Chapter 5

A Generalization of $P_k(r, m)$ and $P'_k(r, m)$ Seminear-Rings

In this Chapter, we introduce the concept of $P_k(r, m)$ and $P'_k(r, m)$ seminear-rings by way of generalizing the concepts of $P_k$ and $P'_k$ seminear-rings. A seminear-ring $R$ is called a $P_k(r, m)$ ($P'_k(r, m)$) seminear-ring if there exist positive integers $k, r, m$ such that for all $x$ in $R$, $x^k R = x^r Rx^m$ ($Rx^k = x^r Rx^m$).

We divide this chapter into three sections. In the first section of this chapter we give some definitions and examples of $P_k(r, m)$ and $P'_k(r, m)$ seminear-rings. We also show by an example that a $P'_k(r, m)$ seminear-ring, where $k > 1$, need not be a $P_k(r, m)$ seminear-ring for any positive integer $k$.

In the other two sections we discuss certain properties of $P_k(r, m)$ and $P'_k(r, m)$ seminear-rings. We prove that if $R$ is a $P(1, 2)$ seminear-ring with a mate function $f$, then $R$ is a
(a) $P_k$ seminear-ring for all positive integers $k$.

(b) $P'_k$ seminear-ring for all positive integers $k$.

We also furnish a characterization of $P_k(r, m)$ and $P'_k(r, m)$ seminear-rings. Let $R$ be a $P(1, 2)$ seminear-ring and let it admit a mate function $f$. Then $R$ is a $P_t(r, m)$ as well as a $P'_t(r, m)$ seminear-ring for all positive integers $t, r, m$ if and only if $R$ is a $P'_k(1, 1)$ seminear-ring for some fixed positive integer $k$.

Many of the results in this chapter are included in the author’s paper “A generalization of $P_k(r, m)$ and $P'_k(r, m)$ seminear-rings,”[41] - published in “The Asian International Journal of Life Sciences”.

5.1 Definitions and Examples

In this section we furnish definitions and examples of $P_k(r, m)$ and $P'_k(r, m)$ seminear-rings.

**Definition 5.1.1.** A seminear-ring $R$ is called a $P_k(r, m)$ ($P'_k(r, m)$) seminear-ring if there exist positive integers $k, r, m$ such that for all $x$ in $R$, $x^kR = x^rRx^m$ ($Rx^k = x^rRx^m$).

**Remark 5.1.2.** A $P_k(1, 1)$ ($P'_k(1, 1)$) seminear-ring is nothing but a $P_k$ ($P'_k$) seminear-ring.

**Example 5.1.3.** (i) The direct product of any two seminear-fields is a $P_k(r, m)$ as well as a $P'_k(r, m)$ seminear-ring.
(ii) Let $R = \{0, a, b, c\}$. We define the semigroup operations “+” and “.” in $R$ as follows:

\[
\begin{array}{cccc}
+ & 0 & a & b \\
0 & 0 & a & b \\
a & a & 0 & c \\
b & b & c & 0 \\
c & c & b & a \\
\end{array}
\quad
\begin{array}{cccc}
. & 0 & a & b \\
0 & 0 & 0 & 0 \\
a & a & 0 & a \\
b & b & 0 & 0 \\
c & c & 0 & 0 \\
\end{array}
\]

Then $(R, +, .)$ is a $P_k(r, m)$ as well as a $P_k'(r, m)$ seminear-ring.

(iii) We consider the seminear-ring $R = \{0, a, b, c\}$ where the semigroup operations “+” and “.” in $R$ are defined as follows:

\[
\begin{array}{cccc}
+ & 0 & a & b \\
0 & 0 & a & b \\
a & a & 0 & c \\
b & b & c & 0 \\
c & c & b & a \\
\end{array}
\quad
\begin{array}{cccc}
. & 0 & a & b \\
0 & 0 & 0 & 0 \\
a & a & 0 & a \\
b & b & 0 & 0 \\
c & c & 0 & 0 \\
\end{array}
\]

Then $(R, +, .)$ is a $P_k'(r, m)$ seminear-ring for $k > 1$ but not a $P_k(r, m)$ seminear-ring for any positive integer $k$.

### 5.2 Properties of $P(1, 2)$ - seminear-rings

In this section we define the notion of $P_k(r, m)$ and $P_k'(r, m)$ seminear-rings by way of generalizing the concepts of $P_k$ and $P_k'$ seminear-rings.
**Proposition 5.2.1.** Let $f$ be a mate function for the seminear-ring $R$. If $E \subseteq C(R)$ then,

(i) $xf(x) = (xf(x))^r = x^r f(x)^r$ and

(ii) $f(x)x = (f(x)x)^r = f(x)^r x^r$ for all $x$ in $R$ and for all positive integers $r$.

**Proof.** (i) We first observe that $xf(x) \in E$ (by Proposition 1.4.31). As $E \subseteq C(R)$ we have $xf(x) = (xf(x))^2 = xf(x)(xf(x))$

$= x(xf(x)f(x)) = x^2 f(x)^2$. Continuing in the same vein we get $(xf(x) =)(xf(x))^r = x^r (f(x))^r$ for all positive integers $r$.

(ii) The proof is similar to the proof of (i). □

**Theorem 5.2.2.** Let $R$ be a $P(1, 2)$ seminear-ring with a mate function $f$. Then $R$ is a

(a) $P_k$ seminear-ring for all positive integers $k$.

(b) $P'_k$ seminear-ring for all positive integers $k$.

**Proof.** Since $R$ is a $P(1, 2)$ seminear-ring. Proposition 3.2.4 demands that every idempotent is central. i.e $E \subseteq C(R)$.

(a) Case (i): Let $k = 1$. For all $x$ in $R$,

$xR = x(f(x)xR) = x(Rf(x)x) \text{ (since } E \subseteq C(R)) = xRx$

(by Proposition 1.4.31(ii)) i.e. $xR = xRx$.

Hence $R$ is a $P_1$ seminear-ring.
Case (ii): For $k > 1$ and for any $x \in R$,

\[ x^k R = x(x^{k-1} R) \subseteq xR = xRx \] (using the result for $k = 1$).

Therefore $x^k R \subseteq xRx$.

Also $xRx = xRx f(x)x = (xRx f(x))x = xf(x) Rx$ (since $E \subseteq C(R)$) = $x(xR)x$ (by Proposition 1.4.31 (ii))

\[ = x^2 Rx = x(xRx) = x(x^2 Rx) = x^3 Rx. \]

Repeating this process,

we obtain $xRx = x^k Rx \subseteq x^k R$ for all positive integers $k$.

Therefore $xRx \subseteq x^k R$. Thus $xRx = x^k R$ for all $x$ in $R$.

Hence $R$ is $P_k$ seminear-ring for any positive integer $k$.

(b) Case (i): Let $k = 1$. For all $x$ in $R$,

\[ Rx = R xf(x)x = (Rx f(x))x = (x f(x))xRx \] (by Proposition 1.4.31(ii)). i.e. $Rx = xRx$. Hence $R$ is a $P'_1$ seminear-ring.

Case (ii): Let $k > 1$. Since $E \subseteq C(R)$ we have for all $x, y$ in $R$,

\[ yx^k = (yx)x^{k-1} = (y xf(x)x)x^{k-1} = (xf(x)y x)x^{k-1} = (xf(x)y x x^{k-1})x \]

\[ \in xRx. \]

Therefore $Rx^k \subseteq xRx$. Also $xyx = (xf(x)x)yRx = xyf(x)x^2$

\[ = (xf(x)x)yf(x)x^2 = xy(f(x))^2 x^3. \]

Repeating this process, we obtain $xyx = xy(f(x))^{k-1} x^k \in Rx^k$ for all positive integers $k$. Therefore $xRx \subseteq Rx^k$. Thus $xRx = Rx^k$ for all $x$ in $R$. Hence $R$ is a $P'_k$ seminear-ring for any positive integer $k$.  

\[ \square \]
5.3 Characterization of $P_k(r, m)$ and $P'_k(r, m)$ - seminear-rings

We furnish below a characterization of $P_k(r, m)$ and $P'_k(r, m)$ seminear-rings.

**Theorem 5.3.1.** Let $R$ be a $P(1, 2)$ seminear-ring admit a mate function $f$. Then $R$ is a $P_t(r, m)$ as well as a $P'_t(r, m)$ seminear-ring for all positive integers $t, r, m$ if and only if $R$ is a $P'_k(1, 1)$ seminear-ring for some fixed positive integer $k$.

**Proof.** Suppose $R$ is a $P'_k(1, 1)$ seminear-ring for a fixed positive integer $k$. Let $t, r, m$ be any three positive integers. For $x \in R$, $x^r Rx^m \subseteq xRx = Rx^k$.

Therefore $x^r Rx^m \subseteq Rx^k \ldots \ldots (1)$

Now let $z \in Rx^k \ (= xRx)$.

Then there exists $y \in R$ such that $z = xyx \ldots \ldots (2)$

$$= (xf(x)x)y(xf(x)x)$$
$$= (xf(x))(xyx)(f(x)x)$$
$$= (xf(x))^r(xyx)(f(x)x)^m$$

as $xf(x), f(x)x \in E$.

Since $R$ is a $P(1, 2)$ seminear-ring, Proposition 3.2.4 demands that $E \subseteq C(R)$. We can make use of Proposition 5.2.1 and get

$z = x^r(f(x)^rxyxf(x)^m)x^m \in x^r Rx^m$. Thus $Rx^k \subseteq x^r Rx^m \ldots \ldots (3)$

From (1) and (3) we get $x^r Rx^m = Rx^k \ (= xRx)$.
Appealing to Theorem 5.2.2, we get $R$ is a $P_t(1, 1)$ (i.e. $P_t$) as well as a $P'_t(1, 1)$ (i.e. $P'_t$) seminear-ring for all positive integers $t$. Collecting all these pieces we get, $x^t R = Rx^t$ ($= xRx = Rx^k = x^r Rx^m$) for all $x$ in $R$. i.e. $R$ is a $P_t(r, m)$ as well as a $P'_t(r, m)$ seminear-ring for all positive integers $t, r, m$.

The converse is obvious.

\begin{proof}
Suppose $R$ is a $P_k(1, 1)$ seminear-ring for a fixed positive integer $k$. Let $t, r, m$ be any three positive integers. For $x \in R$, $x^r Rx^m \subseteq xRx = x^k R$. Therefore $x^r Rx^m \subseteq x^k R$. Consequently $x^k R = x^r Rx^m$. Now let $z \in x^k R$ ($= xRx$). Then there exists $y \in R$ such that $z = xyx$. The rest of the proof is same as in Theorem 5.3.1 from equation (2) onwards.
\end{proof}

\textbf{Theorem 5.3.2.} Let $R$ be a $P(1, 2)$ seminear-ring with a mate function $f$. Then $R$ is a $P_t(r, m)$ as well as a $P'_t(r, m)$ seminear-ring for all positive integers $t, r, m$ if and only if $R$ is a $P_k(1, 1)$ seminear-ring for some fixed positive integer $k$.

\textbf{Proof.} Suppose $R$ is a $P_k(1, 1)$ seminear-ring for a fixed positive integer $k$. Let $t, r, m$ be any three positive integers. For $x \in R$, $x^r Rx^m \subseteq xRx = x^k R$. Therefore $x^r Rx^m \subseteq x^k R$. Consequently $x^k R = x^r Rx^m$. Now let $z \in x^k R$ ($= xRx$). Then there exists $y \in R$ such that $z = xyx$. The rest of the proof is same as in Theorem 5.3.1 from equation (2) onwards. 

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