Chapter 4

Convolution and Neutrix Convolution

This work is built upon the (neutrix)convolution product defined on some generalized Fresnel sine integrals and its associated functions. We define the functions $\sin_+^R x^k$, $\sin_-^R x^k$, $S_{k+}^R(x)$ and $S_{k-}^R(x)$ associated with generalized Fresnel sine integral by using the ramp function $R(x)$. We then discuss its convolution product and neutrix convolution product with x^r, x_+^r and x_-^r. The theory of distribution is extensively used in functional analysis, applied mathematics and engineering sciences.

Neutrix calculus [22],[29],[16] for the product of distributions is further developed to fulfill the needs of physicists and engineers. Evaluating the product of distributions and the convolution product of distribution is at the heart of neutrix calculus. In this chapter we will evaluate some convolution product and Neutrix convolution product of generalized Fresnel sin integrals with the distributions x^r, x_+^r and x_-^r.

Limonka Lazarova and Biljana Jolevska-Tuneska in [47], has given the generalized Fresnel sine integral that was generalized form of the Fresnel sine integral used by Adem Kilicman in[45], is defined as:

$$S_k(x) = \int_0^x \sin(u^k)\, du \qquad \text{for } k = 1, 2, \ldots \tag{4.0.1}$$

Using this, we define associated functions $S_{k+}^R(x)$ and $S_{k-}^R(x)$ with the help of ramp function $R(x)$ as below,

$$\begin{aligned} S_{k+}^R(x) &= R(x) S_k(x) \\ S_{k-}^R(x) &= R(-x) S_k(x) \qquad \text{for } k = 1, 2, \ldots \end{aligned}$$

where $R(x)$ is ramp function which is the anti-derivative of Heaviside step function. It is defined as follows.

$$R(x) = \int_{-\infty}^x H(\xi) d\xi = x H(x) \tag{4.0.2}$$

Thus

$$\frac{d}{dx} R(x) = H(x) \tag{4.0.3}$$

Heaviside step function $H(x)$ is defined as-

$$H(x) = \begin{cases} 1 & for \qquad x > 0 \\ 0 & for \qquad x < 0 \end{cases} \tag{4.0.4}$$

We will use $L_{r,k}(x)$, defined in [47], is given by

$$L_{r,k}(x) = \int_0^x u^r \sin(u^k)\, du \tag{4.0.5}$$

for $r = 0, 1, 2 \ldots$ and for $k = 1, 2 \ldots$

By using the Ramp function, we have defined the following two functions as:

$$\sin_{+}^{R}(x^k) = R(x)\ \sin(x^k) \quad (4.0.6)$$

$$\sin_{-}^{R}(x^k) = R(-x)\ \sin(x^k) \text{ for } k = 1, 2, \ldots. \quad (4.0.7)$$

4.1 Convolution product

In this section we will discuss convolution product of distribution with the functions defined in equation(4.0.6), equation(4.0.7) and $S_{k+}^{R}(x)$ and $S_{k-}^{R}(x)$. For this we first give the following definitions

Definition 4.1.1. Let $f(x)$ and $g(x)$ be two continuous functions with bounded support. Their convolution product $h(x)$ is defined as

$$h(x) = (f * g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)\ dt \quad (4.1.1)$$

The convolution product is commutative, i.e.

$$f * g = g * f \quad (4.1.2)$$

If $(f * g)'$ and $(f * g')$ or $(f' * g)$ exists, then

$$(f * g)' = f * g'(f' * g) \quad (4.1.3)$$

Definition 4.1.2. Let $f(x)$ and $g(x)$ be two distributions in $\mathcal{D}'$ having bounded support. Both the supports are bounded on the left (or right). The convolution product $f * g$ is defined by

$$\langle (f * g)\,, \varphi(x)\rangle = \langle f(y), \langle g(x), \varphi(x+y)\rangle\rangle \tag{4.1.4}$$

Now we are in position to give theorems and proofs on the convolution product of the distributions.

Theorem 4.1.1. *The convolution product* $\left(\sin_+^R(x^k)\right) * x_+^r$ *exists and is given by*

$$\left(\sin_+^R(x^k)\right) * x_+^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i+1,k}(x)\; x_+^i \tag{4.1.5}$$

for $r = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$

Proof. For $x < 0$

$$\left(\sin_+^R(x^k)\right) * x_+^r = 0 \tag{4.1.6}$$

see [40] p.41

for $x > 0$,

$$\begin{aligned} \left(\sin_+^R(x^k)\right) * x_+^r &= \int_{-\infty}^{\infty} \sin_+^R(t^k)\; (x-t)_+^r\; dt \\ &= \int_{-\infty}^{0} \sin_+^R(t^k)\; (x-t)_+^r\; dt + \int_{0}^{\infty} \sin_+^R(t^k)\; (x-t)_+^r\; dt \\ &= \int_{-\infty}^{0} t\; H(t)\;\; \sin(t^k)\; (x-t)_+^r\; dt \\ &\qquad + \int_{0}^{\infty} t\; H(t)\;\; \sin(t^k)\; (x-t)_+^r\; dt \end{aligned}$$

Using equation(4.0.4), above equation gives

$$\begin{aligned}\left(\sin_+^R(x^k)\right) * x_+^r &= \int_0^\infty t\sin t^k\ (x-t)_+^r\ dt \\ &= \int_0^x t\sin t^k\ (x-t)_+^r dt + \int_x^\infty t\sin t^k\ (x-t)_+^r\ dt\end{aligned} \tag{4.1.7}$$

Since

$$(x-t)_+^r = \begin{cases}(x-t)^r & for \quad x>t \\ 0 & for \quad x<t\end{cases} \tag{4.1.8}$$

see in [40]

Obviously second integral in equation (4.1.7) vanishes and we have

$$\begin{aligned}\left(\sin_+^R(x^k)\right) * x_+^r &= \int_0^x t\ \sin t^k\ (x-t)^r\ dt \\ &= \int_0^x t\sin t^k \sum_{i=0}^{r}\binom{r}{i} x^i(-t)^{r-i}\ dt \\ &= \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}\int_0^x t^{r-i+1}\sin t^k\ x^i\ dt \\ &= \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i} L_{r-i+1,k}(x)\ x_+^i\end{aligned} \tag{4.1.9}$$

□

Theorem 4.1.2. *The convolution product* $\left(\sin_-^R(x^k)\right) * x_-^r$ *is given by the following.*

$$\left(\sin_-^R(x^k)\right) * x_-^r = \sum_{i=0}^{r}\binom{r}{i} L_{r-i+1,k}(x)\ x_-^i \tag{4.1.10}$$

for $r = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$

Proof. If $x > 0$,

$$\left(\sin_{-}^{R}(x^{k})\right) * x_{-}^{r} = 0 \tag{4.1.11}$$

see [40]

For $x < 0$, we have

$$\begin{aligned}
\left(\sin_{-}^{R}(x^{k})\right) * x_{-}^{r} &= \int_{-\infty}^{+\infty} \sin_{-}^{R}(t^{k})\ (x-t)_{-}^{r}\ dt \\
&= \int_{-\infty}^{0} \sin_{-}^{R}(t^{k})\ (x-t)_{-}^{r}\ dt\ + \int_{0}^{\infty} \sin_{-}^{R}(t^{k})\ (x-t)_{-}^{r}\ dt \\
&= \int_{-\infty}^{0} (-t)H(-t)\sin t^{k}\ (x-t)_{-}^{r}\ dt \\
&\quad + \int_{0}^{\infty} (-t)H(-t)\sin t^{k}\ (x-t)_{-}^{r}\ dt
\end{aligned}$$

Using definition(4.0.4),

$$\begin{aligned}
\left(\sin_{-}^{R}(x^{k})\right) * x_{-}^{r} &= \int_{-\infty}^{0} (-t)\sin t^{k}\ (x-t)_{-}^{r}\ dt \\
&= \int_{-\infty}^{x} (-t)\sin t^{k}\ (x-t)_{-}^{r}\ dt + \int_{x}^{0} (-t)\sin t^{k}\ (x-t)_{-}^{r}\ dt
\end{aligned} \tag{4.1.12}$$

From [40] page(40) we have

$$(x-t)_{-}^{r} = \begin{cases} (t-x)^{r} & for \quad x < t \\ 0 & for \quad x > t \end{cases} \tag{4.1.13}$$

Using it in (4.1.12), we have

$$\begin{aligned}\left(\sin_{-}^{R}(x^{k})\right)*x_{-}^{r} &= \int_{x}^{0}(-t)\sin t^{k}\ (t-x)^{r}\ dt\\ &= -\int_{x}^{0} t\sin t^{k}\ \sum_{i=0}^{r}\binom{r}{i}(-x)^{i}(t)^{r-i}\ dt\\ &= -\sum_{i=0}^{r}\binom{r}{i}\int_{x}^{0} t^{r-i+1}\sin t^{k}(-x)^{i}\ dt\\ &= \sum_{i=0}^{r}\binom{r}{i}L_{r-i+1,k}(x)\ x_{-}^{i}. \qquad (4.1.14)\end{aligned}$$

□

Theorem 4.1.3. *The convolution product* $\left(S_{k+}^{R}(x)\right)*x_{+}^{r}$ *exists and is given by*

$$\begin{aligned}\left(S_{k+}^{R}(x)\right)*x_{+}^{r} \ &= \ \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r+2-i}\ L_{r-i+2,k}(x)\ x_{+}^{i}\\ &\quad+\frac{1}{(r+1)(r+2)}\sum_{i=0}^{r+2}\binom{r+2}{i}(-1)^{r+2-i}\ L_{r-i+2,k}(x)\ x_{+}^{i}\end{aligned}$$

for $r=0,1,2,\ldots$ *and* $k=1,2,\ldots$

Proof. If $x<0$ then

$$\left(S_{k+}^{R}(x)\right)*x_{+}^{r}=0 \qquad (4.1.15)$$

If $x>0$, then

$$\left(S_{k+}^{R}(x)\right)*x_{+}^{r}=\int_{-\infty}^{+\infty}S_{k+}^{R}(t)\ (x-t)_{+}^{r}\ dt$$

By using similar arguments as in the proof of theorem(4.1.1), we can write

$$
\begin{aligned}
\left(S^R_{k+}(x)\right) * x^r_+ &= \int_0^x t\ (x-t)^r\ S_k(t)\ dt \\
&= \int_0^x t(x-t)^r \int_0^t \ \sin\ u^k\ du\ dt \\
&= \int_0^x \sin u^k \int_u^x t(x-t)^r\ dt\ du \\
&= \frac{1}{r+1} \int_0^x u\ \sin u^k (x-u)^{r+1}\ du \\
&\quad + \frac{1}{(r+1)(r+2)} \int_0^x \ \sin u^k (x-u)^{r+2}\ du \\
&= \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} \int_0^x u^{r+2-i}\ \sin u^k\ (x)^i\ du + \\
&\quad \frac{1}{(r+1)(r+2)} \sum_{i=0}^{r+2} \binom{r+2}{i} (-1)^{r-i+2} \int_0^x u^{r+2-i} \sin u^k (x)^i\ du \\
&= \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} L_{r-i+2,k}(x)\ x^i_+ \\
&\quad + \frac{1}{(r+1)(r+2)} \sum_{i=0}^{r+2} \binom{r+2}{i} (-1)^{r+2-i} L_{r-i+2,k}(x)\ x^i_+ .
\end{aligned}
$$

□

Theorem 4.1.4. *The convolution product* $\left(S^R_{k-}(x)\right) * x^r_-$ *exists and*

$$
\begin{aligned}
\left(S^R_{k-}(x)\right) * x^r_- \;=\; & -\frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} L_{r-i+2,k}(x)\ x^i_- \\
& + \frac{1}{(r+1)(r+2)} \sum_{i=0}^{r+2} \binom{r+2}{i} L_{r-i+2,k}(x)\ x^i_-
\end{aligned}
\tag{4.1.16}
$$

for $r = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$

Proof. If $x > 0$, then using the definition of x_-^r, we have

$$\left(S_{k-}^R(x)\right) * x_-^r = 0$$

For $x < 0$,

$$\left(S_{k-}^R(x)\right) * x_-^r = \int_{-\infty}^{+\infty} S_{k-}^R(t)\ (x-t)_-^r\ dt$$

Similar to the proof of theorem (4.1.2), we get

$$\begin{aligned}
(S_{k-}^R(x)) * x_-^r &= \int_x^0 (-t)\ (t-x)^r\ S_k(t)\ dt \\
&= \int_x^0 (-t)(t-x)^r \int_0^t \sin\ u^k\ du\ dt \\
&= \int_0^x \sin u^k \int_u^x t(t-x)^r\ dt\ du \\
&= -\frac{1}{r+1}\int_0^x u\ \sin u^k\ (u-x)^{r+1}\ du \\
&\quad +\frac{1}{(r+1)(r+2)}\int_0^x \sin u^k\ (u-x)^{r+2}\ du \\
&= -\frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}\int_0^x u^{r+2-i}\sin u^k\ (-x)^i\ du \\
&\quad +\frac{1}{(r+1)(r+2)}\sum_{i=0}^{r+2}\binom{r+2}{i}\int_0^x u^{r+2-i}\sin u^k\ (-x)^i\ du \\
&= -\frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}L_{r-i+2,k}(x)\ x_-^i \\
&\quad +\frac{1}{(r+1)(r+2)}\sum_{i=0}^{r+2}\binom{r+2}{i}L_{r-i+2,k}(x)\ x_-^i.
\end{aligned}$$

□

Theorem 4.1.5. *The convolution product* $\left(\sin_+^R x^k\right) * |x|^r$ *exists and*

$$\begin{aligned}\left(\sin_+^R x^k\right) * |x|^r &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i+1,k}(x)\; x_+^i \\ &\quad + \sum_{i=0}^{r} \binom{r}{i} L_{r-i+1,k}(x)\; x_-^i \qquad (4.1.17)\end{aligned}$$

for $r = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$

Proof. By Gelfand and Shilov in [40]

$$|x|^r = x_+^r + x_-^r \qquad (4.1.18)$$

Using equations (4.1.5), (4.1.10) in equation(4.1.18) we get the result. □

Theorem 4.1.6. *The convolution product* $\left(\sin_+^R x^k\right) * (|x|^r \operatorname{sgn} x)$ *exists and*

$$\begin{aligned}\left(\sin_+^R x^k\right) * (|x|^r \operatorname{sgn} x) &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i+1,k}(x)\; x_+^i \\ &\quad - \sum_{i=0}^{r} \binom{r}{i} L_{r-i+1,k}(x)\; x_-^i \qquad (4.1.19)\end{aligned}$$

for $r = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$

Proof. By Gelfand and Shilov in [40]

$$|x|^r\, sgn\ x = x_+^r - x_-^r \qquad (4.1.20)$$

Using equations (4.1.20), (4.1.5) and (4.1.10) we can find the result. □

Now, by similar process we can give two new functions $\cos_+^R(x^k)$ and $\cos_-^R(x^k)$ using the ramp function.

$$\begin{aligned}\cos_+^R(x^k) &= R(x)\;\cos(x^k) \\ \cos_-^R(x^k) &= R(-x)\;\cos(x^k) \text{ for } k = 1, 2, \ldots\end{aligned}$$

Theorem 4.1.7.

$$\left(\cos_+^R(x^k)\right) * x_+^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \frac{\left[x^{r-i+2}\cos x^k - k\ L_{r-i+k+1,k}(x)\ x_+^i\right]}{(r-i+2)} \tag{4.1.21}$$

for $r = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$

Proof. For $x < 0$

$$\left(\cos_+^R(x^k)\right) * x_+^r = 0 \tag{4.1.22}$$

see page(41) in [40]

for $x > 0$,

$$\begin{aligned}
\left(\cos_+^R(x^k)\right) * x_+^r &= \int_{-\infty}^{\infty} \cos_+^R(t^k)\ (x-t)_+^r\ dt \\
&= \int_{-\infty}^{0} \cos_+^R(t^k)\ (x-t)_+^r\ dt + \int_{0}^{\infty} \cos_+^R(t^k)\ (x-t)_+^r\ dt \\
&= \int_{-\infty}^{0} R(t)\ \cos(t^k)\ (x-t)_+^r\ dt \\
&\quad + \int_{0}^{\infty} R(t)\ \cos(t^k)\ (x-t)_+^r\ dt \\
&= \int_{-\infty}^{0} t\ H(t)\ \cos(t^k)\ (x-t)_+^r\ dt \\
&\quad + \int_{0}^{\infty} t\ H(t)\ \cos(t^k)\ (x-t)_+^r\ dt
\end{aligned}$$

Using equation(4.0.4), above equation gives

$$\begin{aligned}
\left(\cos_+^R(x^k)\right) * x_+^r &= \int_{0}^{\infty} t\cos t^k\ (x-t)_+^r\ dt \\
&= \int_{0}^{x} t\cos t^k\ (x-t)_+^r dt + \int_{x}^{\infty} t\cos t^k\ (x-t)_+^r\ dt
\end{aligned} \tag{4.1.23}$$

Using (4.1.8) in equation (4.1.23), second integral vanishes and we have

$$\begin{aligned}\left(\cos_+^R(x^k)\right) * x_+^r &= \int_0^x t\ \cos t^k\ (x-t)^r\ dt \\ &= \int_0^x t \cos t^k \sum_{i=0}^{r} \binom{r}{i} x^i (-t)^{r-i}\ dt \\ &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \int_0^x t^{r-i+1} \cos t^k\ x^i\ dt \\ &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \frac{\left[x^{r-i+2} \cos x^k - k\ L_{r-i+k+1,k}(x)\ x_+^i\right]}{(r-i+2)}.\end{aligned}$$

□

Theorem 4.1.8. *The convolution product* $\left(\cos_-^R x^k\right) * x_-^r$ *is given by the following.*

$$\left(\cos_-^R x^k\right) * x_-^r = \sum_{i=0}^{r} \binom{r}{i} \frac{\left[x^{r-i+2} \cos x^k - k\ L_{r-i+k+1,k}(x)\ x_-^i\right]}{(r-i+2)} \quad (4.1.24)$$

for $r = 0, 1, 2, \dots$ *and* $k = 1, 2, \dots$

Proof. If $x > 0$,

$$\left(\cos_-^R x^k\right) * x_-^r = 0 \quad (4.1.25)$$

see in [40]

For $x < 0$, we have

$$
\begin{aligned}
(\cos_-^R x^k) * x_-^r &= \int_{-\infty}^{+\infty} \cos_-^R(t^k)\ (x-t)_-^r\ dt \\
&= \int_{-\infty}^{0} \cos_-^R(t^k)\ (x-t)_-^r\ dt\ + \int_{0}^{\infty} \cos_-^R(t^k)\ (x-t)_-^r\ dt \\
&= \int_{-\infty}^{0} R(-t)\sin t^k\ (x-t)_-^r\ dt \\
&\qquad + \int_{0}^{\infty} R(-t)\sin t^k\ (x-t)_-^r\ dt \\
&= \int_{-\infty}^{0} (-t)H(-t)\cos t^k\ (x-t)_-^r\ dt \\
&\qquad + \int_{0}^{\infty} (-t)H(-t)\cos t^k\ (x-t)_-^r\ dt
\end{aligned}
$$

Using definition(4.0.4),

$$
\begin{aligned}
\left(\cos_-^R x^k)\right) * x_-^r &= \int_{-\infty}^{0} (-t)\cos t^k\ (x-t)_-^r\ dt \\
&= \int_{-\infty}^{x} (-t)\cos t^k\ (x-t)_-^r\ dt + \int_{x}^{0} (-t)\cos t^k\ (x-t)_-^r\ dt
\end{aligned}
\tag{4.1.26}
$$

From [40] using result (4.1.13), we have

$$
\begin{aligned}
\left(\cos_-^R x^k\right) * x_-^r &= \int_{x}^{0} (-t)\cos t^k\ (t-x)^r\ dt \\
&= -\int_{x}^{0} t\cos t^k\ \sum_{i=0}^{r} \binom{r}{i} (-x)^i (t)^{r-i}\ dt \\
&= -\sum_{i=0}^{r} \binom{r}{i} \int_{x}^{0} t^{r-i+1}\cos t^k (-x)^i\ dt \\
&= \sum_{i=0}^{r} \binom{r}{i} \frac{\left[x^{r-i+2}\cos x^k - k\ L_{r-i+k+1,k}(x)\ x_-^i\right]}{(r-i+2)}.
\end{aligned}
\tag{4.1.27}
$$

□

4.2 Existence of Neutrix Convolution Product

The range of convolution product to increase in larger class of distributions term is used neutrix convolution. It was first used by Fisher. Adem Kilicman [45] further extended it using the function τ whose properties are given in chapter(1). Related this τ the function τ_ν for $\nu > 0$ defined by Adem Kilicman[45] is given as in chapter(1) In this section we will give theorems and proofs related to neutrix convolution. Certain definitions are required to define neutrix convolution are given below

Definition 4.2.1. Let f and g be distributions in $\mathcal{D}'$ and let $f_\nu = f\tau_\nu$ for $\nu > 0$. The neutrix convolution product $f \circledast g$ is defined as the neutrix limit of the sequence $f_\nu * g$, provided that the limit h exists in the sense that

$$N \underset{\nu\to\infty}{-\lim} \langle f_\nu * g, \varphi\rangle = \langle h, \varphi\rangle \tag{4.2.1}$$

for all φ in $\mathcal{D}$, where N is the neutrix (see van der Corput[59]), having domain natural number and range real numbers, with negligible functions with finite linear sums of the functions $\nu^\lambda \ln^{r-1}\nu, \ln^r \nu, \nu^r \sin\nu^k$ and $\nu^r \cos\nu^k(\lambda \neq 0, r = 1, 2 \ldots)$ and all functions which converge to zero in the normal sense as ν tend to infinity.

Note that if $f * g$ exist by definition (2.1) then $f \circledast g$ exists and

$$f \circledast g = f * g \tag{4.2.2}$$

We now state a theorem proved in [22].

Theorem 4.2.1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution product $f \circledast g$ exists. Then the neutrix convolution product $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g' \tag{4.2.3}$$

Proof. See the paper [22]. □

Now let $L_{r,k} = N - \lim\limits_{\nu\to\infty} L_{r,k}(\nu)$. The following theorem follows,

Theorem 4.2.2. *The neutrix convolution product $(\sin_+^R x^k) \circledast x^r$ exists and*

$$\left(\sin_+^R x^k\right) \circledast x^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i+1,k}\, x^i \tag{4.2.4}$$

for $r = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$

Proof. Let

$$\left(\sin_+^R(x^k)\right)_\nu = \left(\sin_+^R(x^k)\right) \tau_\nu(x) \tag{4.2.5}$$

then the convolution product $\left(\sin_+^R(x^k)\right)_\nu * x^r$ exists by definition (2.2) and we have

$$\begin{aligned}
\left(\sin_+^R(x^k)\right)_\nu * x^r &= \int_{-\infty}^{\infty} \left(\sin_+^R(t^k)\right)_\nu (x-t)^r \, dt \\
&= \int_{-\infty}^{0} \left(\sin_+^R(t^k)\right)_\nu (x-t)^r \, dt \\
&\quad + \int_{0}^{\infty} \left(\sin_+^R(t^k)\right)_\nu (x-t)^r \, dt \\
&= \int_{-\infty}^{0} \left(tH(t)\sin t^k\right) \tau_\nu(t) \; (x-t)^r \, dt \\
&\quad + \int_{0}^{\infty} \left(tH(t)\sin t^k\right) \tau_\nu(t) \; (x-t)^r \, dt \\
&= \int_{0}^{\infty} \left(t\sin t^k\right) \tau_\nu(t) \; (x-t)^r \, dt
\end{aligned}$$

Using properties of $\tau_\nu(x)$ and equation(1.3.1) above integral takes the form

$$\begin{aligned}
\left(\sin_+^R x^k\right)_\nu * x^r &= \int_{0}^{\nu} t\sin t^k \; \tau_\nu(t) \; (x-t)^r \, dt \\
&\quad + \int_{\nu}^{\nu+\nu^{-\nu}} \tau_\nu(t) \; t\sin t^k \; (x-t)^r \, dt \\
&= I_1 + I_2 \quad \text{(say)} \qquad (4.2.6)
\end{aligned}$$

Using $\tau_\nu(t) = 1$ for $t < \nu$ in the integral I_1, we have

$$\begin{aligned}
I_1 &= \int_{0}^{\nu} t\sin t^k \; (x-t)^r \, dt \qquad (4.2.7) \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i+1,k}(\nu) \; x^i
\end{aligned}$$

as in theorem(4.1.1)

It follows that

$$N - \lim_{\nu\to\infty} I_1 = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i+1,k} \; x^i \qquad (4.2.8)$$

Now for fixed x in I_2 we have

$$\lim_{\nu\to\infty} I_2 = 0 \tag{4.2.9}$$

Taking limit $\nu \to \infty$ in equation(4.2.6) and using equations (4.2.8), (4.2.9) we can find required result. □

Theorem 4.2.3. *The neutrix convolution product* $\left(\sin_-^R(x^k)\right) \circledast x^r$ *exists and*

$$\left(\sin_-^R(x^k)\right) \circledast x^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^k L_{r-i+1,k}\, x^i \tag{4.2.10}$$

for $r = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$

Proof.

$$\begin{aligned}
\left(\sin_-^R(x^k)\right)_\nu * x^r &= \int_{-\infty}^{+\infty} \left(\sin_-^R(t^k)\right)_\nu \ (x-t)^r \, dt \\
&= \int_{-\infty}^{0} \left(\sin_-^R(t^k)\right)_\nu \ (x-t)^r \, dt \\
&\qquad + \int_{0}^{+\infty} \left(\sin_-^R(t^k)\right)_\nu \ (x-t)^r \, dt \\
&= \int_{-\infty}^{0} (-t)H(-t)\sin t^k \ \tau_\nu(t) \ (x-t)^r \, dt \\
&\quad + \int_{0}^{+\infty} (-t)H(-t) \ \sin t^k \ \tau_\nu(t) \ (x-t)^r \, dt
\end{aligned} \tag{4.2.11}$$

by definition(4.0.4) and using the properties of $\tau_\nu(x)$ and from equation(1.3.1) above integrals convert as below

$$
\begin{aligned}
\left(\sin_-^R(x^k)\right) * x^r &= \int_{-\infty}^{0} (-t)\sin t^k \tau_\nu(t)\,(x-t)^r\,dt \\
&= \int_{-\nu}^{0} (-t)\sin t^k\,\tau_\nu(t)\,(x-t)^r\,dt \\
&\qquad + \int_{-\nu-\nu^{-\nu}}^{-\nu} (-t)\sin t^k\,\tau_\nu(t)\,(x-t)^r\,dt \\
&= I_1 + I_2 \quad \text{(say)} \qquad (4.2.12)
\end{aligned}
$$

Using $\tau_\nu(t) = 1$ for $t > -\nu$ in I_1,

$$
\begin{aligned}
I_1 &= \int_{-\nu}^{0} (-t)\sin t^k\,(x-t)^r\,dt \\
&= -\sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \int_{-\nu}^{0} t^{r-i+1} \sin t^k\, x^i\,dt \\
&= \sum_{i=0}^{r} \binom{r}{i} \int_{0}^{-\nu} (-1)^{r-i} t^{r-i+1}\ \sin t^k\, x^i\,dt \\
&= -\sum_{i=0}^{r} \binom{r}{i} \int_{0}^{\nu} (-1)^{r-i} (-t)^{r-i+1}\ \sin(-t)^k\, x^i\,dt \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^k \int_{0}^{\nu} t^{r-i+1}\ \sin t^k\, x^i\,dt \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^k L_{r-i+1,k}(\nu)\, x^i
\end{aligned}
$$

$$
N-\lim_{\nu\to\infty} I_1 = \sum_{i=0}^{r} \binom{r}{i} (-1)^k L_{r-i+1,k}\, x^i \qquad (4.2.13)
$$

Now for fixed x in I_2 we have

$$
\lim_{\nu\to\infty} I_2 = 0 \qquad (4.2.14)
$$

Taking limit $\nu \to \infty$ in equation(4.2.12) and using equations(4.2.13), (4.2.14) we can find the result. □

Theorem 4.2.4. *The convolution product* $(S^R_{k+}(x)) \circledast x^r$ *exists and*

$$\begin{aligned}\left(S^R_{k+}(x)\right) \circledast x^r &= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r+1-i}L_{r+2-i,k}\,x^i \\ &\quad +\frac{1}{(r+1)(r+2)}\sum_{i=0}^{r+2}\binom{r+2}{i}(-1)^{r+1-i}L_{r+2-i,k}\,x^i\end{aligned}$$

for $r = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$

Proof. We put $\left(S^R_{k+}(x)\right)_\nu = \left(S^R_{k+}(x)\right)\tau_\nu(x)$

Then the convolution product $\left(S^R_{k+}(x)\right)_\nu * x^r$ exist by definition (4.1.2), we have

$$\begin{aligned}\left(S^R_{k+}(x)\right)_\nu * x^r &= \int_0^\nu t\;S_k(t)\;(x-t)^r dt \;+\; \int_\nu^{\nu+\nu^\nu} \tau_\nu(t)t\;S_k(t)\;(x-t)^r\;dt \\ &= I_1 + I_2 \quad (say) \end{aligned} \tag{4.2.15}$$

Now

$$\begin{aligned} I_1 &= \int_0^\nu (x-t)^r t\int_0^t \sin u^k\;du\;dt \\ &= \int_0^\nu \sin u^k \int_u^\nu t(x-t)^r\;dt\;du \\ &= \int_0^\nu \sin u^k \int_u^\nu t(x-t)^r\;dt\;du \\ &= -\frac{1}{r+1}\int_0^\nu \sin u^k\left[\nu(x-\nu)^{r+1} - u(x-u)^{r+1}\right]du \\ &\quad -\frac{1}{(r+1)(r+2)}\int_0^\nu \sin u^k\left[(x-\nu)^{r+2}-(x-u)^{r+2}\right]du \end{aligned}$$

$$
\begin{aligned}
I_1 \;=\; & -\frac{1}{r+1}\int_0^{\nu} \sin u^k \; \nu \sum_{i=0}^{r+1} \binom{r+1}{i} x^i (-\nu)^{r+1-i} \, du \\
& +\frac{1}{r+1}\int_0^{\nu} \sin u^k \; u \sum_{i=0}^{r+1} \binom{r+1}{i} x^i (-u)^{r+1-i} \, du \\
& -\frac{1}{(r+1)(r+2)}\int_0^{\nu} \sin u^k \; \sum_{i=0}^{r+2} \binom{r+2}{i} x^i (-\nu)^{r+2-i} \, du \\
& +\frac{1}{(r+1)(r+2)}\int_0^{\nu} \sin u^k \; \sum_{i=0}^{r+2} \binom{r+2}{i} x^i (-u)^{r+2-i} \, du
\end{aligned}
$$

$$
\begin{aligned}
I_1 \;=\; & \frac{1}{r+1}\int_0^{\nu} \sum_{i=0}^{r+1} \binom{r+1}{i} x^i (-\nu)^{r+2-i} \sin u^k \, du \\
& +\frac{1}{r+1}\int_0^{\nu} \sum_{i=0}^{r+1} \binom{r+1}{i} x^i (-1)^{r+1-i} (u)^{r+2-i} \sin u^k \, du \\
& -\frac{1}{(r+1)(r+2)}\int_0^{\nu} \sum_{i=0}^{r+2} \binom{r+2}{i} x^i (-\nu)^{r+2-i} \sin u^k \, du \\
& +\frac{1}{(r+1)(r+2)}\int_0^{\nu} \sum_{i=0}^{r+2} \binom{r+2}{i} x^i (-u)^{r+2-i} \sin u^k \, du
\end{aligned}
\tag{4.2.16}
$$

$$
\begin{aligned}
N - \lim_{\nu\to\infty} I_1 \;=\; & \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r+1-i} L_{r+2-i,k} \, x^i \\
& +\frac{1}{(r+1)(r+2)} \sum_{i=0}^{r+2} \binom{r+2}{i} (-1)^{r+2-i} L_{r+2-i,k} \, x^i
\end{aligned}
\tag{4.2.17}
$$

$$
\lim_{\nu\to\infty} I_2 = 0 \tag{4.2.18}
$$

Taking limit $\nu \to \infty$ in equation(4.2.15) and using equations(4.2.18), (4.2.17) gives required result. □

Theorem 4.2.5. *The convolution product* $\left(S^R_{k-}(x)\right) \circledast x^r$ *exists and*

$$\begin{aligned}\left(S^R_{k-}(x)\right) \circledast x^r &= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r-i+1+k}L_{r+2-i,k}\, x^i \\ &+\frac{1}{(r+1)(r+2)}\sum_{i=0}^{r+2}\binom{r+2}{i}(-1)^{r-i+1+k}L_{r+2-i,k}\, x^i\end{aligned} \tag{4.2.19}$$

for $r = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$

Proof. On putting $\left(S^R_{k+}(x)\right)_\nu = \left(S^R_{k+}(x)\right)\tau_\nu(x)$

The convolution product $\left(S^R_{k+}(x)\right)_\nu * x^r$ exist by definition (4.1.2). Using the property of τ_ν and ramp function we can write

$$\begin{aligned}\left(S^R_{k-}(x)\right)_\nu * x^r &= \int_{-nu}^{0}(-t)\ S_k(t)\ (x-t)^r\ dt \\ &\quad - \int_{-\nu-\nu^{-\nu}}^{-\nu}\tau_\nu(t)t\ S_k(t)\ (x-t)^r\ dt \\ &= I_1 + I_2 \quad \text{(say)}\end{aligned} \tag{4.2.20}$$

Now

$$\begin{aligned} I_1 &= -\int_{-\nu}^{0}(x-t)^r t\int_0^t \sin u^k\; du\; dt \\ &= \int_{-\nu}^{0}\sin u^k\int_{-\nu}^{-u} t(x-t)^r\; dt\; du \\ &= \int_{-nu}^{0}\sin u^k\int_{-\nu}^{-u} t(x-t)^r\; dt\; du \\ &= \frac{1}{r+1}\int_{-\nu}^{0}\sin u^k\left[u(x+u)^{r+1}-\nu(x+\nu)^{r+1}\right] du \\ &\quad -\frac{1}{(r+1)(r+2)}\int_{-\nu}^{0}\sin u^k\left[(x+u)^{r+2}-(x+\nu)^{r+2}\right] du \end{aligned}$$

$$\begin{aligned} I_1 &= \frac{1}{r+1}\int_{-\nu}^{0}\sin u^k\; u\sum_{i=0}^{r+1}\binom{r+1}{i}x^i(u)^{r+1-i}\; du \\ &\quad -\frac{1}{r+1}\int_{-\nu}^{0}\sin u^k\; \nu\sum_{i=0}^{r+1}\binom{r+1}{i}x^i(\nu)^{r+1-i}\; du \\ &\quad \frac{1}{(r+1)(r+2)}\int_{-nu}^{0}\sin u^k\; \sum_{i=0}^{r+2}\binom{r+2}{i}x^i(u)^{r+2-i}\; du \\ &\quad +\frac{1}{(r+1)(r+2)}\int_{0}^{\nu}\sin u^k\; \sum_{i=0}^{r+2}\binom{r+2}{i}x^i(\nu)^{r+2-i}\; du \end{aligned}$$

$$\begin{aligned}
I_1 &= \frac{1}{r+1}\int_{-nu}^{0}\sum_{i=0}^{r+1}\binom{r+1}{i}x^i(u)^{r+2-i}\sin u^k\,du \\
&\quad -\frac{1}{r+1}\int_{-\nu}^{0}\sum_{i=0}^{r+1}\binom{r+1}{i}x^i(\nu)^{r+2-i}\sin u^k\,du \\
&\quad -\frac{1}{(r+1)(r+2)}\int_{\nu}^{0}\sum_{i=0}^{r+2}\binom{r+2}{i}x^i(u)^{r+2-i}\sin u^k\,du \\
&\quad +\frac{1}{(r+1)(r+2)}\int_{-\nu}^{0}\sum_{i=0}^{r+2}\binom{r+2}{i}x^i(\nu)^{r+2-i}\sin u^k\,du
\end{aligned} \tag{4.2.21}$$

$$\begin{aligned}
N-\lim_{\nu\to\infty} I_1 &= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r+2-i+k}L_{r+2-i,k}\,x^i \\
&\quad +\frac{1}{(r+1)(r+2)}\sum_{i=0}^{r+2}\binom{r+2}{i}(-1)^{r+1+k-i}L_{r+2-i,k}\,x^i
\end{aligned} \tag{4.2.22}$$

$$\lim_{\nu\to\infty} I_2 = 0 \tag{4.2.23}$$

Taking limit $\nu \to \infty$ in equation(4.2.20) and using equations(4.2.23), (4.2.22) gives required result.

□