Chapter 1

Introduction

This chapter is preliminary in character and contains definitions and properties from such topic of analysis as functional spaces, special functions, special integrals and transforms. which is divided into sections.

1.1 Generalized function or Distribution

In this section we present definition of generalized function and related properties.

We shall consider here generalized function on Ω, where Ω is a domain of R^n. We choose test function on Ω as infinitely differentiable function at the interior point of Ω with prescribed behavior at the boundary point of Ω. We denote by $\langle f, \varphi \rangle$ the value of generalized function as a functional of f on the test function φ.

Definition 1.1.1. A distribution or generalized function f is defined by Gelfand and Shilov and Temple,G. as a continuous linear functional on the space of infinitely differentiable function for each test function φ having compact support. It is given as

$$\langle f, \varphi \rangle = \int_{\Omega} f(x)\varphi(x)dx \tag{1.1.1}$$

where $f(x)$ is locally summable function in every bounded region of R^n. The term compact support we will discuss below with some required definitions.

Those functional which can be expressed by (1.1.1) are called regular and all other including the Dirac delta distribution are called singular.

Definition 1.1.2. A complex generalized function f is defined as a continuous linear functional on the space of infinitely differentiable complex valued function φ having compact support. It is given as

$$\langle f, \varphi \rangle = \int f(x)\bar{\varphi}(x)dx \tag{1.1.2}$$

where bar denote complex conjugate.

Let the space of test function $X = X(\Omega)$ is a topological space and $X' = X'(\Omega)$, the space of continuous linear functional space on X which is called topological dual of X. A generalized function $f \in X'$ is said to be zero on the open set G if $\langle f, \varphi \rangle = 0$ for each test function $\varphi \in X$ beyond the G. The union of all the open set where $f = 0$ is called the null set of the function f. The complement of the null set with respect to Ω is said to be support of generalized function f. We are giving here some points which will be required to understand the generalized function:

(i) An ordinary function is said to be smooth if it is infinitely differentiable.

(ii) An ordinary function φ in R^n is said to be rapid descent if $\varphi(t) = O(|t|)^{-m}$ for every integer $m \in Z = (0, \pm 1, \pm 2, \pm 3, \ldots)$.

(iii) An ordinary function φ is said to be of slow growth if there exist an integer $m \in Z$ such that $\varphi(t) = O(|t|^m)$.

(iv) A generalized function $f \in X'$ is said to be zero on open set Ω if $\langle f, \varphi \rangle = 0$ for every $\varphi \in X(\Omega)$ with $\operatorname{supp}(\varphi) \subset \Omega$. If Ω_f is the largest open set where $f = 0$ then the set Ω/Ω_f is called the support of generalized function f and it is denoted by $\operatorname{supp}(f)$. If $\operatorname{supp}(f) \subset G \subset \Omega$ and G is bounded then generalized function f is said to be of compact support.

1.1.1 Example of Distributions

Following examples will be used for our subsequent work

Example 1.1.1.

$$x_+^r = \begin{cases} x^r & for \quad x > 0 \\ 0 & for \quad x < 0 \end{cases} \tag{1.1.3}$$

for $r = 1, 2 \ldots$.

Example 1.1.2.

$$x_-^r = \begin{cases} 0 & for \quad x > 0 \\ |x|^r & for \quad x < 0 \end{cases} \tag{1.1.4}$$

for $r = 1, 2 \ldots$.

Example 1.1.3.

$$|x|^\lambda \; sgn \; x = x_+^\lambda - x_-^\lambda \tag{1.1.5}$$

. There are so many generalized functions in the mathematics but we will use some of them for our research work.

1.1.2 Derivative of generalized function

Definition 1.1.3. Let f be generalized function and φ defined in Ω then derivative of f can be given as

$$\langle f', \varphi \rangle = \int_{-\infty}^{+\infty} f'(x)\varphi(x)dx$$

on integrating by parts we get

$$\langle f', \varphi \rangle = \langle f, -\varphi' \rangle$$

where f' denotes the derivative of f.

1.2 Product of distribution

In this section we defined the product of distributions. Let $\mathcal{D}$ is the space of infinitely differentiable functions ψ with compact support. Let $\mathcal{D}[a, b]$

is the space of infinitely differentiable functions with support contained in [a,b]. Let $\mathcal{D}'$ is the space of distributions.

Definition 1.2.1. Let f be the distribution in $\mathcal{D}'$ and let g be an infinitely differentiable function. Then the product $f.g$ is defined by

$$\langle f.g, \varphi\rangle = \langle f, g\varphi\rangle$$

for all test function φ with compact support contained in (a, b) and $g\varphi$ is also a test function.

Unfortunately, it is not possible to define the product of two distributions, in general. It turns out that the product does not always exists with in the system of distributions. As an example, for one dimensional variable t, let $f(t) = 1/\sqrt{t}$, then $f(t)$ represents a regular distribution as a locally integrable function. However $f(t).f(t) = [f(t)]^2$, a function of t, defined for all nonzero t, is not integrable over any interval includes the origin. Thus it can not define a distribution through the expression.

$$\left\langle \frac{1}{|t|}, \varphi \right\rangle = \int_{-\infty}^{\infty} \frac{\varphi(t)}{|t|} dt \tag{1.2.1}$$

Since the integral does not converge for every φ in $\mathcal{D}$. In short the product of any function $f(t)$, even with itself need not necessarily exist as a distribution.

It is however possible to define the product of distributions in special cases.

If f is the r^{th} derivative of an ordinary summable function F, then we have the well known result.

$$f.g = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left[F g^{(i)} \right]^{r-i} \tag{1.2.2}$$

where

$$\binom{r}{i} = \frac{r!}{i!(r-i)!}$$

$$\begin{aligned} \langle f.g, \varphi \rangle &= \left\langle \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left[F g^{(i)} \right]^{(r-i)}, \varphi \right\rangle \\ &= (-1)^r \sum_{i=0}^{r} \binom{r}{i} \left\langle F.g^i, \varphi^{(r-i)} \right\rangle \end{aligned} \tag{1.2.3}$$

For all test function φ having support in (a, b). In 1971, Fisher,B. has defined the product of generalized functions f and g on an open interval (a, b), $-\infty \leq a < b \leq \infty$, as the limit of the sequence $\langle f_n g_n \rangle$ provided this sequence is regular on (a, b) where

$$f_n = f * \delta_n = \int_{-1/n}^{1/n} f(x-t)\delta_n(t)dt \tag{1.2.4}$$

$$g_n = g * \delta_n$$

$$\delta_n = n\rho(nx)$$

for $n = 1, 2, 3, \ldots$ and $\rho(x)$ is a fixed infinitely differentiable function having the following properties,

(i) $\rho(x) = 0$ for $|x| \geq 1$

(ii) $\rho(x) \geq 0$

(iii) $\rho(x) = \rho(-x)$

(iv) $\int_{-1}^{+1} \rho(x)dx = 1$

He has also proved a theorem which on using the above definition, shows that the product of two distributions f and g is given by 1.2.3
In series of papers [12], [13], [14], [17], [18], [20], [23], [36], Fisher has illustrated his results by means of a number of examples.
In 1980 he [19] has modified the above definition of product of two distributions as-

Definition 1.2.2. Let f and g be arbitrary distributions and let $g_n = g * \delta$, then the product fg of f and g exists and is equal to h on (a, b) iff

$$\lim_{n\to\infty} \langle f.g_n, \varphi(x)\rangle = \lim_{n\to\infty} \langle f, g_n\varphi(x)\rangle = \langle h(x), \varphi(x)\rangle$$

for all test function φ with support in (a, b).

Fisher, concentrated his study by giving a number of examples of the product of two distributions. However many more problem arises.

(a) Leibnitz theorem for the r^{th} derivative, as well as for the fractional derivative of product of two distribution

(b) Defining the product of three distributions and obtaining expression for it

(c) Proving the distributive laws and forming a ring of generalized functions.

An attempt has been made to solve some of these problems in a series of papers by Ahuja,G. [2], [3], [4], [5], [6], [7]. Fisher [36] has given an expression for the product of distributions $x_+^\lambda \ln^p x$ and $x_+^\mu \ln^q x$.
In our propose of work we have extended it to the product of three distributions and also obtained their neutrix product. Some more examples are also given.

1.3 Neutrix Product

1.3.1 Neutrix

Definition 1.3.1. A neutrix N is defined by Vander Corput in , [59] as a commutative additive group of functions $f(\xi)$ defined on a domain N' with values in an additive group N'', where further if for some f in N, $f(\xi) = \gamma$ for all ξ in N', then $\gamma = 0$. The functions in N are called negligible functions.

1.3.2 Neutrix Limit

Definition 1.3.2. Let N' be a set contained in a topological space with a limit point b, which does not belong to N'. If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant β such that

$f(\xi) - \beta$ is negligible in N, then β is called the neutrix limit or $N - \lim$ of f as ξ tends to b, we can write

$$N - \lim_{\xi \to b} f(\xi) = \beta$$

this β is unique if it exists.

For example, let N be a neutrix having domain $N' = \{\epsilon : \epsilon = 1, 2, \ldots \infty\}$ and range N'', the real numbers, with the negligible functions being finite linear sums of $\epsilon^{\lambda} \ln^{r-1} \epsilon$, $\ln^{r} \epsilon$ $(\lambda < 0, r = 1, 2, \ldots)$ and functions $o(\epsilon)$ which converge to zero in the normal sense as ϵ tends to zero.

Fisher [20] defined the neutrix product of two distribution as-

Definition 1.3.3. Let $g_n(x) = (g * \delta_n)(x)$. The non commutative neutrix product fog of f and g exists and is equal to the distribution h in the interval (a, b) if

$$N - \lim_{n \to \infty} \langle f g_n, \varphi \rangle = \langle h, \varphi \rangle \quad \forall \quad \varphi \in \mathcal{D}$$

where N is the neutrix, having N' a set of natural numbers and range N'' of the real numbers $f(n)$ for which $\lim_{n \to \infty} f(n) = 0$. It is obvious that if the product $f.g$ exists, then the neutrix product fog exists and $f.g = fog$.

The following theorem is stated in [23].

Theorem 1.3.1. *Let f and g be distributions in $\mathcal{D}'$, the space of infinitely differentiable functions with compact support defined in [40] and suppose that the non commutative neutrix products fog and $fog'(or f'og)$ exists then the product $fog'(or f'og)$ exists and $(fog)' = f'og + fog'$.*

We now recall the definition commutative neutrix convolution product of two distributions. We have discussed above the definition of non commutative neutix product using function $\rho(x)$. In a similar manner for commutative neutrix product of distribution function is defined as- Let τ be a function in $\mathcal{D}$ (see D.S. Jones [44]) satisfying following conditions :

(i) $\tau(x) = \tau(-x)$

(ii) $0 \leq \tau(x) \leq 1$

(iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$

(iv) $\tau(x) = 0$ for $|x| \geq 1$

The function τ_ν for $\nu > 0$ is defined as

$$\tau_\nu(x) = \begin{cases} 1, & |x| \leq \nu \\ \tau(\nu^\nu x - \nu^{\nu+1}), & x > \nu \\ \tau(\nu^\nu x + \nu^{\nu+1}), & x < -\nu \end{cases} \tag{1.3.1}$$

Definition 1.3.4. Let f and g be distributions in $\mathcal{D}'$ and let $f_\nu = f\tau_\nu$ for $\nu > 0$. The neutrix convolution product $f \circledast g$ is defined as the neutrix limit of the sequence $f_\nu * g$, provided that the limit h exists in the sense that

$$N - \lim_{\nu \to \infty} \langle f_\nu * g, \varphi \rangle = \langle h, \varphi \rangle \tag{1.3.2}$$

for all φ in $\mathcal{D}$, where N is the neutrix , having domain natural number and range real numbers, with negligible functions with finite linear sums of the functions $\nu^{\lambda}\ln^{r-1}\nu, \ln^{r}\nu, \nu^{r}\sin\nu^{k}$ and $\nu^{r}\cos\nu^{k}(\lambda \neq 0, r = 1, 2\ldots)$ and all functions which converge to zero in the normal sense as ν tends to infinity.

Neutrix convolution of f and g defined by the notation $f \circledast g$.

$$f \circledast g = N - \lim_{\nu\to\infty} \langle f_{\nu} * g, \varphi\rangle \tag{1.3.3}$$

Using above definition, I have given some neutrix convolution product of distributions by some associated functions given by us, will be discuss in next chapters 3.

1.4 Mellin Convoluion Product

Definition 1.4.1. Mellin convolution operator of two functions h and φ defined on R^n can be given as

$$h \circledast_M \varphi = \int_0^{\infty} h(x/t)\, \varphi(t)\, \frac{dt}{t} \tag{1.4.1}$$

1.5 Extended Hankel Transform

1.5.1 Hankel transform

An integral transform of the function $f(t)$ is a mapping of the form

$$\varphi(p) = \int_a^b K(p, x) f(x) dx \tag{1.5.1}$$

provided the integral exists. where $f(t)$ be a function defined on the interval (a, b) and $K(p, x)$ is the described function called the kernel.

We may regard 1.5.1 as an integral transformation between the function φ and f and build up an abstract theory of integral transform in the light of Banach spaces. The simplest of such kernels, with the range of integration from 0 to ∞ are $\exp(-px)$, x^{p-1}, $(px)^{\frac{1}{2}} J_\nu(px)$ and $(p+x)^{-1}$.

The set of these kernels leads to four very important integral transforms, that are listed here.

(i) Laplace Transform

$$\varphi(p) = \int_0^\infty \exp(-px) f(x) dx \tag{1.5.2}$$

(ii) Mellin Transform

$$\varphi(p) = \int_0^\infty x^{p-1} f(x) dx \tag{1.5.3}$$

(iii) Hankel Transform

$$\varphi(p) = \int_0^\infty (px)^{\frac{1}{2}} J_\nu(px) f(x) dx \tag{1.5.4}$$

(iv) Stieltjes Transform

$$\varphi(p) = \int_0^\infty (p+x)^{-1} f(x) dx \tag{1.5.5}$$

Our work is mainly concerned with the Hankel Transform. Generalized Hankel transform given by Mendez Perez Sanchez Quintana in [49] and Molgonde,S.P. etc.

Hankel transform of order μ of a function $f(x)$ is given by

$$F(y) = \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy) f(x)\, dx \tag{1.5.6}$$

where $J_\mu(xy)$ is the Bessel Function of orderμ with $\mu \geq -\frac{1}{2}$. Rooney in [53] has studied the boundedness and range of the transformation

$$(H_{\mu,m}f)(x) = \int_0^\infty \sqrt{xy} J_{\mu,m}(xy) f(y) dy \tag{1.5.7}$$

where,

$$J_{\mu,m}(x) = \sum_{k=m}^{\infty} \frac{(-1)^k (x/2)^{\mu+2k}}{\Gamma(k+1)\Gamma(\mu+k+1)} \tag{1.5.8}$$

on $L_{\mu,p}$ space, whereas [4], studied (1.5.7) on the space H_μ and it's generalized function spaces $H_\mu^{'}$ introduced by [60]. Mendez Perez and Sanchez Quintana in [49], have studied the transformations,

$$(B_{1,\mu}\psi(x))(y) = \int_0^\infty x^{2\mu+1} \mathcal{J}_\mu(xy)\psi(x)dx \tag{1.5.9}$$

$$(B_{2,\mu}\psi(x))(y) = y^{2\mu+1} \int_0^\infty \mathcal{J}_\mu(xy)\psi(x)dx \tag{1.5.10}$$

where $\mathcal{J}_\mu(z) = z^{-\mu} J_\mu(z)$

for the testing function space H and H_μ and their dual spaces. By using the cut Bessel function (1.5.8) we can extend the transformation (1.5.9) & (1.5.10) as

$$(B_{1,\mu,m}\psi(x))(y) = y^{-\mu} \int_0^\infty x^{\mu+1} J_{\mu,m}(xy)\psi(x)dx \tag{1.5.11}$$

$$(B_{2,\mu,m}\psi(x))(y) = y^{\mu+1} \int_0^\infty x^{-\mu} J_{\mu,m}(xy)\psi(x)dx \tag{1.5.12}$$

respectively.

1.6 Generalized Fresnel Integrals

In this section we will discuss the definition of generalized Fresnel integrals. Fresnel integral is defined in [1] as

1. **Fresnel sine integral**

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin u^2 \, du$$

2. **Fresnel cosine integral**

$$C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos u^2 \, du$$

1.6.1 Generalized Fresnel Sine Integral

Kilicman used the Fresnel Sine integral in [45] to find convolution and neutrix convolution with different distributions. Taking similar way Limonka, Biljana in [47] have taken generalized Fresnel sine integral as

$$S_k(x) = \int_0^x \sin u^k \, du \tag{1.6.1}$$

Using generalized Fresnel sine integral and Heaviside step function [47] defined associated functions $S_{k+}(x)$ and $S_{k-}(x)$ where $k = 1, 2 \ldots$. These can be defined as-

$$S_{k+}(x) = H(x)S_k(x) \tag{1.6.2}$$

$$S_{k-}(x) = H(-x)S_k(x) \tag{1.6.3}$$

Where $H(x)$ is Heaviside step function.

Using these associated functions they defined neutrix convolution with distribution x^r where $r = 0, 1, 2, \ldots$.

We defined the associated functions $S^R_{k+}(x)$ and $S^R_{k-}(x)$ with equation(1.6.1) by using ramp function $R(x)$. These functions can be defined as

$$S^R_{k+}(x) = R(x)S_k(x)$$

$$S^R_{k-}(x) = R(-x)S_k(x)$$

for $k = 1, 2 \ldots$ and $r = 0, 1, 2 \ldots$, where $R(x)$ is ramp function which is the anti-derivative of Heaviside step function. It is defined as follows.

$$R(x) = \int_{-\infty}^{x} H(\xi)d\xi = xH(x) \tag{1.6.4}$$

1.6.2 Generalized Fresnel Cosine Integral

In a similar manner we generalized the cosine integral as

$$C_k(x) = \int_0^x \cos u^k \, du. \tag{1.6.5}$$

Using generalized Fresnel cosine integral and Ramp function we defined associated functions $C^R_{k+}(x)$ and $C^R_{k-}(x)$ where $k = 1, 2 \ldots$. These Function can be given as:

$$C_{k+}(x) = R(x)C_k(x) \tag{1.6.6}$$

$$C_{k-}(x) = R(-x)C_k(x). \tag{1.6.7}$$

Using these associated function of Fresnel integrals, we defined convolution product and neutrix product of distribution.