Chapter 4

$P_k$ and $P'_k \Gamma$ - Seminear-rings

In this chapter, we introduce the concepts of $P_k$ and $P'_k \Gamma$ - seminear-rings. We say that $R$ is a $P_k \Gamma$ - seminear-ring ($P'_k \Gamma$ - seminear-ring) if for all $x$ in $R$ there exists a positive integer $k$ such that $x^k \gamma R = x \gamma R \gamma x$ ($R \gamma x^k = x \gamma R \gamma x$).

This chapter is divided into three sections. In the first section, we give examples to show that each of these concepts - $P_k$ and $P'_k$ - are different in general.

In the second section of this chapter, we derive properties of $P_k$ and $P'_k \Gamma$ - seminear-rings. We prove that any homomorphic image of a $P_k (P'_k) \Gamma$ - seminear-ring is a $P_k (P'_k) \Gamma$ - seminear-ring.

We also prove that a left identity (right identity) of a $P_k (P'_k) \Gamma$ - seminear-ring is also a right identity (left identity). We show by an example that a right identity of a $P_k \Gamma$ - seminear-ring need not be a left identity. We establish that a $P_k (P'_k) \Gamma$ - seminear-ring $R$ has a mate function if and only if $R$ is a right-$k$-normal (left-$k$-normal) $\Gamma$. 

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- seminear-ring. We prove a theorem using mate functions. Let $R$ be a $P_k$ or a $P'_k \Gamma$ - seminear-ring. If $R$ admits mate functions then $R$ has no non-zero nilpotent elements i.e. $L = \{0\}$. We prove that every $P_k (P'_k) \Gamma$ - seminear-ring is isomorphic to a sub-direct product of sub-directly irreducible $P_k(P'_k) \Gamma$ - seminear-rings.

If $x \in R \gamma x$ ($x \in x \gamma R$) for all $x \in R$ then $R$ is called a left normal (right normal) $\Gamma$ - seminear-ring. Motivated by this we introduce the concepts of left-$r$-normal and right-$r$-normal $\Gamma$ - seminear-rings, where $r$ is a positive integer. $R$ is called a left-$r$-normal (right-$r$-normal) $\Gamma$ - seminear-ring if $x \in R \gamma x^r$ ($x \in x^r \gamma R$). We have shown that every left-$r$-normal (right-$r$-normal)$\Gamma$ - seminear-ring is a left normal (right normal)$\Gamma$ - seminear-ring.

We discuss the properties of $P_k$ and $P'_k \Gamma$ - seminear-rings when they admit mate functions. We obtain a necessary and sufficient condition for a $P_k (P'_k) \Gamma$ - seminear-ring to admit mate functions vis-a-vis the notion of left-$r$-normal (right-$r$-normal). It is shown that every ideal of a left-$k$-normal $P'_k \Gamma$ - seminear-ring (right-$k$-normal $P_k \Gamma$ - seminear-ring) is also a left-$k$-normal $P'_k \Gamma$ - seminear-ring (right-$k$-normal $P_k \Gamma$ - seminear-ring) in its own right.

In the third section of this chapter, we focus on the properties of $P_1$ and $P'_1 \Gamma$ - seminear-rings. We prove that every left ideal (right ideal) of $R$ is a right ideal (left ideal) if and only if $R$ is a $P_1(P'_1) \Gamma$ - seminear-ring. We also prove that when $R$ is a left normal $P'_1 \Gamma$ -
seminear-ring.

(i) \( M \cap N = M \Gamma N \) where \( M \) and \( N \) are ideals of \( R \)

(ii) Any prime ideal is a completely prime ideal.

(iii) \( R \) has \((\ast, IFP)\)

### 4.1 Definition and Examples

In this section, we define \( P_k \) and \( P'_k \Gamma \) - seminear-rings and furnish examples of these concepts.

**Definition 4.1.1.** A \( \Gamma \) - seminear-ring \( R \) is called a \( P_k \Gamma \) - seminear-ring (\( P'_k \Gamma \) - seminear-ring) if there exists a positive integer \( 'k' \) such that \( x^k \gamma R = x \gamma R \gamma x \) \((R \gamma x^k = x \gamma R \gamma x)\) for all \( x \) in \( R \) and \( \gamma \in \Gamma \).

**Example 4.1.2.** (i) Let \( R = \{0, a, b, c, d\} \). We define the semigroup operations “+” and “\( \gamma \)” in \( R \) as follows.

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Then \((R, +, \gamma)\) is a \( P_k \) as well as a \( P'_k \Gamma \) - seminear-ring for all positive integers \( k \).
(ii) We consider the Γ seminear-ring where $R = \{0, a, b, c, d\}$ and the semigroup operations “+” and “$\gamma$” are defined as follows.

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Then $(R, +, \gamma)$ is a $P_k \Gamma$-seminear-ring for all positive integers $k$ but not a $P'_k \Gamma$-seminear-ring for any positive integer $k$.

(iii) The direct product of any two seminear-fields is a $P_k$ as well as a $P'_k \Gamma$-seminear-ring.

**Definition 4.1.3.** Let ‘$r$’ be a positive integer. We say that $R$ is a left-$r$-normal (right-$r$-normal) $\Gamma$-seminear-ring if $a \in R\gamma a^r (a \in a^r \gamma R)$ for all $a$ in $R$ and $\gamma \in \Gamma$.

**Example 4.1.4.** (i) The $\Gamma$ seminear-ring of example 4.1.2(i) is a left-$r$-normal as well as a right-$r$-normal $\Gamma$ seminear-ring.

(ii) Trivially any Boolean $\Gamma$ seminear-ring is a left-$r$-normal as well as a right-$r$-normal $\Gamma$ seminear-ring.
4.2 Properties of $P_k$ and $P'_k \Gamma$ - Seminear-rings

In this section, we study some of the important properties of $P_k$ and $P'_k \Gamma$ - seminear-rings and give complete characterizations of such seminear-rings.

**Theorem 4.2.1.** Any homomorphic image of a $P_k (P'_k) \Gamma$ - seminear-ring is a $P_k (P'_k) \Gamma$ - seminear-ring.

**Proof.** Let $R$ be a $P_k \Gamma$ - seminear-ring and let $f : R \rightarrow R'$ be a seminear-ring epimorphism. As $R$ is a $P_k \Gamma$ - seminear-ring $x^k \gamma R = x\gamma R \gamma x$, for all $x$ in $R$. Now for every $r' \in R'$ and for every $x$ in $R$ and $\gamma \in \Gamma$, we have $(f(x))^k \gamma r' = f(x^k) \gamma f(r)$ (for some $r$ in $R$) $= f(x^k \gamma r) = f(x \gamma y \gamma x)$ (for some $y$ in $R$) $= f(x) \gamma f(y) \gamma f(x)$. This guarantees, $f(x)^k \gamma R' \subseteq f(x) \gamma R' \gamma f(x)$. Similarly we get $f(x) \gamma R' \gamma f(x) \subseteq f(x)^k \gamma R'$. Hence $f(x^k) \gamma R' = f(x) \gamma R' \gamma f(x)$. i.e. $R'$ is a $P_k \Gamma$ - seminear-ring.

The proof is similer when $R$ is a $P'_k$ seminear-ring also. $\square$

**Theorem 4.2.2.** Every $P_k (P'_k) \Gamma$ - seminear-ring $R$ is isomorphic to a subdirect product of subdirectly irreducible $P_k (P'_k) \Gamma$ - seminear-rings.

**Proof.** By Theorem 1.4.16, $R$ is isomorphic to a subdirect product of subdirectly irreducible $P_k (P'_k) \Gamma$ - seminear-rings. $\square$
Proposition 4.2.3. Every left-$r$-normal (right-$r$-normal) $\Gamma$-seminear-ring is a left (right) normal $\Gamma$-seminear-ring.

**Proof.** Let $R$ be a left-$r$-normal $\Gamma$-seminear-ring with $r \geq 2$. Clearly then for all $a \in R$ and $\gamma \in \Gamma$. Now $a \in R\gamma a^r = (R\gamma a^{r-1})\gamma a \subseteq R\gamma a$. i.e. $a \in R\gamma a$. Therefore $R$ is a left normal $\Gamma$-seminear-ring. $\square$

Proposition 4.2.4. A left identity (right identity) of a $P_k(P'_k)$ $\Gamma$-seminear-ring is also a right identity(left identity).

**Proof.** Case (i) Let $R$ be a $P_k \Gamma$-seminear-ring. Let $e'$ be a left identity of $R$. Then $x = e'x$ for all $x \in R$ and $\gamma \in \Gamma$. Now $e'\gamma R = e\gamma R \Rightarrow x = e'x = e'\gamma y = (e\gamma y)\gamma e = y\gamma e$. Hence $x\gamma e = (y\gamma e)\gamma e = y\gamma e^2 = y\gamma e = x$. i.e. $x = e\gamma x = x\gamma e$. Therefore $e'$ is a right identity as well.

Case (ii) Let $R$ be a $P'_k \Gamma$-seminear-ring and $e$ be a right identity of $R$. Then $x = x\gamma e$ for all $x \in R$ and $\gamma \in \Gamma$. Now $R\gamma e^k = e'dR\gamma e \Rightarrow R\gamma e = e\gamma R\gamma e$. Then there exists $y' \in R$ such that, $x = x\gamma e = e\gamma y' = e\gamma (y'\gamma e) = e\gamma y'$. Hence $x\gamma e = e\gamma (e\gamma y') = e^2\gamma y' = e\gamma y' = x$. i.e. $x = x\gamma e = e\gamma x$. It follows that $e'$ is also a left identity. $\square$

Remark 4.2.5. A right identity of a $P_k \Gamma$-seminear-ring need not be a left identity. The following example substantiates this. We consider the seminear-ring $R = \{0, a, b, c, d\}$ where the semigroup operations “$+$” and “$\gamma$” in $R$ are defined as follows.

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Here $a$, $b$, $c$, $d$ are right identities but none is a left identity.

We furnish below a characterization of $P_k$ ($P'_k$) $\Gamma$-seminear-rings.

**Theorem 4.2.6.** (i) A $P_k$ $\Gamma$-seminear-ring $R$ has a mate function if and only if $R$ is a right-$k$-normal $\Gamma$-seminear-ring.

(ii) A $P'_k$ $\Gamma$-seminear-ring $R$ has a mate function if and only if $R$ is a left-$k$-normal $\Gamma$-seminear-ring.

**Proof.** (i) Let $R$ be a $P_k$ $\Gamma$-seminear-ring. Then $x^{k\gamma}R = x\gamma R\gamma x$ for all $x$ in $R$ and $\gamma \in \Gamma$. If $R$ has a mate function $f$ then $x = x\gamma f(x)\gamma x \in x\gamma R\gamma x (= x^{k\gamma}R)$ and this implies $x \in x^{k\gamma}R$. i.e. $R$ is a right-$k$-normal $\Gamma$-seminear-ring.

Conversely let $R$ be a right-$k$-normal $P_k$ $\Gamma$-seminear-ring. Therefore $x \in x^{k\gamma}R(= x\gamma R\gamma x)$ for all $x$ in $R$ and $\gamma \in \Gamma$. Then there exists some $y$ in $R$ such that $x = x\gamma y\gamma x$. Clearly then $x = x\gamma f(x)\gamma x$ where we set $f(x) = y$. It follows that $f$ is a mate function for $R$.

(ii) Let $R$ be a $P'_k$ $\Gamma$-seminear-ring. Then $R\gamma x^k = x\gamma R\gamma x$ for all $x$ in $R$ and $\gamma \in \Gamma$. If $R$ has a mate function $f$ then $x = x\gamma f(x)\gamma x$
\( x \gamma R \gamma x \) and this implies \( x \in R \gamma x \). i.e. \( R \) is a left-\( k \)-normal \( \Gamma \) - seminear-ring.

Conversely let \( R \) be a left-\( k \)-normal \( P'_k \) \( \Gamma \) - seminear-ring. Therefore \( x \in R \gamma x^k (= x \gamma R \gamma x) \) for all \( x \) in \( R \) and \( \gamma \in \Gamma \). Then there exists some \( y \) in \( R \) such that \( x = x \gamma y \gamma x \). Clearly then \( x = x \gamma f (x) \gamma x \) where we set \( f (x) = y \). It follows that \( f \) is a mate function for \( R \). \( \square \)

**Theorem 4.2.7.** Let \( R \) be a \( P_k \) or a \( P'_k \) \( \Gamma \) - seminear-ring. If \( R \) admits mate functions then \( R \) has no non-zero nilpotent elements i.e. \( L = \{ 0 \} \).

**Proof.** Let \( R \) admit a mate function \( f \). We shall show that

\[ x^2 = 0 \Rightarrow x = 0 \] for some \( x \) in \( R \)..................(1)

**Case(i):** Let \( R \) be a \( P_1 \) \( \Gamma \) - seminear-ring, i.e. \( x \gamma R = x \gamma R \gamma x \) for all \( x \) in \( R \). We have \( x = x \gamma f (x) \gamma x \in x \gamma R \gamma x \). But \( x \gamma R \gamma x = (x \gamma R) \gamma x = (x \gamma R \gamma x) \gamma x = x \gamma R \gamma x^2 = (x \gamma R) x^2 \). Then there exists \( y \in R \) such that \( x = x \gamma y \gamma x^2 \). Consequently (1) holds.

**Case(ii):** Let \( R \) be a \( P_k \) \( \Gamma \) - seminear-ring with \( k > 1 \). Now \( x^k \gamma R = x \gamma R \gamma x \) for all \( x \) in \( R \) and \( \gamma \in \Gamma \). Since \( x = x \gamma f (x) \gamma x \in x \gamma R \gamma x = x^k \gamma R \), \( x = x^k \gamma y \) for some \( y \) in \( R \). If \( k = 2 \), then \( x = x^2 \gamma y \). If \( k > 2 \), we write \( x = x^2 \gamma (x^{k-2} \gamma y) \) and therefore (1) is true.

**Case(iii):** Let \( R \) be a \( P'_1 \) \( \Gamma \) - seminear-ring, i.e. \( R \gamma x = x \gamma R \gamma x \)

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for all \( x \) in \( R \) and \( \gamma \in \Gamma \). Therefore \( x = x\gamma f(x)\gamma x \in x\gamma R\gamma x = x\gamma (R\gamma x) = x\gamma (x\gamma R\gamma x) = x^2\gamma R\gamma x \Rightarrow x = x^2\gamma (R\gamma x) \). Then there exists \( y \in R \) such that \( x = x^2\gamma y\gamma x \). Thus (1) holds good.

**Case(iv):** Let \( R \) be a \( P_k' \) \( \Gamma \) - seminear-ring with \( k > 1 \). Now \( R\gamma x^k = x\gamma R\gamma x \) for all \( x \) in \( R \) and \( \gamma \in \Gamma \). Since \( x = x\gamma f(x)\gamma x \in x\gamma R\gamma x = R\gamma x^k \), we get \( x = y'\gamma x^k \) for some \( y' \) in \( R \). If \( k = 2 \), then \( x = y'\gamma x^2 \). If \( k > 2 \), we write \( x = (y'\gamma x^{k-2})\gamma x^2 \) and again (1) holds.

Now 1.3.9 guarantees that, in all the four cases, \( L = \{0\} \). \( \square \)

**Theorem 4.2.8.** Let \( R \) be a \( P(1,2) \) \( \Gamma \) - seminear-ring with a mate function \( f \). Then \( R \) is a

(a) \( P_k \) \( \Gamma \) - seminear-ring for all positive integers \( k \).

(b) \( P_k' \) \( \Gamma \) - seminear-ring for all positive integers \( k \).

**Proof.** Since \( R \) is a \( P(1,2) \) \( \Gamma \) - seminear-ring. Proposition 3.2.4 demands that every idempotent is central. i.e \( E \subseteq C(R) \).

(a) **Case (i):** Let \( k = 1 \). For all \( x \) in \( R \), \( x\gamma R = x\gamma (f(x)\gamma x\gamma R) = x\gamma (R\gamma f(x)\gamma x) \) (since \( E \subseteq C(R) \)) = \( x\gamma R\gamma x \) (By 1.4.31(ii)) i.e. \( x\gamma R = x\gamma R\gamma x \). Hence \( R \) is a \( P_1 \) \( \Gamma \) - seminear-ring.

**Case (ii):** For \( k > 1 \) and for any \( x \in R \) and \( \gamma \in \Gamma \), \( x^k\gamma R = x\gamma (x^{k-1}\gamma R) \subseteq x\gamma R = x\gamma R\gamma x \) (using the result for \( k = 1 \)).
Therefore $x^k \gamma R \subseteq x \gamma R \gamma x$. Also $x \gamma R \gamma x = x \gamma R \gamma x \gamma f(x) \gamma x = x \gamma (R \gamma x \gamma f(x) \gamma x) = x \gamma (x \gamma f(x) \gamma R) \gamma x$ (since $E \subseteq C(R)$) $= x \gamma (x \gamma R) \gamma x$ (By 1.4.31(ii)) $= x^2 \gamma R = x \gamma (x \gamma R \gamma x) = x \gamma (x^2 \gamma R \gamma x) = x^3 \gamma R \gamma x$. Repeating this process, we obtain $x \gamma R \gamma x = x^k \gamma R \gamma x \subseteq x^k \gamma R$ for all positive integers $k$. Therefore $x \gamma R \gamma x \subseteq x^k \gamma R$. Thus $x \gamma R \gamma x = x^k \gamma R$ for all $x$ in $R$ and $\gamma \in \Gamma$. Hence $R$ is $P_k \Gamma$-seminear-ring for any positive integer $k$.

(b) Case (i): Let $k = 1$. For all $x$ in $R$ and $\gamma \in \Gamma$, $R \gamma x = R \gamma x \gamma f(x) \gamma x = (R \gamma x \gamma f(x)) \gamma x = (x \gamma f(x) \gamma R) \gamma x$ (since $E \subseteq C(R)$) $= x \gamma R \gamma x$ (By 1.4.31(ii)). i.e. $R \gamma x = x \gamma R \gamma x$. Hence $R$ is a $P'_1 \Gamma$-seminear-ring.

Case (ii): Let $k > 1$. Since $E \subseteq C(R)$ we have for all $y, x$ in $R$ and $\gamma \in \Gamma$, $y \gamma x^k = (y \gamma x) \gamma x^{k-1} = (y \gamma x \gamma f(x) \gamma x) \gamma x^{k-1}$ $= (x \gamma f(x) \gamma y \gamma x) \gamma x^{k-1} = (x \gamma f(x) \gamma y \gamma x) \gamma x \gamma x^{k-1} \gamma x \in x \gamma R \gamma x$.

Therefore $R \gamma x^k \subseteq x \gamma R \gamma x$. Also $x \gamma y \gamma x = (x \gamma f(x) \gamma x) \gamma y \gamma x = x \gamma y \gamma f(x) \gamma x^2 = (x \gamma f(x) \gamma x) \gamma y \gamma f(x) \gamma x^2 = x \gamma y \gamma (f(x))^2 \gamma x^3$.

Repeating this process, we obtain $x \gamma y \gamma x = x \gamma y \gamma (f(x))^{k-1} \gamma x^k \in R \gamma x^k$ for all positive integers $k$. Therefore $x \gamma R \gamma x \subseteq R \gamma x^k$.

Thus $x \gamma R \gamma x = R \gamma x^k$ for all $x$ in $R$ and $\gamma \in \Gamma$.

Hence $R$ is $P'_k \Gamma$-seminear-ring for any positive integer $k$. □

Remark 4.2.9. We observe that $P_k$ and $P'_k \Gamma$-seminear-rings need not be a $P(1,2) \Gamma$-seminear-ring. For example, we consider the
seminear-ring $R = \{0, a, b, c, d\}$ where the semigroup operations “+” and “γ” in $R$ are defined as follows.

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<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
</tbody>
</table>

Here $(R, +, γ)$ is a $P_k$ as well as a $P_k' \Gamma$ - seminear-ring for all positive integers $k$. Even though the identity function serves as a mate function for $R$, it is not a $P(1, 2) \Gamma$ - seminear-ring.

**Proposition 4.2.10.** Let $R$ admit a $P_3$ mate function $f$. Then

(a) Every right ideal of a $P_k \Gamma$ - seminear-ring $R$ is a completely semi prime ideal.

(b) Every left ideal of a $(P_k') \Gamma$ - seminear-ring $R$ is a completely semi prime ideal.

**Proof.** (a) **Case (i):** If $k = 1$. Let $I$ be a right ideal of $R$ and let $a^2 \in I$. Then $a = a\gamma f(a)\gamma a = a\gamma(f(a)\gamma a) = a\gamma(a\gamma f(a))$ (since $f$ is a $P_3$ mate function) = $a^2\gamma f(a) \in I \Gamma R \subseteq I$. i.e. $a \in I$ and the result follows.

**Case (ii):** Let $k > 1$. For $a \in R$ and $\gamma \in \Gamma$, $a = a\gamma f(a)\gamma a \in a\gamma R\gamma a = (a^k\gamma R)$ and therefore there exists $y \in R$ such that
a = a^k \gamma y. When k = 2, a^2 \in I \Rightarrow a = a^2 \gamma y \in I \Gamma R \subseteq I. i.e. a^2 \in I \Rightarrow a \in I. When k > 2, a^2 \in I \Rightarrow a = a^2 \gamma (a^{k-2} \gamma y) \in I \Gamma R \subseteq I. i.e. a \in I and the desired result follows.

(b) Case (i): If k = 1. Let I be a right ideal of R and let a^2 \in I. Then a = a\gamma f(a)\gamma a = (a\gamma f(a))\gamma a = (a\gamma a)\gamma f(a) (since f is a P_3 mate function) = f(a)\gamma a^2 \in R\Gamma I \subseteq I. i.e. a \in I and the result follows.

Case (ii): Let k > 1. For a \in R and \gamma \in \Gamma, a = a\gamma f(a)\gamma a \in a\gamma R\gamma a = R\gamma (a^k) and therefore there exists y \in R such that a = y\gamma a^k. When k = 2, a^2 \in I \Rightarrow a = y\gamma a^2 \in R\Gamma I \subseteq I. i.e. a^2 \in I \Rightarrow a \in I. When k > 2, a^2 \in I \Rightarrow a = (y\gamma a^{k-2} \gamma a^2) \in R\Gamma I \subseteq I. i.e. a \in I and the desired result follows. 

\[\square\]

**Theorem 4.2.11.** (a) Any ideal of a left-\(k\)-normal \(P_k'\) \(\Gamma\) - seminear-ring \(R\) is also a left-\(k\)-normal \(P_k'\) \(\Gamma\) - seminear-ring in its own right.

(b) Any ideal of a right-\(k\)-normal \(P_k\) \(\Gamma\) - seminear-ring \(R\) is also a right-\(k\)-normal \(P_k\) \(\Gamma\) - seminear-ring in its own right.

**Proof.** (a) Since \(R\) is a left-\(k\)-normal \(P_k'\) \(\Gamma\) - seminear-ring Theorem 4.2.6 (ii) guarantees the existence of a mate function \(f\) for \(R\). Let \(M\) be an ideal of \(R\). Therefore \(f(x)\gamma x\gamma f(x) \in R\Gamma M\Gamma R \subseteq M\) for all \(x\) in \(M\). Thus we can define a map \(g : M \rightarrow M\) such that \(g(x) = f(x)\gamma x\gamma f(x)\) for all \(x \in M\) and \(\gamma \in \Gamma\). Obviously then \(x\gamma g(x)\gamma x = x\) and therefore \(g\) is a mate function for \(M\). Now let \(x, a \in M\) and \(\gamma \in \Gamma\). Since \(R\gamma x^k = x\gamma R\gamma x\) there exists \(b \in R\) such
that \( a\gamma x^k = x\gamma b\gamma x = x\gamma (b\gamma x\gamma g(x))\gamma x \in x\gamma (R\Gamma M)\gamma x \subseteq x\gamma M\gamma x \).

Therefore \( M\gamma x^k \subseteq x\gamma M\gamma x \)..........................(1)

Also, since \( x\gamma a\gamma x \in x\gamma R\gamma x = R\gamma x^k \), there exists \( y \in R \) such that \( x\gamma a\gamma x = y\gamma x^k \). Again \( x\gamma a\gamma x = x\gamma g(x)\gamma (x\gamma a\gamma x) = x\gamma g(x)\gamma y\gamma x^k = y'\gamma x^k \) where \( y' = x\gamma g(x)\gamma y \in M\Gamma R \subseteq M \).

Therefore \( x\gamma M\gamma x \subseteq M\gamma x^k \).................................(2)

From (1) and (2) we get \( M\gamma x^k = x\gamma M\gamma x \) for all \( x \in M \) and \( \gamma \in \Gamma \).

i.e. \( M \) is a \( P'_k \Gamma \) - seminear-ring. Since \( M \) has a mate function \( g \), \( M \) is a left-\( k \)-normal seminear-ring as well (from Theorem 4.2.6(ii)).

(b) Since \( R \) is a right-\( k \)-normal \( P_k \Gamma \) - seminear-ring Theorem 4.2.6 (i) guarantees the existence of a mate function \( f \) for \( R \). Let \( M \) be an ideal of \( R \). Therefore \( f(x)\gamma x\gamma f(x) \in R\Gamma M\Gamma R \subseteq M \) for all \( x \) in \( M \).

Thus we can define a map \( g : M \to M \) such that \( g(x) = f(x)\gamma x\gamma f(x) \) for all \( x \in M \) and \( \gamma \in \Gamma \). Obviously then \( x\gamma g(x)\gamma x = x \) and therefore \( g \) is a mate function for \( M \).

Now let \( x, a \in M \) and \( \gamma \in \Gamma \). Since \( x^k\gamma R = x\gamma R\gamma x \) there exists \( b \in R \) such that \( x^k\gamma a = x\gamma b\gamma x = x\gamma (b\gamma x\gamma g(x))\gamma x \in x\gamma (R\Gamma M)\gamma x \subseteq x\gamma M\gamma x \). Therefore \( x^k\gamma M \subseteq x\gamma M\gamma x \)..........................(3)

Also, since \( x\gamma a\gamma x \in x\gamma R\gamma x = x^k\gamma R \), there exists \( y \in R \) such that \( x\gamma a\gamma x = x^k\gamma y \). Again \( x\gamma a\gamma x = x\gamma g(x)\gamma (x\gamma a\gamma x) = x\gamma g(x)\gamma x^k\gamma y = x^k\gamma y' \) where \( y' = x\gamma g(x)\gamma y \in M\Gamma R \subseteq M \).

Therefore \( x\gamma M\gamma x \subseteq x^k\gamma M \)..........................(4)

From (3) and (4) we get \( x^k\gamma M = x\gamma M\gamma x \) for all \( x \in M \) and
$\gamma \in \Gamma$. i.e. $M$ is a $P_k \Gamma$ - seminear-ring. Since $M$ has a mate function $g$ then $M$ is a right-k-normal seminear-ring as well (from Theorem 4.2.6 (i)).

\[ \square \]

**Proposition 4.2.12.** (a) Let $R$ be a $P_k \Gamma$ - seminear-ring. Then $R$ satisfies left Ore condition.

(b) Let $R$ be a $P'_k \Gamma$ - seminear-ring. Then $R$ satisfies right Ore condition.

**Proof.** (a) Let $A$ be any subsemigroup of $R$ and let $a \in A$, $r \in R$ and $\gamma \in \Gamma$. Since $a^k \gamma R = a \gamma R \gamma a$ there exists $y \in R$ such that $a^k \gamma r = a \gamma y \gamma a$. i.e. $a_1 \gamma r = a_1 \gamma r_1$ where $a_1 = a^k \in A$ and $r_1 = y \gamma a \in R$ and $R$ fulfills the left Ore condition.

(b) Let $A$ be any subsemigroup of $R$ and let $a \in A$, $r \in R$ and $\gamma \in \Gamma$. Since $R \gamma a^k = a \gamma R \gamma a$ there exists $y \in R$ such that $r \gamma a^k = a \gamma y \gamma a$. i.e. $a_1 \gamma r = a_1 \gamma r_1$ where $a_1 = a^k \in A$ and $r_1 = y \gamma a \in R$ and $R$ fulfills the right Ore condition. \[ \square \]

4.3 **Properties of $P_1$ and $P'_1 \Gamma$ - seminear-rings**

In this section, we prove certain important properties and characterizations of $P_1$ and $P'_1 \Gamma$ - seminear-rings.

**Theorem 4.3.1.** Let $R$ be a $\Gamma$ - seminear-ring with a mate function $f$. Then we have
(i) every left ideal of \( R \) is a right ideal of \( R \) if and only if \( R \) is a \( P_1 \Gamma \) - seminear-ring.

(ii) every right ideal of \( R \) is a left ideal of \( R \) if and only if \( R \) is a \( P'_1 \Gamma \) - seminear-ring.

**Proof.** By hypothesis \( R \) is a \( \Gamma \) - seminear-ring with a mate function \( f \).

(i) Assume that every left ideal of \( R \) is a right ideal of \( R \). By the assumption, \( R\gamma x \), being a left ideal for every \( x \in R \) and \( \gamma \in \Gamma \), is also a right ideal of \( R \). Therefore \( (R\gamma x)\gamma R \subseteq R\gamma x \). Since \( f \) is a mate function \( x = x\gamma f(x)\gamma x \). From this we get

\[
 x\gamma R = x\gamma f(x)\gamma x\gamma R \subseteq x\gamma R\gamma x \subseteq x\gamma R \gamma x \quad \text{(1)}
\]

\[
 x\gamma R\gamma x \subseteq x\gamma R \quad \text{(2)}
\]

From (1) and (2) we get \( x\gamma R = x\gamma R\gamma x \) for all \( x \in R \) and \( \gamma \in \Gamma \). i.e. \( R \) is a \( P_1 \Gamma \) - seminear-ring.

Conversely, let \( A \) be any left ideal of \( R \), then \( R\Gamma A \subseteq A \).

Let \( a \in A \) and \( y \in R \) and \( \gamma \in \Gamma \), we have \( a\gamma y \in a\gamma R = a\gamma R\gamma a \)

\[
 \Rightarrow a\gamma y = a\gamma y'\gamma a \quad \text{(for some} \ y' \text{ in } R) = (a\gamma y')\gamma a \in R\gamma a.
\]

This forces \( a\gamma y \in R\Gamma A \subseteq A \Rightarrow A\Gamma R \subseteq A \) and hence \( A \) is an ideal.

(ii) Assume that every right ideal of \( R \) is a left ideal of \( R \). By the assumption \( x\gamma R \), being a right ideal for every \( x \in R \), is also a left ideal of \( R \). Therefore \( R\gamma (x\gamma R) \subseteq x\gamma R \). Since \( f \) is a mate function \( x = x\gamma f(x)\gamma x \). From this we get \( R\gamma x = R\gamma x\gamma f(x)\gamma x \in R\gamma x\gamma R\gamma x \)
\[ x\gamma R\gamma x \subseteq x\gamma R \gamma x \] (3)
\[ x\gamma R\gamma x \subseteq R\gamma x \] (4)

From (3) and (4) we get \( x\gamma R = x\gamma R\gamma x \) for all \( x \in R \) and \( \gamma \in \Gamma \).

i.e. \( R \) is a \( P'_{1} \Gamma \) - seminear-ring.

Conversely, let \( A \) be a right ideal of \( R \), then \( A\Gamma R \subseteq A \).

Let \( a \in A \) and \( y \in R \) and \( \gamma \in \Gamma \), we have \( ya \in R\gamma a = a\gamma R\gamma a \)
\[ \Rightarrow y\gamma a = a\gamma y'\gamma a \] (for some \( y' \) in \( R \)) = \( a\gamma (y'\gamma a) \in a\gamma R \). This forces \( y\gamma a \in A\Gamma R \subseteq A \Rightarrow R\gamma A \subseteq A \). Hence \( A \) is an ideal. \( \square \)

**Theorem 4.3.2.** Let \( R \) be a left normal \( P'_{1} \Gamma \) - seminear-ring. Then

(i) \( M \cap N = M\Gamma N \) where \( M \) and \( N \) are ideals of \( R \)

(ii) Any prime ideal is a completely prime ideal.

(iii) \( R \) has \((*, IFP)\)

**Proof.** Since \( R \) is left normal \( P'_{1} \Gamma \) - seminear-ring. Theorem 4.2.6 (ii) guarantees that \( R \) has a mate function \( f \).

(i) If \( M, N \) are ideals of \( R \) then \( (M \cap N)^{2} = (M \cap N)\Gamma(M \cap N) \). Also for all \( a \) in \( M \cap N \), \( a = a\gamma(f(a)\gamma a) \in (M \cap N)\Gamma(M \cap N) \). This forces \( (M \cap N) = (M \cap N)^{2} \). Further, \( (M \cap N) = (M \cap N)\Gamma(M \cap N) \subseteq M\Gamma N \).

To prove the reverse inclusion, let us take \( y \in M\Gamma N \). Clearly then \( y \in M\Gamma N \subseteq N \). Also \( y = x\gamma x' \) for some \( x \) in \( M \) and \( x' \) in \( N \). This demands that \( y \in x\gamma R \). Hence \( y \in x\gamma R \subseteq M\Gamma R \subseteq M \). Thus \( y \in M \cap N \) and the desired result follows.
(ii) Let $P$ be a prime ideal of $R$ and let $a\gamma b \in P$. Therefore $R\gamma a\gamma b \subseteq R\Gamma P \subseteq P$.

Since $R\gamma a$ and $R\gamma b$ are ideals of $R$. Then $R\gamma a \cap R\gamma b = R\gamma a\gamma b$ (using the result (i)). Also $R\gamma a = R\gamma a \cap R = R\gamma a\gamma R$. Hence $R\gamma a\gamma b = R\gamma a\gamma R\gamma b = R\gamma a \cap R\gamma b$. This yields $R\gamma a\gamma R\gamma b = (R\gamma a\gamma b) \subseteq P$ and since $P$ is prime, $R\gamma a \subseteq P$ or $R\gamma b \subseteq P$. Therefore $(a =)a\gamma f(a)\gamma a \in P$ or $(b =)b\gamma f(b)\gamma b \in P$ and the desired result follows.

(iii) Since $R$ has a mate function $f$, it follows from Theorem 4.2.7 that $R$ has no non-zero nilpotent elements. If $x\gamma y = 0$ then $(y\gamma x)^2 = (y\gamma x)(y\gamma x) = y\gamma (x\gamma y)\gamma x = 0$.

This implies $y\gamma x = 0$. Again for all $a \in R$, $(x\gamma a\gamma y)^2 = (x\gamma a\gamma y)(x\gamma a\gamma y) = x\gamma a\gamma (y\gamma x)\gamma a\gamma y = 0$. Therefore $x\gamma a\gamma y = 0$. Consequently $R$ has $(\ast, IF P)$.

**Remark 4.3.3.** We observe that, in view of Definition 4.1.3, the three results in Theorem 4.3.2 hold good for a right normal $P_1\Gamma$ - seminear-ring.

**Proposition 4.3.4.** Let $R$ be a $P_1(P_1')\Gamma$ - seminear-ring with a mate function $f$. If $E \subseteq C(R)$ then every principal right ideal of $R$ is $P_1(P_1')\Gamma$ - seminear-ring.

**Proof.** Let $R$ be a $P_1\Gamma$ - seminear-ring with a mate function $f$. We define $g : R \rightarrow (x\gamma R =) x\gamma f(x)\gamma R$, where $x \in R$ and $\gamma \in \Gamma$, such that $g(a) = x\gamma f(x)\gamma a$ for all $a$ in $R$. For all $y$, $z$ in $R$ and $\gamma \in \Gamma$,
\[ g(y + z) = x\gamma f(x)\gamma(y + z) = x\gamma f(x)\gamma y + x\gamma f(x)\gamma z = g(y) + g(z). \]

Also \( g(y\gamma z) = x\gamma f(x)\gamma y\gamma z \)
\[ = x\gamma f(x)\gamma x\gamma f(x)\gamma y\gamma z \text{(since } x\gamma f(x) \in E) \]
\[ = x\gamma f(x)\gamma(x\gamma f(x)\gamma y)\gamma z \]
\[ = x\gamma f(x)\gamma y\gamma x\gamma f(x)\gamma z \text{(since } E \subseteq C(R)) \]
\[ = g(y)\gamma g(z) \]

Also \( g(y\gamma z) = x\gamma f(x)\gamma y\gamma z \)
\[ = (x\gamma f(x)\gamma y)\gamma z \]
\[ = y\gamma x\gamma f(x)\gamma z \text{(since } E \subseteq C(R)) \]
\[ = y\gamma g(z) \]

Thus \( g \) is a seminear-ring \( R \)-homomorphism. Obviously \( g \) is onto and hence \( x\gamma R \) is a homomorphic image of \( R \). Rest of the proof is taken care of by Theorem 4.2.1.

The proof is similar when \( R \) is a \( P'_1 \Gamma \) - seminear-ring. \( \Box \)