Chapter 6

Status Hosoya Polynomial of Some Transmission Regular Graphs
6.1 Introduction

The concept of counting polynomial was first introduced in chemistry by Polya [77] in 1936. However, the subject received little attention from chemists for several decades even though the spectra of the characteristic polynomial of graphs were studied extensively by numerical means in order to obtain the molecular orbitals of unsaturated hydrocarbons.

The Hosoya polynomial of a graph was introduced in the Hosoyas seminal paper [55] in 1988 and received a lot of attention afterwards. The polynomial was later independently introduced and considered by Sagan, Yeh, and Zhang [96] under the name Wiener polynomial of a graph. Both names are still used for the polynomial but the term Hosoya polynomial is nowadays used by the majority of researchers.

The main advantage of the Hosoya polynomial is that it contains a wealth of information about distance based graph invariants. For instance, knowing the Hosoya polynomial of a graph, it is straightforward to determine the Wiener index of a graph as the first derivative of the polynomial at the point \( t = 1 \). Cash [20] noticed that the hyper-Wiener index can be obtained from the Hosoya polynomial in a similar simple manner. Also, Estrada et al. [35] studied several chemical applications of the Hosoya polynomial.
If $G$ is a connected graph with $n$ vertices, and if $d(G,k)$ is the number of (unordered) pairs of its vertices that are at distance $k$, then the Hosoya polynomial of $G$ is defined as

$$H(G,x) = \sum_{k \geq 0} d(G,k)x^k.$$ 

The status Hosoya polynomial is defined as

$$H_S(G,x) = \sum_{uv \in E(G)} x^{\sigma(u)+\sigma(v)}.$$ 

where, $\sigma(u)$ is the status of a vertex $u$.

Consider the Figure 6.1

![Figure 6.1](image)

$\sigma(v_1) = 12, \ \sigma(v_2) = 8, \ \sigma(v_3) = 8, \ \sigma(v_4) = 8, \ \sigma(v_5) = 12$ and $\sigma(v_6) = 8$

$$H_S(G,x) = 2x^{20} + 4x^{16}$$

**Proposition 6.1.1.** Let $G$ be a connected graph with $n$ vertices and $m$ edges. Let $\text{diam}(G) \leq 2$. Then

$$H_S(G,x) = x^{A(n-1)} \sum_{uv \in E(G)} x^{-[d(u)+d(v)]}.$$
Proof. If $diam(G) \leq 2$, then $d(u)$ number of vertices are at distance 1 from the vertex $u$ and the remaining $(n - 1 - d(u))$ vertices are at distance 2. Hence $\sigma(u) = 2(n - 1) - d(u)$, for any vertex $u \in V(G)$, therefore, $\sigma(u) + \sigma(v) = 4(n - 1) - [d(u) + d(v)]$. Hence,

$$H_S(G, x) = \sum_{uv \in E(G)} x^{\sigma(u) + \sigma(v)}$$

$$= \sum_{uv \in E(G)} x^{4(n-1) - [d(u) + d(v)]}$$

$$= x^{4(n-1)} \sum_{uv \in E(G)} x^{-[d(u) + d(v)]}.$$

$\Box$

**Proposition 6.1.2.** Let $G$ be any connected graph on $n$ vertices and $m$ edges. Then the first status connectivity index $S_1(G) = \frac{d}{dx} H_S(G, x) \bigg|_{x=1}$.

**Corollary 6.1.3.** Let $G$ be a connected $r$-regular graph on $n$ vertices and $m$ edges. Let $diam(G) \leq 2$. Then

$$H_S(G, x) = mx^{2[2(n-1)-r]}.$$  

Proof. Since degree of each vertex is $r$, then by Proposition 6.1.1 we have,

$$H_S(G, x) = \sum_{uv \in E(G)} \frac{x^{4(n-1)}}{x^{2r}}$$

$$= mx^{2[2(n-1)-r]}.$$  

$\Box$
Corollary 6.1.4. For a complete bipartite graph $K_{p,q}$ on $n = p + q$ vertices, $H_S(K_{p,q}, x) = pqx^{3(p+q)-4}$.

Proof. Let $V_1$ and $V_2$ be the partite sets of $V(K_{p,q})$ such that $|V_1| = p$, $|V_2| = q$ and each edge of $K_{p,q}$ has one end in $V_1$ and other in $V_2$. Therefore $d(u) = q$ if $u \in V_1$ and $d(u) = p$ if $u \in V_2$. Also $diam(K_{p,q}) = 2$. Therefore by Proposition 6.1.1

$$H_S(K_{p,q}, x) = x^{4(p+q)-1} \sum_{uv \in E} \frac{1}{x^{p+q}} = pqx^{3(p+q)-4}.$$  

$\square$

Proposition 6.1.5. For a cycle $C_n$ on $n \geq 3$ vertices,

$$H_S(C_n, x) = \begin{cases} nx^{\frac{n^2}{2}}, & \text{if } n \text{ is even} \\ nx^{\frac{n^2-1}{2}}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If $n$ is even number then for every vertex $u$ of $C_n$,

$$\sigma(u) = 2 \left[ 1 + 2 + \cdots + \frac{n-1}{2} \right] + \frac{n}{2} = \frac{n^2}{4}.$$  

Therefore

$$H_S(C_n, x) = \sum_{uv \in E(G)} x^{\frac{n^2}{2}} = nx^{\frac{n^2}{2}}.$$  

If $n$ is odd number then for every vertex $u$ of $C_n$,

$$\sigma(u) = 2 \left[ 1 + 2 + \cdots + \frac{n-1}{2} \right] = \frac{n^2-1}{4}.$$
Therefore
\[ H_S(C_n, x) = \sum_{uv \in E(G)} x^{n-1} = nx^{n-1}. \]

\[ \square \]

A wheel \( W_{n+1} \) is a graph obtained from the cycle \( C_n, n \geq 3 \), by adding a new vertex and making it adjacent to all the vertices of \( C_n \). The degree of a central vertex of \( W_{n+1} \) is \( n \) and the degree of all other vertices is 3.

**Proposition 6.1.6.** For a wheel \( W_{n+1} \), \( n \geq 3 \),
\[ H_S(W_{n+1}, x) = n[x^{3(n-1)} + x^{2(2n-3)}]. \]

**Proof.** The wheel graph \( W_{n+1} \) has \( n + 1 \) vertices and \( 2n \) edges. Also \( diam(W_{n+1}) = 2 \). The edge set \( E(W_{n+1}) \) can be partitioned into two sets \( E_1 \) and \( E_2 \), such that \( E_1 = \{uv \mid d(u) = n \text{ and } d(v) = 3\} \) and \( E_2 = \{uv \mid d(u) = 3 \text{ and } d(v) = 3\} \). It is easy to check that \( |E_1| = n \) and \( |E_2| = n \). Therefore from Proposition 6.1.1 we get the result.
\[
\begin{align*}
H_S(W_{n+1}, x) &= x^{4(n+1-1)} \left[ \sum_{uv \in E_1} \frac{1}{x^n + 3} + \sum_{uv \in E_2} \frac{1}{x^6} \right] \\
&= x^{4n} \left[ \frac{n}{x^{n+3}} + \frac{n}{x^6} \right] \\
&= n \left[ x^{3(n-1)} + x^{2(2n-3)} \right].
\end{align*}
\]

\[ \square \]

A friendship graph (or Dutch windmill graph) \( F_n, n \geq 2 \), is a graph that can be constructed by coalescence \( n \) copies of the cycle \( C_3 \) of
length 3 with a common vertex. It has $2n + 1$ vertices and $3n$ edges. The degree of a coalescence vertex of $F_n$ is $2n$ and the degree of all other vertices is 2.

**Proposition 6.1.7.** For a friendship graph $F_n$, $n \geq 2$,

$$H_S(F_n, x) = n[2x^{2(3n-1)} + x^{4(2n-1)}].$$

**Proof.** The edge set $E(F_n)$ can be partitioned into two sets $E_1$ and $E_2$, such that $E_1 = \{uv \mid d(u) = 2n \text{ and } d(v) = 2\}$ and $E_2 = \{uv \mid d(u) = 2 \text{ and } d(v) = 2\}$. It is easy to check that $|E_1| = 2n$ and $|E_2| = n$. Therefore $M_1(F_n) = 4n^2 + 8n$ and $M_2(F_n) = 8n^2 + 4n$. Also $diam(F_n) = 2$. Therefore from Proposition 6.1.1 we get the result.

$$H_S(F_n, x) = x^{4(2n+1-1)} \left[ \sum_{uv \in E_1} \frac{1}{x^{2(n+1)}} + \sum_{uv \in E_2} \frac{1}{x^4} \right]$$

$$= x^{8n} \left[ \frac{2n}{x^{2(n+1)}} + \frac{n}{x^4} \right]$$

$$= n[2x^{2(3n-1)} + x^{4(2n-1)}].$$

\[\square\]

### 6.2 Status Hosoya Polynomial of Some Graphs Obtained from the Complete Graph

**Definition 6.2.1.** [47] Let $e_i$, $i = 1, 2, \ldots, k$, $1 \leq k \leq n - 2$, be the distinct edges of a complete graph $K_n$, $n \geq 3$, all being incident
to a single vertex. The graph $K_n^k(k)$ is obtained by deleting $e_i$, $i = 1, 2, \ldots, k$ from $K_n$. In addition $K_n^0(0) \cong K_n$.

**Definition 6.2.2.** [47] Let $f_i$, $i = 1, 2, \ldots, k$, $1 \leq k \leq \lfloor n/2 \rfloor$ be independent edges of the complete graph $K_n$, $n \geq 3$. The graph $K^b_n(k)$ is obtained by deleting $f_i$, $i = 1, 2, \ldots, k$ from $K_n$. In addition $K^b_n(0) \cong K_n$.

**Definition 6.2.3.** [47] Let $V_k$ be a $k$-element subset of the vertex set of the complete graph $K_n$, $2 \leq k \leq n - 1$, $n \geq 3$. The graph $K^{c_n}(k)$ is obtained by deleting from $K_n$ all the edges connecting pairs of vertices from $V_k$. In addition $K^{c_n}(0) \cong K^{c_n}(1) \cong K_n$.

**Definition 6.2.4.** [47] Let $3 \leq k \leq n$, $n \geq 3$. The graph $K^{d_n}(k)$ is obtained by deleting from $K_n$, the edges belonging to a $k$-membered cycle.

**Proposition 6.2.5.** For $n \geq 3$ and $1 \leq k \leq n - 2$,

$$
H_S(K_n^k(k), x) = \frac{x^{2(n-1)}}{2} \left[(n-k-1)[2x^k + k(2x - 1) + n - 2] + k(k-1)x^2 \right].
$$

**Proof.** The graph $K_n^k(k)$ has $n$ vertices, $\frac{n(n-1)}{2} - k$ edges and $\text{diam}(K_n^k(k)) = 2$. The edge set $E(K_n^k(k))$ can be partitioned into four sets $E_1$, $E_2$, $E_3$ and $E_4$, where $E_1 = \{uv \mid d(u) = n - 1 - k \text{ and } d(v) = n - 1\}$, $E_2 = \{uv \mid d(u) = n - 2 \text{ and } d(v) = n - 2\}$, $E_3 = \{uv \mid d(u) = n - 2 \text{ and } d(v) = n - 1\}$ and $E_4 = \{uv \mid d(u) = n - 1 \text{ and } d(v) = n - 1\}$. It is easy to check that $|E_1| = n - k - 1$, $|E_2| = k(k - 1)/2$, $|E_3| = (n-k-1)(n-k)/2$, $|E_4| = (n-k-1)/2$, $|E_5| = \frac{n(n-1)}{2} - k$. Since $K_n^k(k)$ is not regular, the Tutte polynomial can be written as

$$
H_S(K_n^k(k), x) = \frac{x^{2(n-1)}}{2} \left[(n-k-1)[2x^k + k(2x - 1) + n - 2] + k(k-1)x^2 \right].
$$
\[ |E_3| = (n - k - 1)k \text{ and } |E_4| = (n - k - 1)(n - k - 2)/2. \] Therefore, from Proposition 6.1.1 we get the result.

**Proposition 6.2.6.** For \( n \geq 3 \) and \( 1 \leq k \leq \lfloor n/2 \rfloor \),

\[
H_S(Kb_n(k) , x) = (n-2k) \left[ 2kx^{2n-1} + \frac{n-2k-1}{2}x^{2(n-1)} \right] + \left( \frac{2k(2k-1)}{2} - k \right) x^{2n}.
\]

**Proof.** The graph \( Kb_n(k) \) has \( n \) vertices, \( \frac{n(n-1)}{2} - k \) edges and \( diam(Kb_n(k)) = 2 \). The edge set \( E(Kb_n(k)) \) can be partitioned into three sets \( E_1 \), \( E_2 \), and \( E_3 \), where \( E_1 = \{ uv \mid d(u) = n - 2 \text{ and } d(v) = n - 1 \} \), \( E_2 = \{ uv \mid d(u) = n - 1 \text{ and } d(v) = n - 1 \} \) and \( E_3 = \{ uv \mid d(u) = n - 2 \text{ and } d(v) = n - 2 \} \). It is easy to check that \( |E_1| = 2k(n - 2k) \), \( |E_2| = (n - 2k)(n - 2k - 1)/2 \) and \( |E_3| = (2k(2k - 1)/2) - k \). Therefore, from Proposition 6.1.1 we get the result.

**Proposition 6.2.7.** For \( n \geq 3 \) and \( 1 \leq k \leq \lfloor n/2 \rfloor \),

\[
H_S(Kc_n(k) , x) = (n - k) \left[ kx^{2n+k-3} + \frac{n-k-1}{2}x^{2(n-1)} \right].
\]

**Proof.** The graph \( Kc_n(k) \) has \( n \) vertices and \( \frac{1}{2} (n - k)(n + k - 1) \) edges. Also \( diam(Kc_n(k)) = 2 \). The edge set \( E(Kc_n(k)) \) can be partitioned into two sets \( E_1 \) and \( E_2 \), where \( E_1 = \{ uv \mid d(u) = n - k \text{ and } d(v) = n - 1 \} \) and \( E_2 = \{ uv \mid d(u) = n - 1 \text{ and } d(v) = n - 1 \} \). It is easy to check that \( |E_1| = (n - k)k \) and \( |E_2| = (n - k)(n - k - 1)/2 \). Therefore, from Proposition 6.1.1 we get the result.

**Proposition 6.2.8.** For \( 3 \leq k \leq n \) and \( n \geq 5 \),

\[
H_S(Kd_n(k) , x) = \left( \frac{k(k-1)}{2} - k \right) x^{2(n+1)} + (n-k) \left[ kx^{2n} + \frac{n-k-1}{2}x^{2(n-1)} \right].
\]

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Proof. The graph $Kd_n(k)$ has $n$ vertices and $(n(n-1)/2) - k$ edges. Also $diam(Kd_n(k)) = 2$. The edge set $E(Kd_n(k))$ can be partitioned into three sets $E_1$, $E_2$ and $E_3$, where $E_1 = \{ uv \mid d(u) = n - 3 \text{ and } d(v) = n - 3 \}$, $E_2 = \{ uv \mid d(u) = n - 3 \text{ and } d(v) = n - 1 \}$ and $E_3 = \{ uv \mid d(u) = n - 1 \text{ and } d(v) = n - 1 \}$. It is easy to check that $|E_1| = (k(k-1)/2) - k$, $|E_2| = (n-k)k$ and $|E_3| = (n-k)(n-k-1)/2$. Therefore, from Proposition 6.1.1 we get the result. \qed

6.3 Status Hosoya Polynomial of Some Transmission Regular Graphs

A bijection $\alpha$ on $V(G)$ is called an automorphism of $G$ if it preserves $E(G)$. In other words, $\alpha$ is an automorphism if for each $u, v \in V(G)$, $e = uv \in E(G)$ if and only if $\alpha(e) = \alpha(u)\alpha(v) \in E(G)$. Let

$$Aut(G) = \{ \alpha \mid \alpha : V(G) \to V(G) \text{ is a bijection, which preserves the adjacency} \}.$$ 

It is known that $Aut(G)$ forms a group under the composition of mappings. A graph $G$ is called vertex-transitive if for every two vertices $u$ and $v$ of $G$, there exists an automorphism $\alpha$ of $G$ such that $\alpha(u) = \alpha(v)$. It is known that any vertex-transitive graph is vertex degree regular, transmission regular and self-centred. Indeed, the graph depicted in Figure 6.1 is 14-transmission regular graph but not degree regular and therefore not vertex-transitive (see [2, 4]).
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Figure 6.1: The transmission regular but not degree regular graph with the smallest order.

**Theorem 6.3.1** ([5]). Let $G$ be a connected graph on $n$ vertices with the automorphism group $\text{Aut}(G)$ and the vertex set $V(G)$. Let $V_1, V_2, \ldots, V_t$ be all orbits of the action $\text{Aut}(G)$ on $V(G)$. Suppose that for each $1 \leq i \leq t$, $k_i$ are the transmission of vertices in the orbit $V_i$, respectively. Then

$$W(G) = \frac{1}{2} \sum_{i=1}^{t} |V_i|k_i.$$ 

Specially if $G$ is vertex-transitive (i.e., $t = 1$), then $W(G) = \frac{1}{2}nk$, where $k$ is the transmission of each vertex of $G$ respectively.

Analogous to Theorem 6.3.1 and as a consequence of Proposition 6.1.1, we have the following

**Lemma 6.3.2.** Let $G$ be a connected $k$-transmission regular graph with $m$ edges. Then $H_s(G, x) = mx^{2k}$.

**Theorem 6.3.3.** Let $G$ be a connected graph on $n$ vertices with the automorphism group $\text{Aut}(G)$ and the vertex set $V(G)$. Let $V_1, V_2, \ldots, V_t$
be all orbits of the action $\text{Aut}(G)$ on $V(G)$. Suppose that for each $1 \leq i \leq t$, $d(i)$ and $k_i$ are the vertex degree and the transmission of vertices in the orbit $V_i$, respectively. Then if $G$ is vertex-transitive (i.e., $t = 1$), then

$$H_S(G, x) = \frac{nd}{2}x^{2k}$$

where $d$ and $k$ are the degree and the transmission of each vertex of $G$ respectively.

Following [52] we recall intersection graphs as follows. Let $S$ be a set and $F = \{S_1, \ldots, S_q\}$ be a non-empty family of distinct non-empty subsets of $S$ such that $S = \bigcup_{i=1}^q S_i$. The intersection graph of $S$ which is denoted by $\Omega(F)$ has $F$ as its set of vertices and two distinct vertices $S_i, S_j$, $i \neq j$, are adjacent if and only if $S_i \cap S_j \neq \emptyset$. Here we will consider a set $S$ of cardinality $p$ and let $F$ be the set of all subsets of $S$ of cardinality $t$, $1 < t < p$, which is denoted by $S^{(t)}$. Upon convenience we may set $S = \{1, 2, \ldots, p\}$. Let us denote the intersection graph $\Omega(S^{(t)})$ by $\Gamma^{(t)} = (V, E)$. The number of vertices of this graph is $|V| = \binom{p}{t}$, the degree $d$ of each vertex is as follows:

$$d = \begin{cases} \binom{p}{t} - \binom{p-t}{t} - 1, & p \geq 2t; \\ \binom{p}{t} - 1, & p < 2t. \end{cases}$$

The number of its edges is as follows:

$$|E| = \begin{cases} \frac{1}{2}\binom{p}{t}\left(\binom{p}{t} - \binom{p-t}{t} - 1\right), & p \geq 2t; \\ \frac{1}{2}\binom{p}{t}\left(\binom{p}{t} - 1\right), & p < 2t. \end{cases}$$
Lemma 6.3.4 ([25]). The intersection graph $\Gamma^{[t]}$ is vertex-transitive and for any $t$-element subset $A$ of $S$ we have

$$\sigma_{\Gamma^{[t]}}(A) = \begin{cases} \binom{p}{t} + \binom{p-t}{t} - 1, & p \geq 2t; \\ \binom{p}{t} - 1, & p < 2t. \end{cases}$$

Theorem 6.3.5.

$$H_S(\Gamma^{[t]}, x) = \begin{cases} \frac{1}{2} \binom{p}{t} \binom{p}{t} - \binom{p-t}{t} - 1 \right) x^{2^{\binom{p-t}{t} + \binom{p-t}{t}-1}} , & p \geq 2t; \\ \binom{p}{t} \binom{p}{t} - 1 \right) x^{2^{\binom{p-t}{t}-1}} , & p < 2t. \end{cases}$$

Proof. Follows directly from the consequence of Theorem 6.3.3 and Lemma 6.3.4. \qed

Theorem 6.3.6. For hypercube $H_n$

$$H_S(H_n, x) = n2^{n-1} x^{2^n}.$$ 

Proof. The vertex set of the hypercube $H_n$ consists of all $n$-tuples $(b_1, b_2, \cdots, b_n)$ with $b_i \in \{0, 1\}$. Two vertices are adjacent if the corresponding tuples differ in precisely one place. Moreover, $H_n$ has exactly $2n$ vertices and $n2^{n-1}$ edges. Darafsheh [25] proved that $H_n$ is vertex-transitive and for every vertex $u, \sigma_{H_n}(u) = n2^{n-1}$. Therefore, it follows from Lemma 6.3.2 that

$$H_S(H_n, x) = n2^{n-1} x^{2^n}.$$ \qed
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Figure 6.2: The odd graph $O_3 = KG_{5,2}$ is isomorphic to the Peterson graph

The Kneser graph $KG_{p,k}$ is the graph whose vertices correspond to the $k$-element subsets of a set of $p$ elements, and where two vertices are adjacent if and only if the two corresponding sets are disjoint. Clearly we must impose the restriction $p \geq 2k$. The Kneser graph $KG_{p,k}$ has $\binom{p}{k}$ vertices and it is regular of degree $\binom{p-k}{k}$. Therefore the number of edges of $KG_{p,k}$ is $\frac{1}{2}\binom{p}{k}\binom{p-k}{k}$ (see [71]). The Kneser graph $KG_{n,1}$ is the complete graph on $n$ vertices. The Kneser graph $KG_{2p-1,p-1}$ is known as the odd graph $O_p$. The odd graph $O_3 = KG_{5,2}$ is isomorphic to the Peterson graph (see Figure 6.2).

Lemma 6.3.7 ([71]). The Kneser graph $KG_{p,k}$ is vertex-transitive and for each $k$-subset $A$, $\sigma_{KG_{p,k}}(A) = \frac{2W(KG_{p,k})}{\binom{p}{k}}$.

A nanostructure called achiral polyhex nanotorus (or toroidal fullerenes) of perimeter $p$ and length $q$, denoted by $T[p,q]$ is depicted in Figure 6.3 and its 2-dimensional molecular graph is in Figure 6.4. It is regular
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Figure 6.3: A achiral polyhex nanotorus (or toroidal fullerene) $T[p, q]$

Figure 6.4: A 2-dimensional lattice for an achiral polyhex nanotorus $T[p, q]$ of degree 3 and has $pq$ vertices and $\frac{3pq}{2}$ edges.

The following lemma was proved in [5] and [108].

**Lemma 6.3.8** ([5],[108]). The achiral polyhex nanotorus $T = T[p, q]$ is vertex transitive such that for an arbitrary vertex $u \in V(T)$

\[
\sigma_T(u) = \begin{cases} 
\frac{q}{12}(6p^2 + q^2 - 4), & q < p; \\
\frac{p}{12}(3q^2 + 3pq + p^2 - 4), & q \geq p.
\end{cases}
\]

The following is a direct consequence of Lemma 6.3.2 and Lemma 6.3.8.

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Corollary 6.3.9. Let $T = T[p, q]$ be a achiral polyhex nanotorus. Then

$$H_S(G, x) = \begin{cases} 
\frac{q}{2} x^{\frac{1}{6}(6q^2 + q^2 - 4)}, & q < p; \\
\frac{p}{2} x^{\frac{1}{6}(3q^2 + 3pq + p^2 - 4)}, & q \geq p.
\end{cases}$$

6.4 Conclusion

In this chapter Hosoya polynomial has been studied on the basis of status. This polynomial has been obtained for some standard class of graphs and specially giving importance to transmission regular graphs.