, then $\mathcal{H}_I^3(R)$ is planar.

Proof. Let $e' = \{x_1 + I, x_2 + I, x_3 + I\}$ be an edge in $\mathcal{H}_3(R)$ and $a \in I^*$. Then $Z_I(R, 3) = \{x_1, x_2, x_3, x_1 + a, x_2 + a, x_3 + a\}$ and the edges in $\mathcal{H}_I^3(R)$ are $e_1 = \{x_1, x_2, x_3\}$, $e_2 = \{x_1 + a, x_2, x_3\}$, $e_3 = \{x_1, x_2 + a, x_3\}$, $e_4 = \{x_1, x_2, x_3 + a\}$, $e_5 = \{x_1 + a, x_2 + a, x_3\}$, $e_6 = \{x_1 + a, x_2, x_3 + a\}$, $e_7 = \{x_7, x_2 + a, x_3 + a\}$, $e_8 = \{x_1 + a, x_2 + a, x_3 + a\}$. Then by Figure 5.3, $\mathcal{H}_I^3(R)$ is planar.

\[\text{Figure. 5.3: Planar embedding of } \mathcal{H}_I^3(R)\]

Remark 5.6.4. The converse of Theorem 5.6.3 need not be true. For example, let $R = \mathbb{Z}_{16}$ and $I = (8)$. Here $|I| = 2$ and $\mathcal{H}_I^3(R)$ is planar, but $V(\mathcal{H}_3(R)) = \emptyset$. 

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Remark 5.6.5. Let $R$ be a commutative ring and $I$ be a non-zero ideal. If $\mathcal{H}(R)_I$ has more than one edge, $|I| = 2$ and there is no vertex common to each edge, then $\mathcal{H}_I(R)$ is planar.

The next theorem proves a result for the planarity of $\mathcal{H}_I(R), k \geq 3$.

Theorem 5.6.6. Let $R = F_1 \times \cdots \times F_n$ be a finite commutative ring, where each $F_i$ is a field, $n \geq 3$ and $I$ be a non-zero ideal of $R$. Then,

(i) For $n \geq 5$, $3 \leq k < n$, $\mathcal{H}_I^k(R)$ is non-planar.

(ii) For $n = 4$, $\mathcal{H}_I^3(R)$ is planar if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $I = \prod_{m=k=3}^{m} (0) \times \mathbb{Z}_2$.

Proof. (i) Without loss of generality, assume that $I = \prod_{i=1}^{n} (0) \times \prod_{j=m+1}^{n} R_j$.

Suppose $k \geq 4$. Let $x_1 = (1, \ldots, 1, 0, \ldots, 0)^t$, $x_2 = (1, \ldots, 1, 0, \ldots, 0)^t$, \ldots, $x_k = (0, 1, \ldots, 1, 0, \ldots, 0)^t$.

For $m + 1 \leq j \leq n$, define $\lambda_j = (0, \ldots, 0, 1, 0, \ldots, 0)^t$ and $y_i = x_i + \lambda_{m+1}, 1 \leq i \leq k$. Then $e_1 = \{x_1, \ldots, x_k\}$, $e_2 = \{x_1, \ldots, x_{k-1}, y_k\}$, $e_3 = \{x_1, \ldots, x_{k-2}, y_{k-1}, x_k\}$, $e_4 = \{x_1, \ldots, x_{k-3}, y_{k-2}, x_{k-1}, x_k\} \in E(\mathcal{H}_I^k(R))$. It follows that $\mathcal{I}(\mathcal{H}_I^k(R))$ contains a subdivision of $K_{3,3}$. Hence $\mathcal{H}_I^k(R)$ is non-planar.

For $k = 3$, consider $\Omega = \{x_1, \ldots, x_6\} \subset Z_I(R, 3)$, and $S = \{e_1, \ldots, e_6\} \subset E(\mathcal{H}_I^3(R))$, where $x_1 = (1, 1, 0, 0, \ldots, 0)$, $x_2 = (1, 0, 1, 0, \ldots, 0)$, $x_3 = (0, 1, 1, 0, \ldots, 0)$, $x_4 = (1, 1, 0, 1, 0, \ldots, 0)$, $x_5 = (0, 1, 1, 1, 0, \ldots, 0)$,
$x_6 = (0, 1, 1, 1, 0, \ldots, 0)$, $e_1 = \{x_1, x_2, x_3\}$, $e_2 = \{x_1, x_2, x_5\}$, $e_3 = \{x_1, x_2, x_6\}$, $e_4 = \{x_2, x_4, x_6\}$, $e_5 = \{x_2, x_3, x_4\}$, $e_6 = \{x_2, x_4, x_5\} \in E(H^3_I(R))$. Then $\mathcal{I}(H^3_I(R))$ contains a subdivision of $K_{3,3}$. Hence $H^3_I(R)$ is non-planar.

(ii) Suppose $n = 4$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $I = \prod_{m=k=3} (0) \times \mathbb{Z}_2$, then by Theorem 5.6.3, $H^3_I(R)$ is planar.

Conversely, assume that $H^3_I(R)$ is planar. Suppose $|I| \geq 3$. Then $|F_i| \geq 3$ for some $i$. Without loss of generality, assume that $|F_4| \geq 3$ and $I = (0) \times (0) \times (0) \times F_4$. Consider $\Omega' = \{x_1, \ldots, x_6\} \subset Z_I(R, 3)$, and $S' = \{e_1, \ldots, e_6\} \subset E(H^3_I(R))$, where $x_1 = (1, 1, 0, 0)$, $x_2 = (1, 0, 1, 0)$, $x_3 = (0, 1, 1, 0)$, $x_4 = (1, 0, 1, 1)$, $x_5 = (0, 1, 1, 1)$, $x_6 = (0, 1, 1, a)$, $a \neq 0 \in F_4$ and $e_1 = \{x_1, x_2, x_3\}$, $e_2 = \{x_1, x_2, x_5\}$, $e_3 = \{x_1, x_2, x_6\}$, $e_4 = \{x_1, x_3, x_4\}$, $e_5 = \{x_1, x_4, x_5\}$, $e_6 = \{x_1, x_4, x_6\}$. Then $\mathcal{I}(H^3_I(R))$ contains a subdivision of $K_{3,3}$ and hence $H^3_I(R)$ is non-planar, a contradiction.

Suppose $|F_1| \geq 3$. Let $x_1 = (1, 1, 0, 0)$, $x_2 = (a, 1, 0, 0)$, $x_3 = (1, 1, 0, 1)$, $x_4 = (0, 1, 1, 0)$, $x_5 = (0, 1, 1, 1)$, $x_6 = (1, 0, 1, 0)$, $x_7 = (1, 0, 1, 1) \in Z_I(R, 3)$, where $a \neq 0 \in F_1$. Then $e_1 = \{x_1, x_5, x_6\}$, $e_2 = \{x_2, x_5, x_6\}$, $e_3 = \{x_3, x_5, x_6\}$, $e_4 = \{x_1, x_5, x_7\}$, $e_5 = \{x_2, x_5, x_7\}$, $e_6 = \{x_3, x_5, x_7\}$, $e_7 = \{x_1, x_4, x_7\} \in E(H^3_I(R))$ and $\mathcal{I}(H^3_I(R))$ contains a subdivision of $K_{3,3}$. Hence $H^3_I(R)$ is non-planar. Thus $|F_i| = 2$ for all $i$. \qed

The next theorem proves the equivalent condition for which $\gamma(H^3_I(R)) = 1$. 

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**Theorem 5.6.7.** Let $R$ be a commutative ring and $I$ be a non-zero ideal such that $H_3(R/I)$ is connected. Then $\gamma(H_1^3(R)) = 1$ if and only if $H_3(R/I)$ has exactly two edges and $|I| = 2$.

**Proof.** Suppose $H_3(R/I)$ has exactly two edges with one common vertex and $|I| = 2$. Without loss of generality, assume that $e'_1 = \{x_1 + I, x_2 + I, x_3 + I\}, e'_2 = \{x_1 + I, x_4 + I, x_5 + I\} \in E(H_3(R/I))$. Consider $\Omega = \{x_1, x_2, x_3, x_4, x_5, x_1 + a, x_2 + a, x_3 + a, x_4 + a, x_5 + a\} \subset V(H_3^3(R/I))$ and $E_1 = \{e_1, \ldots, e_7\} \subset E(I(H_3^3(R/I)))$, where $e_1 = \{x_1, x_2, x_3\}, e_2 = \{x_1 + a, x_2, x_3\}, e_3 = \{x_1, x_2 + a, x_3\}, e_4 = \{x_1 + a, x_4, x_5\}, e_5 = \{x_1, x_4, x_5 + a\}, e_6 = \{x_1, x_2 + a, x_3 + a\}, e_7 = \{x_1 + a, x_2, x_3 + a\}$. Then $I(H_3^3(R/I))$ contains a subdivision of $K_{3,3}$, hence $\gamma(I(H_3^3(R/I))) \geq 1$. By Figure. 5.4, $\gamma(H_1^3(R)) = 1$.

![Figure. 5.4: Embedding of $H_1^3(R)$ in $S_1$](image)

Suppose $H_3(R/I)$ has exactly two edges with two common vertices. Without loss of generality, assume that $e'_1 = \{x_1 + I, x_2 + I, x_3 + I\}, e'_2 = \{x_2 + I, x_3 + I, x_4 + I\} \in E(H_3(R/I))$. Let $\Omega' = \{x_1, x_2, x_3, x_4, x_1 + a, x_2 + a, x_3 + a\} \subset V(H_1^3(R))$ and $E_1 = \{e_1, \ldots, e_7\} \subset E(I(H_1^3(R/I)))$,
where \( e_1 = \{x_1, x_2, x_3\}, e_2 = \{x_1 + a, x_2, x_3\}, e_3 = \{x_1, x_2 + a, x_3\}, e_4 = \{x_1, x_2, x_3 + a\}, e_5 = \{x_2 + a, x_3, x_4\}, e_6 = \{x_2 + a, x_3 + a, x_4\}, e_7 = \{x_1 + a, x_2 + a, x_3 + a\}. \) Then \( \mathcal{I}(\mathcal{H}^3_7(R)) \) contains a subdivision of \( K_{3,3} \), hence \( \gamma(\mathcal{I}(\mathcal{H}^3_7(R))) \geq 1 \). By Figure. 5.5, \( \gamma(\mathcal{H}^3_7(R)) = 1 \).

Conversely, assume that \( \gamma(\mathcal{H}^3_7(R)) = 1 \). Suppose \( |I| \geq 3 \). If \( \mathcal{H}_3(R^I) \) has at least one edge, then \( |V(\mathcal{I}(\mathcal{H}^3_7(R)))| \geq 36 \) and \( |E(\mathcal{I}(\mathcal{H}^3_7(R)))| \geq 81 \). Then by Lemma 2.4.4, \( \gamma(\mathcal{I}(\mathcal{H}^3_7(R))) \geq 4 \), which is a contradiction. Hence \( |I| = 2 \).

![Figure. 5.5: Embedding of \( \mathcal{I}(\mathcal{H}^3_7(R)) \) in \( S_1 \)](image)

Suppose \( |E(\mathcal{H}_3(R^I))| \geq 3 \). Consider \( e'_1, e'_2, e'_3 \in E(\mathcal{H}_3(R^I)) \) such that \( x_i + I \in e'_1 \cap e'_2 \cap e'_3 \). Let \( x_1 + I \in e'_1 \cap e'_2 \cap e'_3, S = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_1 + a, x_2 + a, x_3 + a, x_4 + a, x_5 + a, x_6 + a, x_7 + a\} \subseteq V(\mathcal{I}(\mathcal{H}^3_7(R))) \) and \( E = \{e_1, \ldots, e_{12}\} \subseteq E(\mathcal{I}(\mathcal{H}^3_7(R))) \), where \( e_1 = \{x_1, x_2, x_3\}, e_2 = \{x_1 + a, x_2, x_3\}, e_3 = \{x_1, x_2 + a, x_3\}, e_4 = \{x_1 + a, x_4, x_5 + a\}, e_5 = \{x_1, x_4, x_5 + a\}, e_6 = \{x_1 + a, x_2 + a, x_3 + a\}, e_7 = \{x_1, x_2, x_3 + a\}, \).
contains a subdivision of 2 vertex. Therefore contains a subdivision of 2, a contradiction.

Suppose \( x_i + I \in e'_1 \cap e'_2, x_j + I \in e'_2 \cap e'_3 \) for some \( i, j \) and \( i \neq j \), where \( e'_i \in E(\mathcal{H}_3^3(R)) \). Consider \( S' = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_1 + a, x_2 + a, x_3 + a, x_4 + a, x_5 + a, x_6 + a, x_7 + a \} \subset V(H_3^3(R)) \) and \( E' = \{ e_1, \ldots, e_{12} \} \subset E(H_3^3(R)) \), where \( e_1 = \{ x_1, x_2, x_3 \}, e_2 = \{ x_1 + a, x_2 + a, x_3 \}, e_3 = \{ x_1, x_2 + a, x_3 \}, e_4 = \{ x_1, x_2, x_3 + a \}, e_5 = \{ x_1 + a, x_2 + a, x_3 + a \}, e_6 = \{ x_3 + a, x_4, x_5 \}, e_7 = \{ x_3, x_4, x_5 \}, e_8 = \{ x_5, x_6, x_7 \}, e_9 = \{ x_5 + a, x_6, x_7 \}, e_{10} = \{ x_5, x_6 + a, x_7 \}, e_{11} = \{ x_3, x_4 + a, x_5 \}, e_{12} = \{ x_3 + a, x_4 + a, x_5 + a \}, e_{13} = \{ x_5, x_6, x_7 + a \}, e_{14} = \{ x_5 + a, x_6 + a, x_7 + a \} \), and \( a \neq 0 \in I \). Then the incidence graph \( \mathcal{I}(\mathcal{H}_3^3(R)) \) contains a subdivision of \( 2K_{3,3} \). Hence \( \gamma(\mathcal{H}_3^3(R)) \geq 2 \), a contradiction.

Suppose \( x_i + I, x_j + I \in e'_1 \cap e'_2 \) and \( x_i + I \in e'_2 \cap e'_3 \), where \( e'_1, e'_2, e'_3 \in E(\mathcal{H}_3^3(R)) \). Let \( e'_1 = \{ x_1 + I, x_2 + I, x_3 + I \}, e'_2 = \{ x_2 + I, x_3 + I, x_4 + I \} \) and \( e'_3 = \{ x_4 + I, x_5 + I, x_6 + I \} \). Consider \( S'' = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_1+a, x_2+a, x_3+a, x_4+a, x_5+a, x_6+a \} \), where \( a \in I - \{ 0 \} \) and \( E'' = \{ e_1, \ldots, e_{13} \} \subset E(\mathcal{H}_3^3(R)) \), where \( e_1 = \{ x_1, x_2, x_3 \}, e_2 = \{ x_1 + a, x_2, x_3 \}, e_3 = \{ x_1, x_2 + a, x_3 \}, e_4 = \{ x_1, x_2, x_3 + a \}, e_5 = \{ x_1+a, x_2+a, x_3+a \}, e_6 = \{ x_2+a, x_3+a, x_4 \}, e_7 = \{ x_2+a, x_3+a, x_4 \}, e_8 = \{ x_4, x_5, x_6 \}, e_9 = \{ x_4 + a, x_5, x_6 \}, e_{10} = \{ x_4, x_5 + a, x_6 \}, e_{11} = \{ x_4 + a, x_5 + a, x_6 + a \} \). Then \( \mathcal{I}(\mathcal{H}_3^3(R)) \) contains a subdivision of \( 2K_{3,3} \) with \( [x_1, e_6, x_4] \) as a
common vertex. Hence $\gamma(\mathcal{I}(\mathcal{H}_i^3(R))) \geq 2$. 

**Remark 5.6.8.** Suppose $\mathcal{H}_3(R_i)$ has more than two edges such that exactly two edges with one common vertex and the remaining edges have no common vertex, then $\gamma(\mathcal{H}_i^3(R)) = 1$. 