Chapter 3

Group \( \{1, -1, i, -i\} \) cordial labeling of some bipartite graphs and Wheel related graphs

In this Chapter, we discuss the group \( \{1, -1, i, -i\} \) cordial labeling of some Complete Bipartite graphs and Wheel related graphs. We characterize \( K_{2,n}, K_{3,n} \) and \( K_{4,n} \) that are group \( \{1, -1, i, -i\} \) cordial. We prove that Helm, closed Helm, Gear graph, Web graph, Flower graph, Sunflower graph are all group \( \{1, -1, i, -i\} \) cordial. We further characterize Wheels that are group \( \{1, -1, i, -i\} \) cordial.

3.1 Group \( \{1, -1, i, -i\} \) cordial labeling of some bipartite graphs

Theorem 3.1.1. The Complete Bipartite graph \( K_{2,n} \) is group \( \{1, -1, i, -i\} \) cordial if and only if \( n \leq 5 \).

Proof. Let \( V(K_{2,n}) = V_1 \cup V_2 \), where \( V_1 = \{u_1, u_2\} \) and \( V_2 = \{v_1, v_2, ..., v_n\} \). \( E(K_{2,n}) = \{u_1v_i, u_2v_i, 1 \leq i \leq n\} \). The proof is divided into two cases.

Case 1. \( n \leq 5 \)
The group \( \{1, -1, i, -i\} \) cordial labeling of \( K_{2,n} \) is given in Table 3.1. In all these cases, exactly \( n \) edges receive the label 1 and \( n \) edges receive the label 0.

**Case 2.** \( n > 5 \)

Suppose \( f \) is a group \( \{1, -1, i, -i\} \) cordial labeling of \( K_{2,n} \).

**Subcase (i).** \( f(u_1) = 1 \) or \( f(u_2) = 1 \).

In this case, the number of vertices is at least 8. Therefore, every label has to be used at least two times. Since \( f(u_1) = 1 \), already \( n \) edges have received label 1. To maintain the edge condition, no other vertex can be given the label 1.

**Subcase (ii).** \( f(u_1) \neq 1 \) and \( f(u_2) \neq 1 \).

Clearly, \( v_f(1) = \lceil \frac{p}{4} \rceil \) or \( \lceil \frac{p}{4} \rceil - 1 \) and \( e_f(1) = 2 \lceil \frac{p}{4} \rceil \) or \( 2 (\lceil \frac{p}{4} \rceil - 1) \). Therefore, the label 1 should be allocated to the vertices of \( V_2 \). This contradicts the edge condition of group \( \{1, -1, i, -i\} \) cordial labeling.

**Theorem 3.1.2.** The Complete Bipartite graph \( K_{3,n} \) is group \( \{1, -1, i, -i\} \) cordial if and only if either \( n \leq 9 \) or \( n \geq 10 \) and \( n \neq 4k + 2, k \geq 2, k \in \mathbb{Z} \).
**Proof.** Assume that the complete bipartite graph $K_{3,n} = (V_1, V_2)$ is group \{1, -1, i, -i\} cordial. Let $u_1, u_2, u_3$ be the vertices of $V_1$ and let $V_2 = \{v_1, v_2, ..., v_n\}$. Suppose $n \geq 10$. We claim that $n \neq 4k + 2, k \geq 2, k \in \mathbb{Z}$. Suppose $n = 4k + 2, k \geq 2, k \in \mathbb{Z}$. Number of vertices is $n + 3 = 4k + 5$ and number of edges is $3n = 12k + 6$. Let $f$ be a group \{1, -1, i, -i\} cordial labeling of $K_{3,n}$.

We need $6k + 3$ edges with label 1. If $k + 1$ or $k + 2$ vertices that receive label 1 are chosen from $V_2$, we get $3k + 3$ or $3k + 6$ edges with label 1 accordingly. But $k \geq 2$ and so $6k + 3 \geq 3k + 6$ and also $6k + 3 \geq 3k + 3$. Hence at least one vertex from $V_1$ has to be given label 1. Certainly 2 vertices from $V_1$ cannot receive label 1 as that gives $2n = 8k + 4$ edges with label 1. So one vertex in $V_1$ and $k$ or $k + 1$ vertices from $V_2$ are to be given label 1. One vertex in $V_1$ gives $4k + 2$ edges with label 1. As every choice of a vertex in $V_2$ gives only 2 edges with label 1 and as $(6k + 3) - (4k + 2) = 2k + 1$ is odd, there is no group \{1, -1, i, -i\} cordial labeling. Thus $n \neq 4k + 2$.

Conversely, assume that either $n \leq 9$ or $n \geq 10$ and $n \neq 4k + 2, k \geq 2, k \in \mathbb{Z}$. For $n \leq 9$, group \{1, -1, i, -i\} cordial labeling of $K_{3,n}$ is given in Table 3.2.

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<table>
<thead>
<tr>
<th>n</th>
<th>u_1</th>
<th>u_2</th>
<th>u_3</th>
<th>v_1</th>
<th>v_2</th>
<th>v_3</th>
<th>v_4</th>
<th>v_5</th>
<th>v_6</th>
<th>v_7</th>
<th>v_8</th>
<th>v_9</th>
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<td>1</td>
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<td>-i</td>
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<td>-1</td>
<td>i</td>
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<td>i</td>
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<td>-i</td>
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<td>-i</td>
</tr>
</tbody>
</table>
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Table 3.2

Table 3.3 shows that each is a group \{1, -1, i, -i\} cordial labeling. Suppose $n \geq 10$ and $n \neq 4k + 2, k \geq 2, k \in \mathbb{Z}$. Let $f : V(K_{3,n}) \rightarrow \{1, -1, i, -i\}$ be a function.

**Case 1.** $n \equiv 0(mod 4)$
Let $n = 4k$, $k \geq 3$, $k \in \mathbb{Z}$.
Define a labeling $f$ as follows:
\[
\begin{align*}
    f(u_1) &= f(v_1) = f(v_2) = \cdots = f(v_k) = 1, \\
    f(u_2) &= f(v_{k+1}) = \cdots = f(v_{2k}) = -1, \\
    f(u_3) &= f(v_{2k+1}) = \cdots = f(v_{3k}) = i, \\
    f(v_{3k+1}) &= \cdots = f(v_{4k}) = -i.
\end{align*}
\]

**Case 2.** $n \equiv 1(\text{mod } 4)$

Let $n = 4k + 1$, $k \geq 3$, $k \in \mathbb{Z}$.
Define a labeling $f$ as follows:
\[
\begin{align*}
    f(u_1) &= f(v_1) = f(v_2) = \cdots = f(v_k) = 1, \\
    f(u_2) &= f(v_{k+1}) = \cdots = f(v_{2k}) = -1, \\
    f(u_3) &= f(v_{2k+1}) = \cdots = f(v_{3k}) = i, \\
    f(v_{3k+1}) &= \cdots = f(v_{4k}) = f(v_{4k+1}) = -i.
\end{align*}
\]

**Case 3.** $n \equiv 3(\text{mod } 4)$

Let $n = 4k + 3$, $k \geq 2$, $k \in \mathbb{Z}$.
Define a labeling $f$ as follows:
\[
\begin{align*}
    f(u_1) &= f(v_1) = f(v_2) = \cdots = f(v_{k+1}) = 1, \\
    f(u_2) &= f(v_{k+2}) = f(v_{k+3}) = \cdots = f(v_{2k+2}) = -1, \\
    f(u_3) &= f(v_{2k+3}) = \cdots = f(v_{3k+2}) = i, \\
    f(v_{3k+3}) &= \cdots = f(v_{4k}) = f(v_{4k+3}) = -i.
\end{align*}
\]

Table 3.4 shows that in all 3 cases, $f$ is a group $\{1, -1, i, -i\}$ cor-
dial labeling.

<table>
<thead>
<tr>
<th>nature of n</th>
<th>$v_f(1)$</th>
<th>$v_f(-1)$</th>
<th>$v_f(i)$</th>
<th>$v_f(-i)$</th>
<th>$e_f(0)$</th>
<th>$e_f(1)$</th>
</tr>
</thead>
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<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$k$</td>
<td>$6k$</td>
<td>$6k$</td>
</tr>
<tr>
<td>$n = 4k + 1, k \geq 3$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$6k + 2$</td>
<td>$6k + 1$</td>
</tr>
<tr>
<td>$n = 4k + 3, k \geq 2$</td>
<td>$k + 2$</td>
<td>$k + 2$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$6k + 4$</td>
<td>$6k + 5$</td>
</tr>
</tbody>
</table>

Table 3.4

Example 3.1.3. An illustration for $K_{3,7}$ is given in Fig 3.1.

Theorem 3.1.4. $K_{4,n}$ is group $\{1,-1,i,-i\}$ cordial if and only if $n \in \{4,5,6,7,9\}$.

Proof. Assume that the complete bipartite graph $K_{4,n} = (V_1, V_2)$ is group $\{1,-1,i,-i\}$ cordial. Let $u_1, u_2, u_3, u_4$ be the vertices of $V_1$ and let $V_2 = \{v_1, v_2, \ldots, v_n\}$. Suppose $f$ is a group $\{1,-1,i,-i\}$ cordial labeling of $K_{4,n}$.

Case 1. $n \equiv 0 (mod 4)$

Let $n = 4k, k \geq 1, k \in \mathbb{Z}$. Each vertex label should appear $k + 1$ times and each edge label should appear $8k$ times. As $k \geq 1$, we have $k + 1 \geq 2$. If $k + 1 > 2$ and if $k + 1$ vertices to get label 1 are chosen in $\{v_1, \ldots, v_n\}$, we need to have $4(k + 1) = 8k \Rightarrow k = 1$, which is a contradiction. So at least $v_1$ should be given label 1. If $v_2$ is also given label 1, we have 2 vertices of label 1 yielding $8k$ edges of label 1, which
is again a contradiction. So \( k \) vertices of label 1 are to be chosen from \( \{v_1, \ldots, v_n\} \). This gives \( 3k \) edges with label 1. But \( 3k = 4k \) is impossible. So \( k+1 > 2 \) is not possible. Thus \( k+1 = 2 \Rightarrow k = 1 \Rightarrow n = 4 \).

**Case 2.** \( n \equiv 1(\text{mod } 4) \)

Let \( n = 4k + 1, k \geq 1, k \in \mathbb{Z} \). We claim that \( k+1 \leq 3 \). Suppose \( k+1 \geq 4 \). If 2 \( u_i \)'s are given label 1, we get \( 8k + 2 \) edges with label 1 by giving label 1 to just 2 vertices which contradicts the necessary condition. So at most one \( u_i \) is given label 1. If \( k+2 \) or \( k+1 \) vertices of \( \{v_1, v_2, \ldots, v_n\} \) are given label 1, we get \( 4(k+2) \) or \( 4(k+1) \) edges with label 1. So \( 4k+8 = 8k+2 \) or \( 4k+4 = 8k+2 \). Thus \( 4k = 6 \) or \( 4k = 2 \), both impossible. So one \( u_i \) is given label 1 and \( k \) or \( k+1 \) vertices of \( \{v_1, v_2, \ldots, v_n\} \) are given label 1. Accordingly, \( 3k = 4k+1 \) or \( 3(k+1) = 4k+1 \Rightarrow k = -1 \) or \( k = 2 \), both impossible. Thus \( k+1 \leq 3 \). If \( k+1 = 3, n = 9 \) and if \( k+1 = 2, n = 5 \).

**Case 3.** \( n \equiv 2(\text{mod } 4) \)

Let \( n = 4k + 2, k \geq 1, k \in \mathbb{Z} \). We claim that \( k+1 \leq 2 \). Suppose \( k+1 \geq 3 \). As in Cases 1 and 2, we need to give label 1 to exactly one \( u_i \). So \( k+1 \) or \( k \) vertices are to be chosen from \( \{v_1, v_2, \ldots, v_n\} \). Accordingly, \( 3(k+1) = 4k+2 \) or \( 3k = 4k+2 \) and so \( k = 1 \) or \( k = -2 \), both impossible. Thus \( k+1 = 2 \) and so \( n = 6 \).

**Case 4.** \( n \equiv 3(\text{mod } 4) \)

Let \( n = 4k + 3, k \geq 1, k \in \mathbb{Z} \). Suppose \( k+1 \geq 3 \). As in Cases 1 and 2, we need to give label 1 to exactly one \( u_i \). So \( k+1 \) or \( k \) vertices are to be chosen from \( \{v_1, v_2, \ldots, v_n\} \). Accordingly, \( 3(k+1) = 4k+3 \) or \( 3k = 4k+3 \) and so \( k = 0 \) or \( k = -3 \), both impossible. Thus \( k+1 = 2 \) and so \( n = 7 \). Thus \( n \in \{4, 5, 6, 7, 9\} \).

Conversely, assume that \( n \in \{4, 5, 6, 7, 9\} \). A group \( \{1, -1, i, -i\} \) cordial labeling of \( K_{4,n} \) in each case is given in Table 3.5.
3.2 Group \( \{1, -1, i, -i\} \) cordial labeling of Wheel related graphs

Theorem 3.2.1. The Wheel \( W_n \) is group \( \{1, -1, i, -i\} \) cordial if and only if \( 3 \leq n \leq 6 \).

Proof. Let \( W_n = C_n + K_1 \), where \( C_n \) is the cycle \( u_1, u_2, ..., u_n, u_1 \) and \( V(K_1) = \{u\} \). For \( 3 \leq n \leq 6 \), the group \( \{1, -1, i, -i\} \) cordial labeling of \( W_n \) is given in Table 3.6.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( u )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
<th>( u_5 )</th>
<th>( u_6 )</th>
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<td>-i</td>
<td>-1</td>
<td>-i</td>
<td>i</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 3.6

Assume \( n > 6 \). Let \( n = 2t \). Suppose \( f \) is a group \( \{1, -1, i, -i\} \) cordial labeling of \( W_n \), \( n > 6 \).

Case 1. \( f(u) = 1 \).

Subcase(i). \( n \equiv 0, 1, 2 \pmod{4} \).
Let \( n = 4t \) or \( 4t + 1 \) or \( 4t + 2 \), \( v_f(1) = t \) or \( t + 1 \). The maximum possible edges with label 1 occurs only when we assign the label 1 to the cycle vertices alternatively. In this way, the possible number of cycle edges with label 1 is \( 2t \) and the spokes with label 1 is \( t \). Thus \( e_f(1) \leq 3t \), a contradiction, since the size of the wheel is \( 8t \).

**Subcase(ii).** \( n \equiv 3 \pmod{4} \).

Let \( n = 4t + 3 \). Then \( v_f(1) = t \). In this case, \( e_f(1) \leq 2(t - 1) + t - 1 \). So \( e_f(1) \leq 3t - 3 \), again a contradiction.

**Case 2.** \( f(u) \neq 1 \).

In this case, the possible edges with label 1 is only the cycle edges.

**Subcase(i).** \( n \equiv 0, 1, 2 \pmod{4} \).

Let \( n = 4t \) or \( 4t + 1 \) or \( 4t + 2 \) according as \( n \equiv 0(\text{mod} \ 4) \) or \( n \equiv 1(\text{mod} \ 4) \) or \( n \equiv 2(\text{mod} \ 4) \). Then \( v_f(1) = t \) or \( t + 1 \). Therefore \( e_f(1) \leq 2(t + 1) = 2t + 2 \), a contradiction.

**Subcase(ii).** \( n \equiv 3 \pmod{4} \).

Let \( n = 4t + 3 \). Then \( v_f(1) = t + 1 \). Here also \( e_f(1) \leq 2t + 2 \), a contradiction.

\(\Box\)

**Theorem 3.2.2.** All Helms are group \( \{1, -1, i, -i\} \) cordial.

**Proof.** Let \( H_n \) be the Helm with \( V(H_n) = \{u, u_i, v_i : 1 \leq i \leq n\} \) and \( E(H_n) = \{uu_i, u_iu_{i-1}, u_iv_i, u_nu_1 : 1 \leq i \leq n\} \). Note that \( H_n \) has \( 2n + 1 \) vertices and \( 3n \) edges. Let \( f : V(H_n) \rightarrow \{1, -1, i, -i\} \) be a function. As \( 2n + 1 \) is always odd, we need to consider the following two cases.

**Case 1.** \( 2n + 1 \equiv 1 \pmod{4} \).
Now $2n \equiv 0 \pmod{4}$ and $\lceil \frac{2n+1}{4} \rceil = \frac{2n}{4} + 1$. Moreover, $2n \equiv 0 \pmod{4}$ and so $n$ is even. Let $i = \frac{2n}{4} - 1$. Assign the label 1 to the vertices $u_1, u_2, u_3, \ldots, u_i, v_{i+1}, v_{i+2}$. Label the remaining vertices arbitrarily in such a way that $i + 1$ vertices get label $-1$, $i + 1$ vertices get label $i$ and $i + 1$ vertices get label $-i$. Number of edges with label $1 = 4 + 3(i - 1) + 2 = 3i + 3 = \frac{3n}{2}$.

![Graph](image)

**Fig. 3.2**

**Case 2.** $2n + 1 \equiv 3 \pmod{4}$.

Now $2n \equiv 2 \pmod{4}$ and so $n$ is odd. As $\lceil \frac{2n+1}{4} \rceil = \frac{2n-2}{4} + 1$, 3 of the vertex labels should occur $\frac{2n-2}{4} + 1$ times and 1 vertex label should occur $\frac{2n-2}{4}$ times. Let $i = \frac{2n-2}{4}$. Assign the label 1 to the vertices $u_1, u_2, u_3, \ldots, u_i$. Label the remaining vertices arbitrarily in such a way that $i + 1$ vertices get label $-1$, $i + 1$ vertices get label $i$ and $i + 1$ vertices get label $-i$. Number of edges with label
1 = 4 + 3(i - 1) = 3i + 1 = \frac{3n-1}{2}.

That this vertex labeling is a group \{1, -1, i, -i\} cordial labeling follows from Table 3.7.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Parity of n & \(v_f(1)\) & \(v_f(-1)\) & \(v_f(i)\) & \(v_f(-i)\) & \(e_f(1)\) & \(e_f(0)\) \\
\hline
\text{even} & \frac{2n}{4} + 1 & \frac{2n}{4} & \frac{2n}{4} & \frac{2n}{4} & \frac{3n}{2} & \frac{3n}{2} \\
\text{odd} & \frac{2n-2}{4} & \frac{2n+2}{4} & \frac{2n+2}{4} & \frac{2n+2}{4} & \frac{3n-1}{2} & \frac{3n+1}{2} \\
\hline
\end{tabular}
\caption{Table 3.7}
\end{table}

\textbf{Example 3.2.3.} Group \{1, -1, i, -i\} cordial labelings of \(H_6\) and \(H_7\) are given in Fig.3.2

\textbf{Theorem 3.2.4.} Closed Helm \(CH_n\) is group \{1, -1, i, -i\} cordial for every \(n\).

\textbf{Proof.} Number of vertices in \(CH_n = 2n + 1\) and number of edges in \(CH_n\) is \(4n\). Label the vertices of \(CH_n\) as follows:
Label of the center of the Wheel is \(u\). Label the \(n\) vertices of the rim of the Wheel by \(u_1, u_2, \ldots, u_n\) in order. Label the \(n\) pendent vertices of the Helm as \(v_1, v_2, \ldots, v_n\) in order so that \(u_i v_i\) is an edge for every \(i\), \(1 \leq i \leq n\).
Let \(f : V(CH_n) \to \{1, -1, i, -i\}\) be a function.

\textbf{Case 1.} \(n\) is odd.

Let \(n = 2k + 1\), \(k \geq 1\). Label the vertices \(u, v_1, v_2, \ldots, v_k\) by \(1\). Label the remaining vertices arbitrarily so that \(k + 1\) of them get label \(-1\), \(k + 1\) of them get label \(i\) and \(k\) of them get label \(-i\).

\textbf{Case 2.} \(n\) is even.

Let \(n = 2k\), \(k \geq 2\).
Label the vertices $u_1, u_3, u_5, ..., u_{2k-1}$ by 1. Label the remaining vertices arbitrarily so that $k$ of them get label $-1$, $k$ of them get label $i$ and $k + 1$ of them get label $-i$.

It follows from Table 3.8 that all these labelings are group $\{1, -1, i, -i\}$ cordial. □

<table>
<thead>
<tr>
<th>Parity of $n$</th>
<th>$v_f(1)$</th>
<th>$v_f(-1)$</th>
<th>$v_f(i)$</th>
<th>$v_f(-i)$</th>
<th>$e_f(0)$</th>
<th>$e_f(1)$</th>
</tr>
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<tbody>
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<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$k$</td>
<td>$4k + 2$</td>
<td>$4k + 2$</td>
</tr>
<tr>
<td>even, $n = 2k$</td>
<td>$k$</td>
<td>$k$</td>
<td>$k$</td>
<td>$k + 1$</td>
<td>$4k$</td>
<td>$4k$</td>
</tr>
</tbody>
</table>

Table 3.8

**Example 3.2.5.** A group $\{1, -1, i, -i\}$ cordial labeling of $CH_5$ is given in Fig. 3.3.

![Fig. 3.3](image)

**Theorem 3.2.6.** The Web graph $W(2, n)$ is group $\{1, -1, i, -i\}$ cordial for every $n$.

**Proof.** $W(2, n)$ has $3n + 1$ vertices and $5n$ edges. Label the center vertex of the wheel as $u$, the rim vertices of the wheel as $u_1, u_2, ..., u_n$ in
order, the vertices of the outer cycle of the closed Helm as \( v_1, v_2, \ldots, v_n \)
in order and the pendent vertices as \( w_1, w_2, \ldots, w_n \) in order so that for
\( 1 \leq i \leq n \), \( u_i \) is adjacent with \( v_i \) and \( v_i \) is adjacent with \( w_i \).
Let \( f : V(W(2, n)) \to \{1, -1, i, -i\} \) be a function.

**Case 1.** \( n \) is odd.

Let \( n = 2k + 1(k \geq 1, k \in \mathbb{Z}) \).

**Subcase(i).** \( k \) is even.

Now each vertex label should appear \((3k/2)+1\) times. Label \( u, u_1, u_2, \ldots, u_{3k/2} \) with 1. Label the remaining vertices arbitrarily so that \( \frac{3k}{2} + 1 \) of them get label \(-1\), \( \frac{3k}{2} + 1 \) of them get label \( i \) and \( \frac{3k}{2} + 1 \) of them get label \(-i\).

**Subcase(ii).** \( k \) is odd.

Label \( u, u_1, u_2, \ldots, u_{3k+1}/2 \) with 1. Label the remaining vertices arbitrarily so that \( \frac{3k+1}{2} \) of them get label \(-1\), \( \frac{3k+1}{2} \) of them get label \( i \) and \( \frac{3k+1}{2} \) of them get label \(-i\).

**Case 2.** \( n \) is even.

Let \( n = 2k(k \geq 2, k \in \mathbb{Z}) \).

**Subcase(i).** \( k \) is even.

As \( k \) is even, \( 6k \equiv 0(mod\ 4) \) and so \( 6k + 1 \equiv 1(mod\ 4) \). Label \( u, u_1, u_2, \ldots, u_{3k}/2-1, u_{3k}/2 \) with 1. Label the remaining vertices arbitrarily so that \( \frac{3k}{2} \) of them get label \(-1\), \( \frac{3k}{2} \) of them get label \( i \) and \( \frac{3k}{2} \) of them get label \(-i\).

**Subcase(ii).** \( k \) is odd.
As $6k \equiv 2 \mod 4$, $6k + 1 \equiv 3 \mod 4$. Label $u, u_1, u_2, \ldots, u_{\frac{3k-1}{2}}$ with 1. Label the remaining vertices arbitrarily so that $\frac{3k+1}{2}$ of them get label $-1$, $\frac{3k+1}{2}$ of them get label $i$ and $\frac{3k-1}{2}$ of them get label $-i$. Table 3.9 and Table 3.10 shows that in all cases, the labelings are group \{1, -1, i, -i\} cordial.

\[
\begin{array}{|c|c|c|c|c|}
\hline
n & v_f(1) & v_f(-1) & v_f(i) & v_f(-i) \\
\hline
n = 2k + 1, k \text{ is even} & \frac{3k}{2} + 1 & \frac{3k}{2} + 1 & \frac{3k}{2} + 1 & \frac{3k}{2} + 1 \\
\hline
n = 2k + 1, k \text{ is odd} & \lfloor \frac{3k+2}{2} \rfloor & \lfloor \frac{3k+2}{2} \rfloor & \lfloor \frac{3k+2}{2} \rfloor & \lfloor \frac{3k+2}{2} \rfloor \\
\hline
n = 2k, k \text{ is even} & \frac{3k}{2} + 1 & \frac{3k}{2} & \frac{3k}{2} & \frac{3k}{2} \\
\hline
n = 2k, k \text{ is odd} & \frac{3k+1}{2} & \frac{3k+1}{2} & \frac{3k+1}{2} & \frac{3k-1}{2} \\
\hline
\end{array}
\]

Table 3.9

\[
\begin{array}{|c|c|c|}
\hline
n & e_f(0) & e_f(1) \\
\hline
n = 2k + 1, k \text{ is even} & 5k + 3 & 5k + 2 \\
\hline
n = 2k + 1, k \text{ is odd} & 5k + 2 & 5k + 3 \\
\hline
n = 2k, k \text{ is even} & 5k & 5k \\
\hline
n = 2k, k \text{ is odd} & 5k & 5k \\
\hline
\end{array}
\]

Table 3.10

**Example 3.2.7.** A group \{1, -1, i, -i\} cordial labeling of $W(2, 5)$ is given in Fig. 3.4.
Theorem 3.2.8. The Gear graph $G_n$ is group $\{1, -1, i, -i\}$ cordial for every $n$.

Proof. Number of vertices of $G_n$ is $2n + 1$ and number of edges is $3n$. Let $f : V(G_n) \rightarrow \{1, -1, i, -i\}$ be a function.

Label the center vertex of the wheel by $u$, the $n$ vertices on the rim of the wheel by $u_1, u_2, ..., u_n$ in order and the $n$ newly added vertices by $v_1, v_2, ..., v_n$ in order so that for $1 \leq i \leq n - 1$, $v_i$ subdivides the edge $u_iu_{i+1}$ and $v_n$ subdivides the edge $u_nu_1$.

Case 1. $n$ is odd.

Let $n = 2k + 1, k \geq 1$. Label the vertices $u_1, u_2, u_3, ..., u_k$ and $v_{k+1}$ by 1. Label the remaining vertices arbitrarily so that $k + 1$ of them get label $-1$, $k + 1$ of them get label $i$ and $k + 1$ of them get label $-i$.

Fig. 3.4
Case 2. $n$ is even.

Let $n = 2k, k \geq 2$.
Label the vertices $u_1, u_2, u_3, ..., u_k$ by 1. Label the remaining vertices arbitrarily so that $k$ of them get label $-1$, $k$ of them get label $i$ and $k + 1$ of them get label $-i$.

Table 3.11 shows that $f$ is a group \( \{1, -1, i, -i\} \) cordial labeling.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Parity of } n & v_f(1) & v_f(-1) & v_f(i) & v_f(-i) & e_f(0) & e_f(1) \\
\hline
\text{odd, } n = 2k + 1, k \geq 1 & k + 1 & k + 1 & k + 1 & k & 3k + 1 & 3k + 2 \\
\text{even, } n = 2k, k \geq 2 & k & k & k & k + 1 & 3k & 3k \\
\hline
\end{array}
\]

Table 3.11

Example 3.2.9. A group \( \{1, -1, i, -i\} \) cordial labeling of $G_b$ is given in Fig. 3.5.
Theorem 3.2.10. The Flower graph $F_l\, n$ is group $\{1, -1, i, -i\}$ cordial for every $n$.

Proof. Let $u$ be the center of the wheel $W\, n$. Let $u_1, u_2, ..., u_n$ be the vertices on the cycle of $W\, n$ and $v_1, v_2, ..., v_n$ be the pendent vertices of the Helm such that $v_i$ is adjacent to $u_i$ for $1 \leq i \leq n$. Number of vertices in $F_l\, n$ is $2n + 1$ and number of edges is $4n$.

Let $f : V (F_l\, n) \to \{1, -1, i, -i\}$ be a function.

Case 1. $n \equiv 0 (mod 4)$.

Let $n = 4k, k \geq 1, k \in \mathbb{Z}$.

In a group $\{1, -1, i, -i\}$ cordial labeling, three of the vertex labels should appear $2k$ times and one vertex label should appear $2k + 1$ times. Each edge label should appear $8k$ times. Define a labeling $f$ as follows:

Label $u_1, u_3, u_5, ..., u_{2k}$ with 1. Label the remaining vertices arbitrarily so that $2k$ of them get label $-1$, $2k$ of them get label $i$ and $2k + 1$ of them get label $-i$. Number of edges with label 1 is $4(2k) = 8k$.

Case 2. $n \equiv 1 (mod 4)$.

Let $n = 4k + 1, k \geq 1, k \in \mathbb{Z}$.

In a group $\{1, -1, i, -i\}$ cordial labeling, one vertex label should appear $2k$ times and three other vertex labels should appear $2k + 1$ times. Each edge label should appear $8k + 2$ times. Define a labeling $f$ as follows:

Label $u_1, u_3, u_5, ..., u_{2k}, v_2$ with 1. Label the remaining vertices arbitrarily so that $2k + 1$ of them get label $-1$, $2k + 1$ of them get label $i$ and $2k$ of them get label $-i$. Number of edges with label 1 is $4(2k) + 2 = 8k + 2$.

Case 3. $n \equiv 2 (mod 4)$.

Let $n = 4k + 2, k \geq 1, k \in \mathbb{Z}$. 

40
In a group \(\{1, -1, i, -i\}\) cordial labeling, three of the vertex labels should appear \(2k + 1\) times and one vertex label should appear \(2k + 2\) times. Each edge label should appear \(8k + 4\) times. Define a labeling \(f\) as follows:
Label \(u_1, u_3, u_5, \ldots, u_{n-1}\) with 1. Label the remaining vertices arbitrarily so that \(2k + 1\) of them get label \(-1\), \(2k + 1\) of them get label \(i\) and \(2k + 2\) of them get label \(-i\). Number of edges with label 1 is \(4(2k + 1) = 8k + 4\).

**Case 4.** \(n \equiv 3 (mod \ 4)\).

Let \(n = 4k + 3, k \geq 0, k \in \mathbb{Z}\).
In a group \(\{1, -1, i, -i\}\) cordial labeling, three vertex labels should appear \(2k + 2\) times and one vertex label should appear \(2k + 1\) times. Each edge label should appear \(8k + 6\) times. Define a labeling \(f\) as follows:
Label \(u_1, u_3, u_5, \ldots, u_{n-2}, v_2\) with 1. Label the remaining vertices arbitrarily so that \(2k + 2\) of them get label \(-1\), \(2k + 2\) of them get label \(i\) and \(2k + 1\) of them get label \(-i\). Number of edges with label 1 is \(4(2k + 1) + 2 = 8k + 6\).

Table 3.12 shows that in all cases, the given labeling is group \(\{1, -1, i, -i\}\) cordial.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(v_f(1))</th>
<th>(v_f(-1))</th>
<th>(v_f(i))</th>
<th>(v_f(-i))</th>
<th>(e_f(0))</th>
<th>(e_f(1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4k, k \geq 1, k \in \mathbb{Z})</td>
<td>(2k)</td>
<td>(2k)</td>
<td>(2k)</td>
<td>(2k + 1)</td>
<td>(8k)</td>
<td>(8k)</td>
</tr>
<tr>
<td>(4k + 1, k \geq 1, k \in \mathbb{Z})</td>
<td>(2k + 1)</td>
<td>(2k + 1)</td>
<td>(2k + 1)</td>
<td>(2k)</td>
<td>(8k + 2)</td>
<td>(8k + 2)</td>
</tr>
<tr>
<td>(4k + 2, k \geq 1, k \in \mathbb{Z})</td>
<td>(2k + 1)</td>
<td>(2k + 1)</td>
<td>(2k + 1)</td>
<td>(2k + 2)</td>
<td>(8k + 4)</td>
<td>(8k + 4)</td>
</tr>
<tr>
<td>(4k + 3, k \geq 0, k \in \mathbb{Z})</td>
<td>(2k + 2)</td>
<td>(2k + 2)</td>
<td>(2k + 2)</td>
<td>(2k + 1)</td>
<td>(8k + 6)</td>
<td>(8k + 6)</td>
</tr>
</tbody>
</table>

Table 3.12

**Example 3.2.11.** A group \(\{1, -1, i, -i\}\) cordial labeling of \(Fl_6\) is given in Fig.3.6.
Theorem 3.2.12. The Sunflower graph \( SF_n \) is group \( \{ 1, -1, i, -i \} \) cordial for every \( n \).

Proof. Let \( u \) be the center of the wheel and \( u_1, u_2, ..., u_n \) be the vertices on the cycle of the wheel. Let \( v_1, v_2, ..., v_n \) be the additional vertices so that \( v_i \) is joined by edges to \( u_i, u_{i+1} \) where \( i + 1 \) is taken modulo \( n \). Number of vertices in \( SF_n \) is \( 2n + 1 \) and number of edges is \( 4n \).

Let \( f : V(SF_n) \to \{ 1, -1, i, -i \} \) be a function.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( v_f(1) )</th>
<th>( v_f(-1) )</th>
<th>( v_f(i) )</th>
<th>( v_f(-i) )</th>
<th>( e_f(0) )</th>
<th>( e_f(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4k, k \geq 1, k \in \mathbb{Z} )</td>
<td>2k</td>
<td>2k</td>
<td>2k</td>
<td>2k + 1</td>
<td>8k</td>
<td>8k</td>
</tr>
<tr>
<td>( 4k + 1, k \geq 1, k \in \mathbb{Z} )</td>
<td>2k + 1</td>
<td>2k + 1</td>
<td>2k + 1</td>
<td>2k</td>
<td>8k + 2</td>
<td>8k + 2</td>
</tr>
<tr>
<td>( 4k + 2, k \geq 1, k \in \mathbb{Z} )</td>
<td>2k + 1</td>
<td>2k + 1</td>
<td>2k + 1</td>
<td>2k + 2</td>
<td>8k + 4</td>
<td>8k + 4</td>
</tr>
<tr>
<td>( 4k + 3, k \geq 0, k \in \mathbb{Z} )</td>
<td>2k + 2</td>
<td>2k + 2</td>
<td>2k + 2</td>
<td>2k + 1</td>
<td>8k + 6</td>
<td>8k + 6</td>
</tr>
</tbody>
</table>

Table 3.13

Case 1. \( n \equiv 0 \pmod{4} \)
Let $n = 4k$, $k \geq 1, k \in \mathbb{Z}$. Label $u, u_1, v_2, v_3, \ldots, v_{2k-1}$ with 1. This induces edge label 1 to $4k + 4 + (2k - 2)2 = 8k$ edges. Label the remaining vertices arbitrarily so that $2k$ vertices get label $-1$, $2k$ vertices get label $i$ and $2k + 1$ vertices get label $-i$.

**Case 2.** $n \equiv 1(\text{mod } 4)$

Let $n = 4k + 1$, $k \geq 1, k \in \mathbb{Z}$. Label $u, u_1, v_1, v_2, v_3, \ldots, v_{2k-1}$ with 1. This induces edge label 1 to $4k + 1 + 4 + 1 + (2k - 2)2 = 8k + 2$ edges. Label the remaining vertices arbitrarily so that $2k + 1$ vertices get label $-1$, $2k + 1$ vertices get label $i$ and $2k$ vertices get label $-i$.

**Case 3.** $n \equiv 2(\text{mod } 4)$

Let $n = 4k + 2$, $k \geq 1, k \in \mathbb{Z}$. Label $u, u_1, v_2, v_3, \ldots, v_{2k}$ with 1. This induces edge label 1 to $4k + 2 + 4 + (2k - 1)2 = 8k + 4$ edges. Label the remaining vertices arbitrarily so that $2k + 1$ vertices get label $-1$, $2k + 1$ vertices get label $i$ and $2k + 2$ vertices get label $-i$.

**Case 4.** $n \equiv 3(\text{mod } 4)$

Let $n = 4k + 3$, $k \geq 1, k \in \mathbb{Z}$. Label $u, u_1, v_1, v_2, v_3, \ldots, v_{2k}$ with 1. This induces edge label 1 to $4k + 3 + 4 + 1 + (2k - 1)2 = 8k + 6$ edges. Label the remaining vertices arbitrarily so that $2k + 2$ vertices get label $-1$, $2k + 2$ vertices get label $i$ and $2k + 1$ vertices get label $-i$.

Table 3.13 shows that the above labelings are group $\{1, -1, i, -i\}$ cordial.

Example 3.2.13. A group $\{1, -1, i, -i\}$ cordial labeling of $SF_3$ is given in Fig.3.7.
Example 3.2.14. A group \( \{1, -1, i, -i\} \) cordial labeling of \( SF_5 \) is given in Fig. 3.8.