Chapter 5

Mellin Convolution product

This chapter is based on Mellin convolution. In this chapter, the concept of neutrix and neutrix limit is applied to generalize the definition of Mellin convolution product. Using this, some results on the Mellin convolution product of distributions are proposed. Further some results are also given with the help of gamma function for negative integers using neutrix limit.

In the classical sense, gamma function is not defined for the negative integers. With the help of neutrix limit the gamma function is defined as follows.

$$\Gamma(m) = \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \exp(-t) t^{m-1} \, dt \quad \text{for } m \neq -1, -2, \ldots . \tag{5.0.1}$$

Using this definition Adem kilicman, in [28], defined gamma function for negative integers as well.

$$\Gamma(-m) = N - \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \exp(-t) t^{-m-1} \, dt \quad \text{for } m = 1, 2, \ldots. \tag{5.0.2}$$

Using the concept of gamma function for negative integers, we have generalized the definition of Mellin convolution as under.

Definition 5.0.1. The Mellin convolution product of the functions f and g can be defined by using neutrix limit as

$$(f \circledast_M g)(t) = N - \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} f(t) g(x/t) \frac{dt}{t} \tag{5.0.3}$$

In the following, we will study the Mellin convolution product of distributions, with the help of definition 5.0.1. Consider a function $E_n(z)$ defined in [1].

$$E_n(z) = \int_1^{\infty} \frac{\exp(-zt)}{t^n} \, dt \text{ for } n = 0, 1, 2, \ldots \text{ and } Re(z) > 0 \tag{5.0.4}$$

Theorem 5.0.1. *The Mellin convolution of x_+^r and $E_n(x)$ exist and is given by*

$$x_+^r \circledast_M E_n(x) = x^r \, \Gamma(-r) \, \frac{1}{(-n + r + 1)} \tag{5.0.5}$$

for $r = 1, 2, \ldots$, $n = 1, 2, \ldots$ and $n - 1 > r$.

Proof. The Mellin convolution product can be given as

$$\begin{aligned} x_+^r \circledast_M E_n(x) &= N - \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} (x/t)^r E_n(t) \frac{dt}{t} \\ &= N - \lim_{\epsilon \to 0} x^r \int_{\epsilon}^{\infty} (t)^{(-r-1)} E_n(t) \, dt \\ &= N - \lim_{\epsilon \to 0} x^r \int_{\epsilon}^{\infty} (t)^{(-r-1)} \int_1^{\infty} \frac{\exp(-ut)}{u^n} \, du \, dt \end{aligned}$$

On changing the order of integration, we get.

$$\begin{aligned} x_+^r \circledast_M E_n(x) &= N-\lim_{\epsilon\to 0} x^r \int_1^\infty u^{(-n+r)} \int_\epsilon^\infty \exp(-tu)(ut)^{-r-1}\, udt\, du \\ &= x^r\, \Gamma(-r) \int_1^\infty u^{(-n+r)}\, du \\ &= x^r\, \Gamma(-r)\, \frac{1}{(-n+r+1)}. \end{aligned}$$

□

Theorem 5.0.2. *The Mellin convolution of* $\ln x$ *and* $\exp(-x)$ *exist and given by*

$$\ln x \ \circledast_M \exp(-x) = \ln x\, \Gamma(0) + \Gamma'(0) \tag{5.0.6}$$

Proof. The Mellin convolution product is

$$\begin{aligned} &\ln x \circledast_M \exp(-x) = N-\lim_{\epsilon\to 0} \int_\epsilon^\infty \exp(-t)\, \ln(x/t)\, \frac{dt}{t} \\ &= N-\lim_{\epsilon\to 0} \int_\epsilon^\infty \exp(-t) t^{-1} \ln x\, dt - N-\lim_{\epsilon\to 0} \int_\epsilon^\infty t^{-1} \exp(-t) \ln t\, dt \\ &= N-\lim_{\epsilon\to 0} \ln x \int_\epsilon^\infty \exp(-t) t^{-1}\, dt - N-\lim_{\epsilon\to 0} \int_\epsilon^\infty t^{-1} \exp(-t) \ln t\, dt \\ &= \ln x\, \Gamma(0) + \Gamma'(0). \end{aligned}$$

□

Theorem 5.0.3. *The Mellin convolution of* $\exp(-x)$ *and* x_+^r *exist and given by*

$$\exp(-x) \circledast_M \ x_+^r = x^r\, \Gamma(-r) \ \textit{for}\ r = 1, 2, \ldots. \tag{5.0.7}$$

Proof.

$$\begin{aligned}
\exp(-x) \circledast_M \; x_+^r &= N-\lim_{\epsilon\to 0} \int_\epsilon^\infty (x/t)^r \; \exp(-t) \, \frac{dt}{t} \\
&= N-\lim_{\epsilon\to 0} x^r \int_\epsilon^\infty \exp(-t) \, t^{-r-1} \, dt \\
&= N-\lim_{\epsilon\to 0} x^r \int_\epsilon^\infty \exp(-t) \, t^{-r-1} \, dt \\
&= x^r \, \Gamma(-r).
\end{aligned}$$

□

The functions $\ln(1+x_+)$ and $\ln(1+x_-)$ are defined as

$$\begin{aligned}
\ln(1+x_+) &= H(x)\ln(1+x) \\
\ln(1+x_-) &= H(-x)\ln(1+x). \qquad (5.0.8)
\end{aligned}$$

We have the following theorems in order.

Theorem 5.0.4. *The neutrix Mellin convolution product of* $\exp(-x)$ *and* $\ln(1+x_+)$ *exist and given by*

$$\exp(-x) \circledast_M \ln(1+x_+) = \sum_{m=1}^{\infty} \frac{x^m(-1)^{m+1}}{m} \, \Gamma(-m) \qquad (5.0.9)$$

for $r = 1, 2, \ldots$

Proof.

$$\begin{aligned}
\exp(-x) \circledast_M \ x_+^r &= N-\lim_{\epsilon\to 0}\int_\epsilon^\infty \ln(1+x/t)\exp(-t)\ \frac{dt}{t} \\
&= N-\lim_{\epsilon\to 0}\ \int_\epsilon^\infty \sum_{m=1}^{\infty}\frac{(x/t)^m(-1)^{m+1}}{m}t^{-1}\exp(-t)\ dt \\
&= N-\lim_{\epsilon\to 0}\sum_{m=1}^{\infty}\frac{x^m(-1)^{m+1}}{m}\ \int_\epsilon^\infty \exp(-t)\ t^{-m-1}\ dt \\
&= \sum_{m=1}^{\infty}\frac{x^m(-1)^{m+1}}{m}\ \Gamma(-m).
\end{aligned}$$

□

Similarly we can define the Mellin convolution product of other distributions.