Chapter 2

Generalized Hankel Transformation

This chapter deals with the study of generalized extended Hankel Transformations

$$(B_{1,\mu,m}\psi(x))(y) = y^{-\mu}\int_0^\infty x^{\mu+1}J_{\mu,m}(xy)\psi(x)dx \tag{2.0.1}$$

and

$$(B_{2,\mu,m}\psi(x))(y) = y^{\mu+1}\int_0^\infty x^{-\mu}J_{\mu,m}(xy)\psi(x)dx \tag{2.0.2}$$

on the testing function space H and H_μ and also on their dual spaces. If we take $m = 0$ the equations (2.0.1) and (2.0.2) reduce the transformations studied by Mendez [50].

2.1 Behavior of Generalized Extended Hankel Transformation

In this section we will study the behavior of generalized extended Hankel Transformation $B_{1,\mu,m}\psi(x)$ and $B_{2,\mu,m}\psi(x)$ on the spaces H and H_μ and it's dual $H_\mu{}'$. For our convenience, we recall briefly the necessary definitions and important results which will be useful for proving result.

Let μ be arbitrary real number, H_μ is the space of all infinitely differentiable complex valued functions $\psi(x)$ defined on I, for which

$$\rho^{\mu}_{n,k} = \sup_{x \in I} \mid x^n x^{-1} D^k x^{-2\mu-1} \psi(x) \mid \tag{2.1.1}$$

exists for each pair of non negative integers n and k with topology generated by the multinorm $\rho^{\mu}_{n,k}$. H_μ is a Frechet space. Now suppose that $\psi(x)$ admits the expansion

$$\psi(x) = x^{2\mu+1}\left[b_0 + b_1 x^2 + \ldots + b_k x^{2k} + o(x^{2k})\right] \tag{2.1.2}$$

in some vicinity of the origin. Obviously function $\psi(x)$, $x \in I$ belongs to the space H_μ if and only if $\psi(x)$ is infinitely differentiable, has the form (2.1.2) at the origin and $D^k\psi(x)$ is of rapid descent as $x \to \infty$ for each $k = 0, 1, 2, \ldots$

$H_\mu{}'$ denote the dual space of H_μ and it's members are generalized functions of slow growth. The Altenburg space H turns to be particular case of H_μ when $\mu = -1/2$ that is $H = H_{-\frac{1}{2}}$.

Following differential operators are defined as

$$P_\mu\psi(x) = x^{-2\mu-1}Dx^{2\mu+2}\psi(x) \tag{2.1.3}$$

$$T\psi(x) = x^{-1}D\psi(x) \tag{2.1.4}$$

$$P_\mu^*\psi(x) = -x^{2\mu+2}Dx^{-2\mu-1}\psi(x) \tag{2.1.5}$$

$$T^*\psi(x) = -Dx^{-1}\psi(x). \tag{2.1.6}$$

Now we will study the operators (2.1.3) and (2.1.4) for the transformations $B_{1,\mu,m}$ and $B_{2,\mu,m}$.

Theorem 2.1.1. *For $\mu + 2m \geq -\frac{1}{2}$ and $\mu \in H$*

$$B_{1,\mu+1,m}T\psi = -B_{1,\mu+1,m}\psi \tag{2.1.7}$$

$$TB_{1,\mu,m}\psi = -B_{1,\mu+1,m-1}\psi \tag{2.1.8}$$

$$B_{1,\mu,m}(P_\mu T\psi) = -y^2 B_{1,\mu,m-1}\psi \tag{2.1.9}$$

$$P_\mu TB_{1,\mu,m}\psi = B_{1,\mu,m-1}\left(-x^2\psi\right) \tag{2.1.10}$$

$$B_{1,\mu,m}(P_\mu\psi) = y^2 B_{1,\mu+1,m-1}\psi \tag{2.1.11}$$

$$P_\mu B_{1,\mu+1,m}\psi = B_{1,\mu,m-1}\psi \tag{2.1.12}$$

Proof. We can write L.H.S. of (2.1.7) as

$$\begin{aligned} B_{1,\mu+1,m}(T\psi(x)) &= y^{-(\mu+1)} \int_0^\infty x^{\mu+2} J_{\mu+1,m-1}(xy) x^{-1}\ D\psi(x)\ dx \\ &= y^{-(\mu+1)} \int_0^\infty x^{\mu+1} J_{\mu+1,m-1}(xy)\ D\psi(x)\ dx \end{aligned}$$

which on integrating by parts gives

$$B_{1,\mu+1,m}(T\psi(x)) = -y^{-(\mu+1)} \int_0^\infty D\left[x^{\mu+1} J_{\mu+1,m-1}(xy)\right] \psi(x)\ dx \tag{2.1.13}$$

Using

$$\frac{d}{dx}[x^\nu J_{\nu,m}(x)] = x^\nu J_{\nu-1,m+1}(x) \tag{2.1.14}$$

(see [46] [p.186])

in right hand side of equation(2.1.13)

$$\begin{aligned} B_{1,\mu+1,m}(T\psi(x)) &= -y^{-\mu} \int_0^\infty x^{\mu+1} J_{\mu,m}(xy)\psi(x)dx \\ &= -B_{1,\mu,m}\psi(x). \end{aligned}$$

The L.H.S. of (2.1.8) can be written as-

$$\begin{aligned} TB_{1,\mu,m}\psi &= y^{-1} D\left[y^{-\mu} \int_0^\infty x^{\mu+1} J_{\mu,m}(xy)\psi(x)dx\right] \\ &= y^{-1} \int_0^\infty x^{\mu+1} \frac{d}{dy}[y^{-\mu} J_{\mu,m}(xy)]\psi(x)\ dx \end{aligned}$$

which on using

$$\frac{d}{dx}[x^{-\nu} J_{\nu,m}(x)] = x^{-\nu} J_{\nu+1,m-1}(x) \tag{2.1.15}$$

see [46], [p.186]

yields

$$\begin{aligned} TB_{1,\mu,m}\psi(x) &= y^{-(\mu+1)}\int_0^\infty x^{\mu+2}J_{\mu+1,m-1}(xy)\psi(x)dx \\ &= -B_{1,\mu+1,m-1}\psi. \end{aligned} \tag{2.1.16}$$

We can write L.H.S. of (2.1.9) as

$$\begin{aligned} B_{1,\mu,m}(P_\mu T\psi) &= y^{-\mu}\int_0^\infty x^{\mu+1}J_{\mu,m}(xy)[P_\mu T\psi](x)\ dx \\ &= y^{-\mu}\int_0^\infty x^{\mu+1}J_{\mu,m}(xy)x^{-2\mu-1}Dx^{2\mu+2}T\psi(x)\ dx \\ &= y^{-\mu}\int_0^\infty x^{\mu+1}J_{\mu,m}(xy)x^{-2\mu-1}Dx^{2\mu+2}x^{-1}D\psi(x)dx \\ &= y^{-\mu}\int_0^\infty x^{-\mu}J_{\mu,m}(xy)D\left[x^{2\mu+1}D\psi(x)\right]dx \\ &= -y^{-\mu}\int_0^\infty D\left[x^{-\mu}J_{\mu,m}(xy)\right]x^{2\mu+1}D\psi(x)dx \end{aligned}$$

using (2.1.15) we get

$$B_{1,\mu,m}(P_\mu T\psi) = y^{-\mu+1}\int_0^\infty x^{\mu+1}J_{\mu+1,m-1}(xy)D\psi(x)dx$$

Again integrating by parts

$$B_{1,\mu,m}(P_\mu T\psi) = -y^{-\mu+1}\int_0^\infty D\left[x^{\mu+1}J_{\mu+1,m-1}(xy)\right]\psi(x)dx$$

using equation (2.1.14) in above equation we get

$$\begin{aligned} B_{1,\mu,m}(P_\mu T\psi) &= -y^{-\mu+2}\int_0^\infty x^{\mu+1}J_{\mu,m-1}(xy)\psi(x)dx \\ &= -y^2B_{1,\mu,m-1}\psi. \end{aligned}$$

We can write L.H.S. of (2.1.10) as

$$\begin{aligned} P_\mu T B_{1,\mu,m}\psi &= P_\mu T y^{-\mu}\int_0^\infty x^{\mu+1}J_{\mu,m}(xy)\psi(x) \\ &= y^{-2\mu-1}Dy^{2\mu+1}Dy^{-\mu}\int_0^\infty x^{\mu+1}J_{\mu,m}(xy)\psi(x) \\ &= -y^{-(2\mu+1)}Dy^{2\mu+1}\int_0^\infty Dy^{-\mu}x^{\mu+1}J_{\mu,m}(xy)\psi(x)dx \end{aligned}$$

using(2.1.15)

$$= -y^{-(2\mu+1)}Dy^{2\mu+1}\int_0^\infty x^{2\mu+2}y^{-\mu}J_{\mu+1,m-1}(xy)\psi(x)dx$$

which on using (2.1.14) gives

$$\begin{aligned} P_\mu T B_{1,\mu,m}\psi &= -y^{-\mu}\int_0^\infty x^2 J_{\mu,m-1}(xy)\psi(x)dx \\ &= B_{1,\mu,m-1}(-x^2\psi). \end{aligned}$$

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We can write L.H.S. of (2.1.11) as

$$B_{1,\mu,m}(P_\mu\psi) = y^{-\mu}\int_0^\infty x^{\mu+1}J_{\mu,m}(xy)x^{-2\mu-1}Dx^{2\mu+2}\psi(x)dx$$

which on integrating by parts & using (2.1.15) gives

$$\begin{aligned} B_{1,\mu,m}(P_\mu\psi) &= -y^{-\mu}\int_0^\infty D\left[x^{-\mu}J_{\mu+1,m-1}(xy)\right]x^{2\mu+2}\psi(x)dx \\ &= y^{-\mu+1}\int_0^\infty x^{\mu+2}J_{\mu+1,m-1}(xy)\psi(x)dx \\ &= y^2 B_{1,\mu+1,m-1}\psi. \end{aligned}$$

L.H.S. of (2.1.12) can be written as

$$\begin{aligned} P_\mu B_{1,\mu+1,m}\psi &= y^{-2\mu-1}Dy^{2\mu+2}y^{-\mu-1}\int_0^\infty x^{\mu+2}J_{\mu+1,m}(xy)\psi(x)dx \\ &= B_{1,\mu,m-1}(x^2\psi). \end{aligned}$$

□

Theorem 2.1.2. *If $\psi \in H_\mu$ then*

$$B_{2,\mu+1,m}(P_\mu^* \psi) = y^2 B_{2,\mu,m}\psi \tag{2.1.17}$$

$$P_\mu^* B_{2,\mu,m}\psi = B_{2,\mu+1,m-1}(x^2\psi) \tag{2.1.18}$$

$$B_{2,\mu,m}(T^* P_\mu^* \psi) = -y^2 B_{2,\mu,m-1}\psi \tag{2.1.19}$$

$$T^* P_\mu^* B_{2,\mu,m}\psi = B_{2,\mu,m-1}(-x^2\psi) \tag{2.1.20}$$

Proof. The proof follows as theorem (2.1.1). □

Theorem 2.1.3. *Let $\mu + 2m \geq -\frac{1}{2}$ and if $\psi \in H_{\mu+1}$ then*

$$B_{2,\mu,m}(T^*\psi) = -B_{2,\mu+1,m-1}\psi \tag{2.1.21}$$

$$T^* B_{2,\mu+1,m}\psi = -B_{1,\mu,m}\psi \tag{2.1.22}$$

Proof. The proof is similar as theorem (2.1.1). □

Theorem 2.1.4. *If $m \geq 0$ and $Re(\mu + 2m) \geq -\frac{1}{2}$ then $B_{1,\mu,m}$ is an automorphism on H.*

Proof. Repeating (2.1.12) k times and multiplying by $(y^2)^n$ we get

$$(y^2)^n P_{\mu+k+1}.....P_{\mu+1}.P_{\mu}.B_{1,\mu+k,m+k-1}\psi = (y^2)^n B_{1,\mu+k-1,m+k-2}(x^2)^k\psi$$

which on using (2.1.11) n times gives

$$\begin{aligned}&(y^2)^n P_{\mu+k+1}.....P_{\mu+1}.P_{\mu}.B_{1,\mu+k,m+k-1}\psi \\ &= B_{1,\mu+k+n,m+k-n-1}(P_{\mu+n-1}....P_{\mu})(x^2)^k\psi\end{aligned} \tag{2.1.23}$$

since

$$P_{\mu+k-1}...P_{\mu+1}P_{\mu}\psi(x) = x^{-2\mu+2k-2}(x^{-1}D)^k x^{2\mu+2}\psi \tag{2.1.24}$$

thus (2.1.23) becomes

$$\begin{aligned}&(x^{2n}x^{-2\mu+2(k-1)}(x^{-1}D)^k x^{2\mu+2}B_{1,\mu+k,m+k-1}\psi = \\ &x^{-\mu-k+n}\int_0^\infty y^{\mu+k-n+1}J_{\mu+k-n,m+k-n-1}(xy) \\ &y^{-2\mu+2n-2}(y^{-1}D)^n y^{2\mu+2}\psi(y)dy\end{aligned} \tag{2.1.25}$$

or

$$\begin{aligned}&x^{-2\mu+n+2k-2}(x^{-1}D)^k x^{2\mu+2}B_{1,\mu+k,m+k-1\psi}(x) = \\ &\int_0^\infty y^{2k+n-1}(y^{-1}D)^n y^{2\mu+2} \\ &\psi(y)(xy)^{-\mu-k}B_{1,\mu+k-n,m+k-n-1}(xy)dy < \infty \\ &\qquad \text{for } \mu = -1/2\end{aligned} \tag{2.1.26}$$

which implies that $B_{1,\mu,m}$ is an automorphism on H. □

2.2 The Generalized Schwartz's Hankel Transformation $B'_{1,\mu,m}$

Let μ be arbitrary real number such that $\mu + 2m \geq -\frac{1}{2}$. The generalized Hankel transformation $B'_{1,\mu,m}$ is defined on H'_μ as the adjoint operator $B_{2,\mu,m}$ on H_μ that is

$$\langle B_{1,\mu,m} f, \varphi \rangle = \langle f, B_{2,\mu,m} \varphi \rangle \tag{2.2.1}$$

Theorem 2.2.1. *The generalized Schwartz's Hankel transformation $B'_{1,\mu,m}$ of order $\mu + 2m \geq -\frac{1}{2}$ is an automorphism on H'_μ.*

Proof. proof will be similar (2.1.4). □

Theorem 2.2.2. *Let $\mu + 2m \geq -1/2$ for every $f \in H'_\mu$, we obtain*

$$B'_{1,\mu+1,m}(Tf) = -B'_{1,\mu,m} f \tag{2.2.2}$$

$$TB'_{1,\mu,m}(f) = -B'_{1,\mu+1,m} f \tag{2.2.3}$$

$$B'_{1,\mu,m}(P_\mu Tf) = -y^2 B'_{1,\mu,m-1} f \tag{2.2.4}$$

$$P_\mu TB'_{1,\mu,m} f = B'_{1,\mu,m-1}(-x^2 f) \tag{2.2.5}$$

Proof. L.H.S. of (2.2.2) may be written as

$$\begin{aligned}\langle B'_{1,\mu+1,m}Tf,\varphi\rangle &= \langle Tf, B_{2,\mu+1,m}\varphi\rangle\\ &= \langle f, T^* B_{2,\mu+1,m}\varphi\rangle\\ &= \langle f, -B_{1,\mu,m}\varphi\rangle\\ &= \langle -B'_{1,\mu,m}f,\varphi\rangle\end{aligned}$$

Thus

$$B'_{1,\mu+1,m}Tf = -B'_{1,\mu,m}f \tag{2.2.6}$$

Now from equation(2.2.3) to equation(2.2.5) can be proved in a similar manner. □

Theorem 2.2.3. *If* $\mu + 2m \geq -1/2$

$$\langle B'_{1,\mu,m}(P_\mu Tf),\varphi\rangle = \langle P_\mu Tf, B_{2,\mu,m}\varphi\rangle \tag{2.2.7}$$

$$\langle P_\mu Tf, B_{2,\mu,m}\varphi\rangle = \langle f, T^* P^*_\mu B_{2,\mu,m}\varphi\rangle \tag{2.2.8}$$

$$\langle f, B_{2,\mu,m}(-y^2\varphi)\rangle = \langle -y^2 B'_{1,\mu,m}f,\varphi\rangle \tag{2.2.9}$$

$$\langle B'_{1,\mu,m}(P_\mu f),\varphi\rangle = \langle P_\mu f, B_{2,\mu,m}\varphi\rangle \tag{2.2.10}$$

Proof. Proof will be similar as (2.1.1). □