Chapter 3

Harmonic Status Index and Coindex of Derived Graphs
3.1 Introduction

In this chapter we obtain the harmonic status coincides of graphs. Let $G$ be a connected graph of with $n$ vertices and $m$ edges. Let $V(G)$ be the vertex set and $E(G)$ be the vertex set and $E(G)$ be the edge set of $G$. The Status\cite{53} of a vertex $u \in V(G)$, denoted by $\sigma(u)$ is defined as the sum of its distance from every other vertex in $V(G)$, that is

$$\sigma(u) = \sum_{v \in V(G)} d_G(u, v).$$

Where, $d(u, v)$ is the distance between $u$ and $v$ in $G$.

The Harmonic Index of a graph $G$ is defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)},$$

where $d(u)$ and $d(v)$ are the degrees of vertices $u$ and $v$ respectively.

Inspired from this definition we have defined in Chapter 2, the Harmonic Status Index as

$$HS(G) = \sum_{uv \in E(G)} \frac{2}{\sigma(u) + \sigma(v)}.$$

The first and second Zagreb indices of a graph $G$ are defined as \cite{49}

$$M_1(G) = \sum_{uv \in E(G)} [d(u) + d(v)] \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$
The first and second Zagreb co-indices of a graph $G$ are defined as [32]

$$
\overline{M}_1(G) = \sum_{uv \notin E(G)} [d(u) + d(v)] \quad \text{and} \quad \overline{M}_2(G) = \sum_{uv \notin E(G)} [d(u)d(v)].
$$

More results on Zagreb coindices can be found in [9, 7, 6].

In this chapter we study the harmonic status coindex of graphs. The *Harmonic status coindex* of a graph $G$ is defined as

$$
\overline{HS}(G) = \sum_{uv \notin E(G)} \frac{2}{\sigma(u) + \sigma(v)}.
$$

For a graph given in *Figure 3.1*, $HS(G) = 1.408$, $\overline{HS}(G) = 0.530$.

### 3.2 Bounds for Harmonic Status Coindices of Graphs

**Theorem 3.2.1.** Let $G$ be any $r$-regular graph with $n$- vertices and $\text{diam}(G) = D$. Then

$$
\sum_{uv \notin E(G)} \frac{2}{2D(n-1) - (D-1)[d(u) + d(v)]} \leq \overline{HS}(G)
$$

$$
\leq \sum_{uv \notin E(G)} \frac{2}{4n - 4 - [d(u) + d(v)]}.
$$

(3.1)
Equality on both sides holds if and only if \( \text{diam}(G) \leq 2 \).

Proof. For any vertex \( u \) of graph \( G \) there are \( d(u) \) vertices which are at distance 1 and the remaining \( n - 1 - d(u) \) vertices are at distance atmost \( D \). Therefore,

\[
\sigma(u) \leq d(u) + D(n - 1 - d(u)) = D(n - 1) - (D - 1)d(u).
\]

Therefore, \( \sigma(u) + \sigma(v) \leq 2D(n - 1) - (D - 1)[d(u) + d(v)] \).

Hence,

\[
\overline{HS}(G) = \sum_{u,v \in E(G)} \frac{2}{\sigma(u) + \sigma(v)} \geq \sum_{u,v \in E(G)} \frac{2}{2D(n - 1) - (D - 1)[d(u) + d(v)]}
\]

which forms the lower bound.

For any vertex \( u \) of graph \( G \) there are \( d(u) \) vertices which are at distance 1 and the remaining \( n - 1 - d(u) \) vertices are at distance atleast 2. Therefore,

\[
\sigma(u) \geq d(u) + 2(n - 1 - d(u)) = 2n - 2 - d(u).
\]

Therefore, \( \sigma(u) + \sigma(v) \geq 4n - 4 - [d(u) + d(v)] \).
Hence,

$$\overline{HS}(G) = \sum_{uv \notin E(G)} \frac{2}{\sigma(u) + \sigma(v)}$$

$$\leq \sum_{uv \notin E(G)} \frac{2}{4n - 4 - [d(u) + d(v)]}.$$ 

For equality if $\text{diam}(G) = D \geq 3$, then there exists at least one pair of vertices $u$ and $w$ such that $d(u, w) \geq 3$. Therefore,

$$\sigma(u) \geq d(u) + 3 + 2[n - 2 - d(u)]$$

$$= 2n - 1 - d(u).$$

$\sigma(w) \geq 2n - 1 - d(w)$ and $\sigma(v) \geq 2n - 1 - d(v)$, $v \neq u, w$. Therefore for $uv \notin E(G)$,

$$\sigma(u) + \sigma(v) \geq 4n - 2 - [d(u) + d(v)]$$

$$> 4n - 4 - [d(u) + d(v)].$$

Therefore,

$$\overline{HS}(G) = \sum_{uv \notin E(G)} \frac{2}{\sigma(u) + \sigma(v)}$$

$$< \sum_{uv \notin E(G)} \frac{2}{4n - 4 - [d(u) + d(v)]}$$

a contradiction. Hence $\text{diam}(G) \leq 2$. \hfill \Box

**Corollary 3.2.2.** Let $G$ be a graph with $n$-vertices $m$-edges and $\text{diam}(G) = D$. Let $\Delta$ and $\delta$ are the maximum and minimum degrees of the vertices of graph $G$ respectively. Then
\[
\frac{n(n-1)-2m}{2D(n-1)-2(D-1)\delta} \leq \overline{HS}(G) \leq \frac{n(n-1)-2m}{4n-4-2\Delta}.
\]

**Proof.** For any vertex \( u \notin V(G) \),
\[
\delta \leq d(u) \leq \Delta.
\]
Therefore,
\[
2\delta \leq d(u) + d(v) \leq 2\Delta.
\]
There are \( \frac{n(n-1)}{2} - m \) pair of vertices in \( G \) which are not adjacent.
Substituting \( d(u) + d(v) \geq 2\delta \) on left side of Eqn. 3.1 and \( d(u) + d(v) \leq 2\Delta \) on right side of Eqn. 3.1 we get the results. \( \square \)

**Corollary 3.2.3.** Let \( G \) be an \( r \)-regular connected graph with \( n \) vertices, and \( \text{diam}(G) = D \). Then
\[
\frac{n(n-1)-nr}{2D(n-1)-2(D-1)r} \leq \overline{HS}(G) \leq \frac{n(n-1)-nr}{4n-4-2r}.
\]
Equality on both sides holds if and only if \( \text{diam}(G) \leq 2 \).

**Proof.** Substituting \( d(u) = r \) for all \( u \in V(G) \) in Theorem 3.2.1 we get the result. \( \square \)

**Proposition 3.2.4.** For a complete graph \( K_n \),
\[
\overline{HS}(K_n) = 0.
\]

**Proposition 3.2.5.** For a complete bipartite graph \( K_{p,q} \),
\[
\overline{HS}(K_{m,n}) = \frac{p(p-1)}{2q + 4(p-1)} + \frac{q(q-1)}{2p + 4(q-1)}.
\]
Proof. Let $V_1$, $V_2$ be the partite sets of $V(K_{p,q})$. Where $|V_1| = p$ and $|V_2| = q$. If $u \in V_1$ then $\sigma(u) = q + 2(p - 1)$ and if $u \in V_2$ then $\sigma(u) = p + 2(q - 1)$.

Therefore for $u, v \in V_1$ then $\sigma(u) + \sigma(v) = 2q + 4(p - 1)$ and for $u, v \in V_2$ then $\sigma(u) + \sigma(v) = 2p + 4(q - 1)$.

Therefore,

$$\overline{HS}(G) = \sum_{uv \in E(K_{p,q})} \frac{2}{\sigma(u) + \sigma(v)}$$

$$= \sum_{\{u,v\} \subseteq V_1} \frac{2}{2q + 4(p - 1)} + \sum_{\{u,v\} \subseteq V_2} \frac{2}{2p + 4(q - 1)}$$

$$= \frac{p(p - 1)}{2q + 4(p - 1)} + \frac{q(q - 1)}{2p + 4(q - 1)}.$$

$\square$

Proposition 3.2.6. For a cycle $C_n$ on $n \geq 3$ vertices,

$$\overline{HS}(C_n) = \begin{cases} \frac{2(n-3)}{n} & \text{if } n \text{ is even} \\ \frac{2n(n-3)}{n^2-1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If $n$ is even number, then for every vertex $u$ of $C_n$,

$$\sigma(u) = 2 \left[ 1 + 2 + \cdots + \frac{n-1}{2} \right] + \frac{n}{2} = \frac{n^2}{4}.$$
Therefore

\[
\overline{HS}(C_n) = \sum_{uv \in E(C_n)} \frac{2}{\sigma(u) + \sigma(v)}
\]

\[
= \sum_{uv \in E(C_n)} \frac{2}{\frac{n^2}{2}}
\]

\[
= \frac{2(n - 3)}{n}
\]

If \( n \) is odd number, then for every vertex \( u \) of \( C_n \),

\[
\sigma(u) = 2 \left[ 1 + 2 + \ldots + \frac{n-1}{2} \right] = \frac{n^2 - 1}{4}.
\]

Therefore

\[
\overline{HS}(C_n) = \sum_{uv \in E(C_n)} \frac{2}{\sigma(u) + \sigma(v)}
\]

\[
= \sum_{uv \in E(C_n)} \frac{2}{\frac{n^2 - 1}{2}}
\]

\[
= \frac{2n(n - 3)}{n^2 - 1}.
\]

\[
\square
\]

**Proposition 3.2.7.** For a path \( P_n \) on \( n \geq 2 \) vertices,

\[
\overline{HS}(P_n) = \sum_{i=1}^{n-1} \sum_{j=i+2}^{n} \frac{2}{(n+1)(n-i-j) + i^2 + j^2}.
\]

*Proof.* Let \( v_1, v_2, \ldots, v_n \) be the vertices of the path \( P_n \), such that \( v_i \) is adjacent to \( v_{i+1} \), where \( i = 1, 2, \ldots, n - 1 \).
Therefore for $i = 1, 2, \ldots, n$

$$
\sigma(v_i) = (i - 1) + (i - 2) + \cdots + 1 + 2 + \cdots + (n - i) \\
= \frac{n^2 + n}{2} + i(n - i - 1).
$$

The vertex $v_i$ is not adjacent to $v_j$, where $j = i + 2, \ldots, n$.

Therefore,

$$
\overline{HS}(P_n) = \sum_{uv \notin E(P_n)} \frac{2}{\sigma(u) + \sigma(v)} \\
= \sum_{j=3}^{n} \frac{2}{\sigma(v_1) + \sigma(v_j)} + \sum_{j=4}^{n} \frac{2}{\sigma(v_2) + \sigma(v_j)} + \cdots + \sum_{j=n}^{n} \frac{2}{\sigma(v_{n-2}) + \sigma(v_j)} \\
= \sum_{i=1}^{n-2} \sum_{j=i+2}^{n} \frac{2}{\sigma(v_i) + \sigma(v_j)} \\
= \sum_{i=1}^{n-2} \sum_{j=i+2}^{n} \left[ \frac{2}{\frac{n^2 + n}{2} + i(n - i - 1) + \frac{n^2 + 2}{2} + j(n - j - 1)} \right] \\
= \sum_{i=1}^{n-2} \sum_{j=i+2}^{n} \frac{2}{(n + 1)(n - i - j) + i^2 + j^2}.
$$

\[\square\]

**Proposition 3.2.8.** For wheel $W_{n+1}$, $n \geq 3$,

$$
\overline{HS}(W_{n+1}) = \frac{n^2 - 3n}{4n - 6}.
$$

**Proof.** The non adjacent pairs of vertices of the wheel $W_{n+1}$ has degree 3 and there are $\frac{(n+1)n}{2} - 2n$ pairs of non adjacent vertices in $W_{n+1}$. Also $diam(W_{n+1}) = 2$. 

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Therefore by the equality part of Theorem 3.2.1,
\[
\overline{HS}(W_{n+1}) = \sum_{uv \notin E(W_{n+1})} \frac{2}{4(n + 1) - 4 - [d(u) + d(v)]} 
\]
\[
= \sum_{uv \notin E(W_{n+1})} \frac{2}{4(n + 1) - 4 - 6} 
\]
\[
= \left[ \frac{n(n + 1)}{2} - 2n \right] \left[ \frac{2}{4(n + 1) - 4 - 6} \right] 
\]
\[
= \frac{n^2 - 3n}{4n - 6}. 
\]

\[\Box\]

**Proposition 3.2.9.** For a friendship graph \( F_n \), \( n \geq 2 \)
\[
\overline{HS}(F_n) = \frac{n(n - 1)}{2n - 1}. 
\]

Proof. The non adjacent pairs of vertices of the friendship graph \( F_n \) has degree 2 and these are \( \frac{2n(2n+1)}{2} - 3n \) pairs of non adjacent vertices in \( F_n \). Also \( \text{diam}(F_n) = 2 \). Therefore by the equality part of Theorem 3.2.1,
\[
\overline{HS}(F_n) = \sum_{uv \notin E(F_n)} \frac{2}{4(2n + 1) - 4 - [d(u) + d(v)]} 
\]
\[
= \sum_{uv \notin E(F_n)} \frac{2}{4(2n + 1) - 8} 
\]
\[
= \left[ \frac{2n(2n + 1)}{2} - 3n \right] \left[ \frac{2}{8n - 4} \right] 
\]
\[
= \frac{n(n - 1)}{2n - 1}. 
\]

\[\Box\]

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3.3 Harmonic Status Indices and Coindices of Line Graphs

Theorem 3.3.1. [86] For a connected graph $G$, $\text{diam}(L(G)) \leq 2$ if and only if none of the three graphs $F_1$, $F_2$ and $F_3$ of Figure 3.2 is an induced subgraph of $G$.

![F1_F2_F3](image)

*Figure 3.2: The forbidden induced subgraphs*

Theorem 3.3.2. Let $G$ be a connected $r$-regular graph on $n$ vertices and $m$ edges with $\text{diam}(G) \leq 2$ and none of $F_i = 1, 2, 3$ of Figure 3.2 is an induced subgraph of $G$ then

$$H(S(L(G))) = \frac{nr^2 - 2m}{4(m-r)}$$

and

$$\overline{H}(S(L(G))) = \frac{m(m+1) - nr^2}{4(m-r)}.$$

*Proof.* The number if vertices of $L(G)$ is $n' = m$ and the number of edges is $m' = -m + \frac{1}{2} \sum_{i=1}^{n} d_i^2 \ i, e \ m' = -m + \frac{1}{2} nr^2$ from [52].

We know that, for regular graphs

$$H(S(G)) = \frac{m}{2n-2-r}. \quad (3.2)$$
Therefore since $diam(G) \leq 2$ and $G$ has no $F_i = 1, 2, 3$ is its induced subgraph and if $r$ is the regularity of $G$, then regularity of $L(G)$ is [85] $r' = d(u) + d(v) - 2$ where $uv = e \in E(G)$ i.e. $r' = 2r - 2$. From Eqn.(3.2)

$$HS(L(G)) = \frac{m'}{2n' - 2 - r'} = \frac{\frac{1}{2}nr^2 - m}{2m - 2(2r - 2)} = \frac{nr^2 - 2m}{4(m - r)}.$$ 

Continuing on the similar lines we have,

$$\overline{HS}(L(G)) = \sum_{uv \notin E(L(G))} \frac{2}{2n' - 2 - r'} = \frac{m(m + 1) - nr^2}{4(m - r)}.$$  

\[\square\]

**Theorem 3.3.3.** Let $G$ be any connected $(n,m)$ graph and $\triangle$ be the maximum degree of $G$. Then

$$HS(L(G)) \leq \frac{n \triangle^2 - 2m}{4m - 4\triangle},$$

$$\overline{HS}(L(G)) \leq \frac{m(m + 1) - n \triangle^2}{4(m - \triangle)}.$$ 

**Proof.** We know that from the definition of line graph [34] the number of vertices of $L(G)$, $n' = m$. Consider an edge $e = uv \in E(G)$ which is
adjacent to \( d(u) + d(v) - 2 = d(e) \) edges at \( u \) and \( v \) taken together in \( L(G) \). Hence the edge \( e \) is not adjacent to remaining \( m - 1 - d(e) \) edges of \( G \). In \( L(G) \) the distance between \( e \) and the remaining \( m - 1 - d(e) \) vertices is more than 1 i.e. atleast 2. Hence for any graph \( G \) the \( \sigma(e) \) in \( L(G) \) is

\[
\sigma(e) \geq [d(u) + d(v) - 2] + 2(m - 1 - [d(u) + d(v) - 2]).
\]

Since \( \triangle \) is the maximum degree, \( d(u) + d(v) \leq 2\triangle \).

Therefore \( \sigma(e) \geq 2m - 2\triangle \). Now suppose \( e \) and \( f \) are any two vertices of \( L(G) \), \( \sigma(e) + \sigma(f) \geq 4m - 4 \triangle \). Implies that,

\[
HS(L(G)) = \sum_{ef \in E(L(G))} \frac{2}{\sigma(e) + \sigma(f)} \\
\leq \sum_{ef \in E(L(G))} \frac{2}{4m - 4\triangle} \\
\leq \frac{nr^2 - 2m}{4(m - \triangle)}.
\]

For regular graphs, \( m' = -m + \frac{1}{2}nr^2 \) and \( r = \triangle \) implies that, \( m' = -m + \frac{1}{2}n\triangle^2 \). Therefore,

\[
HS(L(G)) \leq \frac{n \triangle^2 - 2m}{4m - 4\triangle}.
\]

Now, from the definition of harmonic status coindices we have,

\[
\overline{HS}(L(G)) = \sum_{ef \notin E(L(G))} \frac{2}{\sigma(e) + \sigma(f)} \\
\leq \frac{m(m + 1) - nr^2}{4m - 4\triangle}.
\]
Thus, for regular graphs,
\[
\overline{HS}(L(G)) \leq \frac{m(m+1) - n\Delta^2}{4(m - \triangle)}.
\]

Equality in above theorem holds for regular graph with \(\text{diam}(G) \leq 2\).

### 3.4 Harmonic Status Coincides of Some Graphs Obtained from the Complete Graph

In this section we find the harmonic status coincides of some graphs obtained in [47].

**Definition 3.4.1.** Let \(e_i, i = 1, 2, \ldots, k, 1 \leq k \leq n-2\), be the distinct edges of a complete graph \(K_n, n \geq 3\), all being incident to a single vertex. The graph \(K_a_n(k)\) is obtained by deleting \(e_i, i = 1, 2, \ldots, k\) from \(K_n\). In addition \(K_a_n(0) \cong K_n\).

**Definition 3.4.2.** Let \(f_i, i = 1, 2, \ldots, k, 1 \leq k \leq \lfloor n/2 \rfloor\) be independent edges of the complete graph \(K_n, n \geq 3\). The graph \(K_b_n(k)\) is obtained by deleting \(f_i, i = 1, 2, \ldots, k\) from \(K_n\). In addition \(K_b_n(0) \cong K_n\).

**Definition 3.4.3.** Let \(V_k\) be a \(k\)-element subset of the vertex set of the complete graph \(K_n, 2 \leq k \leq n - 1, n \geq 3\). The graph \(K_c_n(k)\) is
obtained by deleting from $K_n$ all the edges connecting pairs of vertices from $V_k$. In addition $Kc_n(0) \cong Kc_n(1) \cong K_n$.

**Definition 3.4.4.** Let $3 \leq k \leq n$, $n \geq 3$. The graph $Kd_n(k)$ is obtained by deleting from $K_n$, the edges belonging to a $k$-membered cycle.

**Proposition 3.4.5.** For $n \geq 3$ and $1 \leq k \leq n - 2$,

$$\overline{HS}(Ka_n(k)) = \frac{2k}{2n - 1 + k}.$$ 

**Proof.** There exist a edge set say $E_1 = \{uv \mid d(u) = n - 1 - k, \ d(v) = n - 2 \text{ and } uv \notin E(Ka_n(k))\}$. It is easy to check that $|E_1| = k$. Also $diam(Ka_n(k)) = 2$.

Therefore by the equality part of Theorem 3.2.1

$$\overline{HS}(Ka_n(k)) = \sum_{uv \notin E(Ka_n(k))} \frac{2}{4(n - 1) - [d(u) + d(v)]}$$

$$= \sum_{uv \in E_1} \frac{2}{4(n - 1) - [d(u) + d(v)]}$$

$$= \sum_{uv \in E_1} \frac{2}{4(n - 1) - (2n - 3 - k)}$$

$$= \frac{2k}{2n - 1 + k}.$$

**Proposition 3.4.6.** For $n \geq 3$ and $1 \leq k \leq \lfloor n/2 \rfloor$,

$$\overline{HS}(Kb_n(k)) = \frac{1}{n} \lfloor \frac{n}{2} \rfloor.$$
Proof. There exist a set say $E_1 = \{uv \mid d(u) = n - 2 \text{ and } uv \notin E(Kb_n(k))\}$ and $|E_1| = \left\lceil \frac{n}{2} \right\rceil$. Also $diam(Kb_n(k)) = 2$.

Therefore by the equality part of Theorem 3.2.1

$$
\overline{HS}(Kb_n(k)) = \sum_{uv \in E(Kb_n(k))} \frac{2}{4(n - 1) - [d(u) + d(v)]}
$$

$$
= \sum_{uv \in E_1} \frac{2}{4(n - 1) - [d(u) + d(v)]}
$$

$$
= \sum_{uv \in E_1} \frac{2}{4(n - 1) - (2n - 4)}
$$

$$
= \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil.
$$

\[\square\]

**Proposition 3.4.7.** For $n \geq 3$ and $2 \leq k \leq n - 1$,

$$
\overline{HS}(Kc_n(k)) = \frac{k(k - 1)}{2n - 4 + 2k}.
$$

Proof. There exist a set say $E_1 = \{uv \mid d(u) = n - k, \ d(v) = n - k \text{ and } uv \notin E(Kc_n(k))\}$ and $|E_1| = \frac{k(k-1)}{2}$. Also $diam(Kc_n(k)) = 2$.

Therefore by the equality part of Theorem 3.2.1

$$
\overline{HS}(Kc_n(k)) = \sum_{uv \notin E(Kc_n(k))} \frac{2}{4(n - 1) - [d(u) + d(v)]}
$$

$$
= \sum_{uv \in E_1} \frac{2}{4(n - 1) - [d(u) + d(v)]}
$$

$$
= \sum_{uv \in E_1} \frac{2}{4(n - 1) - (2n - 2k)}
$$

$$
= \frac{k(k - 1)}{2n - 4 + 2k}.
$$

\[\square\]
Proposition 3.4.8. For $3 \leq k \leq n$ and $n \geq 5$,

$$\overline{HS}(Kd_n(k)) = \frac{k}{n + 1}.$$ 

Proof. There exist a edge set say $E_1 = \{uv \mid d(u) = n - 3, \ d(v) = n - 3 \text{ and } uv \notin E(Kd_n(k))\}$ and $|E_1| = k$. Also $\text{diam}(Kd_n(k)) = 2$ Therefore by the equality part of Theorem 3.2.1

$$\overline{HS}(Kd_n(k)) = \sum_{uv \notin E(Kd_n(k))} \frac{2}{4(n - 1) - [d(u) + d(v)]}$$

$$= \sum_{uv \in E_1} \frac{2}{4(n - 1) - [d(u) + d(v)]}$$

$$= \sum_{uv \in E_1} \frac{2}{4(n - 1) - (2n - 6)}$$

$$= \frac{k}{n + 1}.$$

\[ \square \]

3.5 Conclusion

In this chapter we study the harmonic status coindex of some standard class of graphs. Further we have studied the harmonic status coindex of line graph,. We obtained the bounds for harmonic status coindex of line graphs of some standard class of graphs.