Chapter 2

Group A cordial labeling

In this Chapter, for any group $A$ we define a new labeling, called as Group $A$ cordial labeling and give a necessary and sufficient condition for a graph $G$ to be group $(\mathbb{Z}_p, \oplus)$ cordial where $\mathbb{Z}_p$ is the group of integers modulo $p$ under addition.. We then investigate the group \{1, −1, i, −i\} cordial labeling of some graphs. This is the group of fourth roots of unity, which is cyclic with generators $i$ and $−i$. We prove that the Bistar, Path, Cycle, Friendship graph and $W(n, n)$ are group \{1, −1, i, −i\} cordial. We also characterize Star and Complete graphs that are group \{1, −1, i, −i\} cordial.

2.1 Group A cordial labeling of a graph

**Definition 2.1.1.** Let $G$ be a $(p,q)$ graph and let $A$ be a group. Let $f : V(G) \rightarrow A$ be a function. For each edge $uv$ assign the label 1 if $(o(f(u)), o(f(v))) = 1$ or 0 otherwise. $f$ is called a group $A$ cordial labeling if $|v_f(a) - v_f(b)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(x)$ and $e_f(y)$ respectively denote the number of vertices labeled with an element $x$ and number of edges labeled with $y (y = 0, 1)$. A graph which admits a group $A$ cordial labeling is called a group $A$ cordial graph.

**Example 2.1.2.** A simple example of a group $(\mathbb{Z}_5, \oplus)$ cordial graph $G$ is given in Fig.2.1.
We have \( \mathbb{Z}_5 = \{0, 1, 2, 3, 4\} \).
Also \( o(0) = 1, o(1) = 5, o(2) = 5, o(3) = 5 \) and \( o(4) = 5 \).
Define \( f : V(G) \to \mathbb{Z}_5 \) by \( f(u) = 1, f(v) = 0, f(w) = 2, f(x) = 3 \) and \( f(y) = 4 \).
So for every \( n \in \mathbb{Z}_5 \) we have \( v_f(n) = 1 \). As \( (1, 5) = 1 \), the edges \( uv, vw, vy \) get label 1 and the edges \( wx, xy \) get label 0.
Thus \( e_f(0) = 2 \) and \( e_f(1) = 3 \). Since we have \( |v_f(a) - v_f(b)| = 0 \) for every \( a, b \in \mathbb{Z}_5 \) and \( |e_f(0) - e_f(1)| = 1 \), \( f \) is a group \( (\mathbb{Z}_5, +) \) cordial labeling and \( G \) is a group \( (\mathbb{Z}_5, +) \) cordial graph.

![Figure 2.1](image.png)

We now give a necessary and sufficient condition for a graph \( G \) with \( p \) vertices where \( p \) is the power of a single prime, to have a group \( (\mathbb{Z}_p, +) \) cordial labeling.

2.2 Some results on Group A cordial labeling

**Theorem 2.2.1.** Let \( G \) be a \((p, q)\) graph with \( p = z^\alpha \) where \( z \) is a prime. Then \( G \) is a group \((\mathbb{Z}_p, +)\) cordial graph if and only if \( G \) has a vertex \( v \) with \( \text{deg } v = d \) where \( d = \frac{q}{2}, \lceil \frac{q}{2} \rceil \) or \( \lfloor \frac{q}{2} \rfloor \).

**Proof.** Let \( V(G) = \{v, v_1, v_2, ..., v_{p-1}\} \). Assume that \( G \) has a group \((\mathbb{Z}_p, +)\) cordial labeling \( g \). Choose \( v \) as the vertex with \( g(v) = 0 \). Let \( \text{deg } v = d \). As \( g \) is a group \((\mathbb{Z}_p, +)\) cordial labeling, \( |e_g(0) - e_g(1)| \leq 1 \).
As $p = z^\alpha$, by Theorem 1.3.3, order of the labels of all other vertices is $z^j$ for some $j \leq \alpha$. So the $d$ edges incident at $v$ alone have label 1 and all other edges have label 0.

Thus $|e_g(0) - e_g(1)| \leq 1$

$\Rightarrow |(q - d) - d| \leq 1$

$\Rightarrow |q - 2d| \leq 1$

$\Rightarrow q = 2d, 2d + 1$ or $2d - 1$

$\Rightarrow d = \frac{q}{2}, \left\lceil \frac{q}{2} \right\rceil$ or $\left\lfloor \frac{q}{2} \right\rfloor$.

Conversely, assume that $G$ has a vertex $v$ with $\text{deg } v = d$ where $d = \frac{q}{2}, \left\lceil \frac{q}{2} \right\rceil$ or $\left\lfloor \frac{q}{2} \right\rfloor$. Define $f : V(G) \rightarrow \mathbb{Z}_p$, by $f(v) = 0$ and $f(v_i) = i$ for $1 \leq i \leq p - 1$. The $d$ edges incident with $v$ have label 1 and all other edges have label 0. As $d = \frac{q}{2}, \left\lceil \frac{q}{2} \right\rceil$ or $\left\lfloor \frac{q}{2} \right\rfloor$, $|e_f(0) - e_f(1)| \leq 1$ and so $f$ is a group $(\mathbb{Z}_p, \oplus)$ cordial labeling.

**Corollary 2.2.2.** If $p$ is a power of a single prime, then $P_p$ is group $(\mathbb{Z}_p, \oplus)$ cordial if and only if $p \leq 5$.

**Proof.** For $p \leq 5$, a group $(\mathbb{Z}_p, \oplus)$ cordial labeling of $P_p$ is given in Fig.2.2.

For $p \geq 5$, the proof follows from Theorem 2.2.1.

**Corollary 2.2.3.** If $n$ is a prime, then the Star $K_{1,n^\alpha-1}$ is group $\mathbb{Z}_{n^\alpha}$ cordial if and only if $n^\alpha \leq 3$.

**Proof.** The proof follows from Theorem 2.2.1.
Corollary 2.2.4. If $n$ is a prime, then the Bistar $B_{\frac{n^\alpha-3}{2}, \frac{n^\alpha-1}{2}}$ is group $\mathbb{Z}_{n^\alpha}$ cordial.

**Proof.** The order and size of the Bistar $B_{\frac{n^\alpha-3}{2}, \frac{n^\alpha-1}{2}}$ are $n^\alpha$ and $n^\alpha - 1$ respectively. Note that the degree of one of the central vertices is $\frac{n^\alpha-1}{2}$. Hence the proof follows from Theorem 2.2.1. \qed

2.3 Group \{1, -1, i, -i\} cordial labeling of some graphs

We now investigate the group \{1, -1, i, -i\} cordial labeling of some graphs. This is the group of fourth roots of unity, which is cyclic with generators $i$ and $-i$. We prove that the Bistar, Path, Cycle, Friendship graph and $W(n,n)$ are group \{1, -1, i, -i\} cordial. We also characterize Star and Complete graphs that are group \{1, -1, i, -i\} cordial.

**Definition 2.3.1.** Let $G$ be a $(p, q)$ graph and consider the group \{1, -1, i, -i\} with multiplication. Let $f : V(G) \to \{1, -1, i, -i\}$ be a function. For each edge $uv$ assign the label 1 if $(o(f(u)), o(f(v))) = 1$ or 0 otherwise. $f$ is called a group \{1, -1, i, -i\} cordial labeling if $|v_f(a) - v_f(b)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(x)$ and $e_f(y)$ respectively denote the number of vertices labeled with an element $x$ and number of edges labeled with $y$ ($y = 0, 1$). A graph which admits a group \{1, -1, i, -i\} cordial labeling is called a group \{1, -1, i, -i\} cordial graph.

**Example 2.3.2.** A group \{1, -1, i, -i\} cordial labeling of $C_3$ is given in Fig. 2.3.

![Fig. 2.3](image-url)
Remark 2.3.3. Let \( A \) be any group of order 4. As identity is the only element of order 1, by Theorem 1.3.3, every other element of \( A \) is of order 2 or 4. Thus a graph \( G \) is group \( \{1, -1, i, -i\} \) cordial if and only if \( G \) is group \( A \) cordial for any group of order 4.

Theorem 2.3.4. Every graph is a subgraph of a connected group \( \{1, -1, i, -i\} \) cordial graph.

Proof. Let \( G \) be a \((p, q)\) graph and \( G_i(1 \leq i \leq 4) \) be four copies of the complete graph \( K_p \). Let \( u_1^i, u_2^i, \ldots, u_p^i \) be the vertices of \( G_i(1 \leq i \leq 4) \).

Let \( m = 2\binom{p}{2} \). Let \( G^* \) be obtained from \( G_i(1 \leq i \leq 4) \) as follows:

The vertex set of \( G^* \) is \( V(G_1) \cup V(G_2) \cup V(G_3) \cup V(G_4) \cup \{v_i : 1 \leq i \leq m\} \) and the edge set is given by \( E(G^*) = E(G_1) \cup E(G_2) \cup E(G_3) \cup E(G_4) \cup \{u_1^1u_1^2, u_2^2u_3^3, u_3^3u_1^1\} \cup \{u_1^iv_j : 1 \leq j \leq m\} \). Clearly \( G^* \) has \( 4p + m \) vertices and \( 4\binom{p}{2} + m + 3 \) edges. As \( G^* \) contains 4 copies of \( K_p \), \( G \) is clearly a subgraph of \( G^* \).

Let \( f : V(G^*) \rightarrow \{1, -1, i, -i\} \) be any function. Assign label 1 to all the vertices of \( G_1 \), \( i \) to all the vertices of \( G_2 \), \( -i \) to all the vertices of \( G_3 \) and \(-i\) to all the vertices of \( G_4 \).

Case 1. \( m = 4t, t \in \mathbb{Z} \).

Assign label 1 to the vertices \( v_1, v_2, \ldots, v_t \), \( i \) to the vertices \( v_{t+1}, \ldots, v_{2t} \), \(-i\) to the vertices \( v_{2t+1}, \ldots, v_{3t} \) and finally assign the label \(-i\) to the vertices \( v_{3t+1}, v_{3t+2}, \ldots, v_{4t} \). In this case \( v_f(1) = v_f(-1) = v_f(i) = v_f(-i) = p + t \).

Case 2. \( m = 4t + 1, t \in \mathbb{Z} \).

As in Case 1, assign labels to the vertices \( v_i(1 \leq i \leq m - 1) \) and assign 1 to the vertex \( v_m \). Now \( v_f(1) = p + t + 1 \) and \( v_f(-1) = v_f(i) = v_f(-i) = p + t \).
Case 3. \( m = 4t + 2, t \in \mathbb{Z} \).

Assign labels to the vertices \( v_i (1 \leq i \leq m - 1) \) as in Case 2. Finally assign the label \(-1\) to the vertex \( v_m \). Here \( v_f(1) = v_f(-1) = p + t + 1 \) and \( v_f(i) = v_f(-i) = p + t \).

Case 4. \( m = 4t + 3, t \in \mathbb{Z} \).

As in Case 3 , assign labels to the vertices \( v_i (1 \leq i \leq m - 1) \) and assign \( i \) to the vertex \( v_m \). In this case \( v_f(1) = v_f(-1) = v_f(i) = p + t + 1 \) and \( v_f(-i) = p + t \).

In all cases, \( e_f(0) = 3\binom{p}{2} + 2 \) and \( e_f(1) = 3\binom{p}{2} + 1 \). So \( f \) is a group \( \{1, -1, i, -i\} \) cordial labeling of \( G^* \).

\[ \square \]

Theorem 2.3.5. The Star \( K_{1,n} \) is group \( \{1, -1, i, -i\} \) cordial if and only if \( n \leq 5 \).

**Proof.** Let \( V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\} \) and \( E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\} \). Suppose \( n \leq 5 \). Group \( \{1, -1, i, -i\} \) cordial labelings of \( K_{1,n} \) are given in Table 2.1.

Conversely, suppose that \( K_{1,n} \) is group \( \{1, -1, i, -i\} \) cordial. Let \( f \) be a group \( \{1, -1, i, -i\} \) cordial labeling of \( K_{1,n} \).

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<td>1</td>
<td>-i</td>
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<td>-1</td>
</tr>
</tbody>
</table>

Table 2.1

Case 1. \( f(u) = 1 \)

In this case all the edges receive the label 1. Thus \( e_f(1) = n \) and
$e_f(0) = 0$, a contradiction.

**Case 2.** $f(u) \in \{-1, i, -i\}$

Clearly $v_f(1) = \lceil \frac{p}{4} \rceil$ or $\lceil \frac{p+1}{4} \rceil$. This implies $e_f(1) \leq \lceil \frac{p+1}{4} \rceil$. This contradicts the edge condition of group $\{1, -1, i, -i\}$ cordial labeling. \hfill \Box

**Theorem 2.3.6.** The Bistar $B_{n,n}$ is group $\{1, -1, i, -i\}$ cordial for every $n$.

**Proof.** Let $V(B_{n,n}) = \{u, v\} \cup \{u_i v_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uv\} \cup \{uu_i, vv_i : 1 \leq i \leq n\}$. Let $f : V(B_{n,n}) \rightarrow \{1, -1, i, -i\}$ be a function. Assign the label $1$, $-1$ to the vertices $u$ and $v$ respectively.

**Case 1.** $n \equiv 0 \pmod{4}$.

Let $n = 4t$. Assign the label $1$ to the vertices $u_1, u_2, ..., u_{2t}$ and $-1$ to the vertices $u_{2t+1}, u_{2t+2}, ..., u_{4t}$. Assign the label $i$ to the vertices $v_1, v_2, ..., v_{2t}$ and assign the label $-i$ to the vertices $v_{2t+1}, v_{2t+2}, ..., v_{4t}$.

**Case 2.** $n \equiv 1 \pmod{4}$.

Let $n = 4t + 1$. As in Case 1, assign labels to the vertices $u_i, v_i (1 \leq i \leq n - 1)$. Finally assign the label $i, -i$ respectively to the vertices $u_n, v_n$.

**Case 3.** $n \equiv 2 \pmod{4}$.

Let $n = 4t + 2$. Assign labels to the vertices $u_i, v_i (1 \leq i \leq n - 1)$ as in Case 2. Then assign the label $1, -1$ to the vertices $u_n, v_n$ respectively.

**Case 4.** $n \equiv 3 \pmod{4}$.
Let \( n = 4t + 3 \). As in Case 3, assign labels to the vertices \( u_i, v_i (1 \leq i \leq n - 1) \). Finally assign the label \( i, -i \) to the vertices \( u_n, v_n \) respectively. That this vertex labeling \( f \) is a group \( \{1, -1, i, -i\} \) cordial labeling follows from Table 2.2.

<table>
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<th>( n )</th>
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<th>( v_f(-1) )</th>
<th>( v_f(i) )</th>
<th>( v_f(-i) )</th>
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<td>4t + 3</td>
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Table 2.2

**Example 2.3.7.** Group \( \{1, -1, i, -i\} \) cordial labelings of \( B(4, 4) \) and \( B(6, 6) \) are given in Fig. 2.4.

\[\begin{array}{ccccccc}
 n = 4 & 1 & i & 1 & i & 1 & -1
\end{array}\]

\[\begin{array}{ccccccc}
 n = 6 & 1 & i & 1 & i & 1 & -1
\end{array}\]

**Theorem 2.3.8.** Any Path \( P_n \) is group \( \{1, -1, i, -i\} \) cordial.
Proof. Let \( P_n \) be the path \( u_1, u_2, u_3, ..., u_n \). Clearly \( P_n \) is group \( \{1, -1, i, -i\} \) cordial if \( n \leq 3 \). Assume \( n \geq 4 \).

Let \( f : V(P_n) \rightarrow \{1, -1, i, -i\} \) be a function.

**Case 1.** \( n \equiv 0(\text{mod } 4) \)

Let \( n = 4t \). Assign the label \( i \) to the vertices \( u_i \) (\( 1 \leq i \leq t \)). Then assign the label \( -i \) to the vertices \( u_{t+i} \) (\( 1 \leq i \leq t \)). Next assign the label \( -1 \) to the vertices \( u_{2t+1}, u_{2t+3}, ..., u_{4t-1} \). Finally assign the label 1 to the vertices \( u_{2t+2}, u_{2t+4}, ..., u_{4t} \).

**Case 2.** \( n \equiv 1(\text{mod } 4) \)

Let \( n = 4t + 1 \). Assign labels to the vertices \( u_i \) (\( 1 \leq i \leq n-1 \)) as in Case 1. Finally assign the label 1 to the vertex \( u_n \).

**Case 3.** \( n \equiv 2(\text{mod } 4) \)

Let \( n = 4t + 2 \). Assign the label 1 to the vertices \( u_1, u_3, u_5, ..., u_{2t+1} \) and assign the label \( -1 \) to the vertices \( u_2, u_4, ..., u_{2t-2}, u_{2t+2} \). Then assign the label \( -i \) to the vertices \( u_{2t+3}, u_{2t+5}, ..., u_{4t+1} \). Finally assign the label \( i \) to the vertices \( u_{3t+2}, u_{3t+4}, ..., u_{4t+1} \) and \( u_{4t+2} \).

**Case 4.** \( n \equiv 3(\text{mod } 4) \)

Let \( n = 4t + 3 \). As in Case 3, assign labels to the vertices \( u_i \) (\( 1 \leq i \leq n-1 \)). Finally assign \( i \) to the vertex \( u_n \).

Table 2.3 and Table 2.4 establish that the above vertex labeling \( f \) is a group \( \{1, -1, i, -i\} \) cordial labeling of the path \( P_n \).

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<th>( v_f(-1) )</th>
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Table 2.3
\begin{array}{|c|c|c|}
\hline
n & e_f(0) & e_f(1) \\
\hline
4t & 2t & 2t - 1 \\
4t + 1 & 2t & 2t \\
4t + 2 & 2t & 2t + 1 \\
4t + 3 & 2t + 1 & 2t + 1 \\
\hline
\end{array}

Table 2.4

\textbf{Example 2.3.9.} Group \( \{1, -1, i, -i\} \) cordial labelings of \( P_8 \) and \( P_9 \) are given in Fig. 2.5.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example2.3.9}
\caption{Fig. 2.5}
\end{figure}

\textbf{Corollary 2.3.10.} Any Cycle \( C_n \) is group \( \{1, -1, i, -i\} \) cordial.

\textbf{Proof.} Let the Cycle \( C_n \) be \( u_1, u_2, u_3, ..., u_n, u_1 \).
Let \( f : V(C_n) \rightarrow \{1, -1, i, -i\} \) be a function.

\textbf{Case 1.} \( n \equiv 0, 1, 3(\text{mod } 4) \)

The group \( \{1, -1, i, -i\} \) cordial labeling given in Theorem 2.3.8 is also a group \( \{1, -1, i, -i\} \) cordial labeling of \( C_n \).

\textbf{Case 2.} \( n \equiv 2(\text{mod } 4) \)

Assign labels to the vertices \( u_i(1 \leq i \leq n) \) as in Theorem 2.3.8.
Finally relabel the vertex $u_2$ with 1 and relabel $u_3$ with $-1$. 

\[
\square
\]

**Example 2.3.11.** Group $\{1, -1, i, -i\}$ cordial labelings of $C_{10}$ and $C_{11}$ are given in Fig. 2.6.

![Figure 2.6](image)

**Theorem 2.3.12.** The Complete graph $K_n$ is group $\{1, -1, i, -i\}$ cordial if and only if $n \in \{1, 2, 3, 4, 7, 14, 21\}$.

**Proof.** Let $V(K_n) = \{u_i : 1 \leq i \leq n\}$.

Tables 2.5 and 2.6 give a group $\{1, -1, i, -i\}$ cordial labeling of $K_n$, $n \in \{1, 2, 3, 4, 7, 14, 21\}$.

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Table 2.5
Table 2.6

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<tr>
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<th>u_{11}</th>
<th>u_{12}</th>
<th>u_{13}</th>
<th>u_{14}</th>
<th>u_{15}</th>
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<th>u_{18}</th>
<th>u_{19}</th>
<th>u_{20}</th>
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<td>−i</td>
<td>−i</td>
<td>−i</td>
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<tr>
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<td>−1</td>
<td>i</td>
<td>i</td>
<td>i</td>
<td>−i</td>
<td>−i</td>
<td>−i</td>
<td>−i</td>
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</tbody>
</table>

Assume \( n > 4 \). Suppose \( f \) is a group \{1, −1, i, −i\} cordial labeling of \( K_n \).

**Case 1.** \( n \equiv 0 \pmod{4} \)

Let \( n = 4t, t \in \mathbb{N} \) and \( t > 1 \). Obviously \( v_f(1) = v_f(-1) = v_f(i) = v_f(-i) = t \). It is easy to verify that \( e_f(1) = \binom{t}{2} + t(3t) \). Thus \( e_f(1) = \frac{t(t-1)}{2+3t^2} \). Also \( e_f(0) = \binom{t}{2} + \binom{t}{2} + \binom{t}{2} + t(t) + t(t) = 3 \binom{t}{2} + 3t^2 = \frac{3t(t-1)}{2+3t^2} \). So \( e_f(0) - e_f(1) = t^2 - t > 1 \), a contradiction.

**Case 2.** \( n \equiv 1 \pmod{4} \)

Let \( n = 4t + 1, t \in \mathbb{N}, t \neq 5 \). Then \( v_f(1) = t \) or \( t + 1 \).

**Subcase (i).** \( v_f(1) = t \).

In this case, \( e_f(1) = \binom{t}{2} + t^2 + t^2 + t(t + 1) = \frac{t(t-1)}{2} + 3t^2 + t \).

Then \( e_f(0) = \binom{t}{2} + \binom{t}{2} + \binom{t+1}{2} + t(t) + t(t+1) + t(t+1) = 2 \binom{t}{2} + \binom{t+1}{2} + t^2 + 2t(t+1) = \frac{2t(t-1)}{2} + \frac{t(t+1)}{2} + 3t^2 + 2t \).

Hence \( e_f(0) - e_f(1) = \frac{t(t-1)}{2} + \frac{t(t+1)}{2} + t = t^2 + t > 1 \), a contradiction.

**Subcase (ii).** \( v_f(1) = t + 1 \).
Here $e_f(1) = \binom{t+1}{2} + 3t(t+1)$ and $e_f(0) = 3\binom{t}{2} + t^2 + t^2 + t^2 = 3\binom{t}{2} + 3t^2$. Hence $e_f(0) - e_f(1) = t^2 - 5t$, a contradiction.

**Case 3.** $n \equiv 2 (\text{mod } 4)$

Let $n = 4t + 2, t \in \mathbb{N}, t \neq 3$. Then $v_f(1) = t$ or $t + 1$.

**Subcase (i).** $v_f(1) = t$.

In this case, $e_f(1) = \binom{t}{2} + t^2 + t(t + 1) + t(t + 1)$ and $e_f(0) = \binom{t}{2} + \binom{t+1}{2} + \binom{t+1}{2} + t(t + 1) + t(t + 1) + (t + 1)(t + 1)$

$= \binom{t}{2} + 2\binom{t+1}{2} + 3t^2 + 4t + 1$. Hence $e_f(0) - e_f(1) = t^2 + 3t + 1 > 1$, a contradiction.

**Subcase (ii).** $v_f(1) = t + 1$.

In this case, $e_f(1) = \binom{t+1}{2} + 2t(t + 1) + (t + 1)^2$ and $e_f(0) = 2\binom{t}{2} + \binom{t+1}{2} + t^2 + 2t(t + 1)$.

Hence $e_f(0) - e_f(1) = t^2 - 3t - 1 > 1$, a contradiction.

**Case 4.** $n \equiv 3 (\text{mod } 4)$

Let $n = 4t + 3, t \in \mathbb{N}, t \neq 1$. Then $v_f(1) = t$ or $t + 1$.

**Subcase (i).** $v_f(1) = t$.

In this case, $e_f(1) = \binom{t}{2} + 3t(t + 1)$ and $e_f(0) = 3\binom{t+1}{2} + 3(t + 1)^2$.

Thus $e_f(0) - e_f(1) = t^2 + 5t + 3 > 1$, a contradiction.

**Subcase (ii).** $v_f(1) = t + 1$.

In this case, $e_f(1) = \binom{t+1}{2} + 2(t + 1)^2 + t(t + 1)$ and $e_f(0) = \binom{t}{2} + 2\binom{t+1}{2} + (t + 1)^2 + 2t(t + 1)$.

Hence $e_f(0) - e_f(1) = t^2 + t - 1 > 1$, a contradiction.

\[\square\]
**Theorem 2.3.13.** The Friendship graph $C_3^{(t)}$ is group \{1, −1, i, −i\} cordial if and only if $t \leq 4$.

**Proof.** Let the vertices of $C_3^{(t)}$ be labeled as follows: Let $u$ be the vertex common to all 3-cycles and let every 3-cycle be labeled as $uu_i v_i (1 \leq i \leq t)$. The group \{1, −1, i, −i\} cordial labeling of $C_3^{(t)}, t \leq 4$ is given in Table 2.7.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$u$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>−1</td>
<td></td>
<td></td>
<td>i</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>−1</td>
<td>1</td>
<td>i</td>
<td></td>
<td>1</td>
<td>−i</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>−1</td>
<td>i</td>
<td>−i</td>
<td>1</td>
<td>1</td>
<td>−i</td>
<td>i</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>−i</td>
<td>−1</td>
<td>i</td>
</tr>
</tbody>
</table>

Table 2.7

Assume $t > 4$. Suppose $f$ is a group \{1, −1, i, −i\} cordial labeling of $C_3^{(t)}$.

**Case 1.** $f(u) = 1$.

Then $e_f(1) \geq 2t$ and $e_f(0) \leq t$. Thus $e_f(1) − e_f(0) \geq t$, a contradiction.

**Case 2.** $f(u) \neq 1$.

**Subcase(i).** $t$ is even.

Let $t = 2m$. Clearly two 1’s contribute at most 4 edges with label 1. Therefore $m$ 1’s contribute at most $2m$ edges with label 1. Hence $e_f(1) \leq 2(m+1) = 2m+2$. But the size of $C_3^{(t)}$ is $6m$, a contradiction.

**Subcase(ii).** $t$ is odd.

Let $t = 2m + 1$. In this case also, $e_f(1) \leq 2m + 2$, a contradiction. □

**Theorem 2.3.14.** $W(n, n)$ is group \{1, −1, i, −i\} cordial for every $n$. 

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Proof. Let $V(W_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(W_{n,n}) = \{uu_i, vv_i : 1 \leq i \leq n\} \cup \{u_i u_{(i+1) \mod n}, v_i v_{(i+1) \mod n} : 1 \leq i \leq n\} \cup \{uv\}$. Note that $W(n, n)$ has $2n + 2$ vertices and $4n + 1$ edges. Let $f : V(W_{n,n}) \to \{1, -1, i, -i\}$ be a function.

Case 1. $n$ is odd.

Let $n = 2k + 1$, $k \geq 1$, $k \in \mathbb{Z}$. Label the vertices $u, u_1, u_3, \ldots, u_{n-2}$ by 1. Label the remaining vertices arbitrarily so that $k + 1$ of them get label $-1$, $k + 1$ of them get label $i$ and $k + 1$ of them get label $-i$.

Case 2. $n$ is even.

Let $n = 2k$, $k \geq 2$, $k \in \mathbb{Z}$.
Label the vertices $u, u_1, u_3, u_5, \ldots, u_{n-1}$ by 1. Label the remaining vertices arbitrarily so that $k + 1$ of them get label $-1$, $k$ of them get label $i$ and $k$ of them get label $-i$.

Table 2.8 shows that $f$ is a group $\{1, -1, i, -i\}$ cordial labeling.

<table>
<thead>
<tr>
<th>Parity of $n$</th>
<th>$v_f(1)$</th>
<th>$v_f(-1)$</th>
<th>$v_f(i)$</th>
<th>$v_f(-i)$</th>
<th>$e_f(0)$</th>
<th>$e_f(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>odd, $n = 2k + 1$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$4k + 3$</td>
<td>$4k + 2$</td>
</tr>
<tr>
<td>even, $n = 2k$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$k$</td>
<td>$k$</td>
<td>$4k$</td>
<td>$4k + 1$</td>
</tr>
</tbody>
</table>

Table 2.8

Example 2.3.15. A group $\{1, -1, i, -i\}$ cordial labeling of $W(5, 5)$ is given in Fig. 2.7.
Fig 2.7