Chapter 3

Neutrix Product of Distribution

In this chapter, we propose some generalized results on the product of distributions $x_+^\lambda \ln^p x_+,\ x_+^\mu \ln^q x_+,\ x_+^\nu \ln^r x_+,\ x_-^\lambda \ln^p x_-,\ x_-^\mu \ln^q x_-,\ x_-^\nu \ln^r x_-$ and $sgn\ x\,|x|^\lambda \ln^p x_+$ given by Fisher, B.

Let the space $\mathcal{D}$ be the space of infinitely differentiable functions with compact support in $[a, b]$ and $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Locally summable functions $x_+^\lambda \ln^p x_+$ and $x_-^\lambda \ln^p x_-$ for $\lambda \geq -1$ and $p = 0, 1, 2, ..$ defined by Fisher are

$$x_+^\lambda \ln^p x_+ = \begin{cases} x^\lambda \ln^p x & x > 0 \\ 0 & x < 0 \end{cases} \tag{3.0.1}$$

$$x_-^\lambda \ln^p x_- = \begin{cases} |x|^\lambda \ln^p |x| & x > 0 \\ 0 & x < 0 \end{cases} \tag{3.0.2}$$

Derivatives of x_+^λ and x_-^λ are given by

$$(x_+^\lambda)' = \quad \lambda\, x_+^{\lambda-1} \tag{3.0.3}$$

$$(x_-^\lambda)' = -\lambda\, x_-^{\lambda-1} \tag{3.0.4}$$

for $\lambda < -1$ and $\lambda \neq -2, -3, \ldots$

If r is a positive integer and $-r-1 < \lambda < -r$ then for arbitrary φ in $\mathcal{D}$, we can define the inner product as follows

$$\langle x_+^\lambda, \varphi(x)\rangle = \int_0^\infty x^\lambda \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i\right] dx \tag{3.0.5}$$

$$\langle x_-^\lambda, \varphi(x)\rangle = \int_{-\infty}^0 |x|^\lambda \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i\right] dx \tag{3.0.6}$$

For $p = 0, 1, \ldots$ the distributions $x_+^\lambda \ln^p x_+$ and $x_-^\lambda \ln^p x_-$ are defined as,

$$\begin{aligned}\langle x_+^\lambda \ln^p x_+, \varphi(x)\rangle &= \frac{\partial^p}{\partial \lambda^p}\langle x_+^\lambda, \varphi(x)\rangle \\ &= \int_0^\infty x^\lambda \ln^p x \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i\right] dx \end{aligned} \tag{3.0.7}$$

$$\langle x_-^\lambda \ln^p x_-, \varphi(x)\rangle = \int_{-\infty}^0 |x|^\lambda \ln^p(|x|) \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i\right] dx \tag{3.0.8}$$

If φ is a function whose support is contained in the interval $[-1, +1]$, then

$$\langle x_+^\lambda \ln^p x_+, \varphi(x)\rangle = \int_0^1 x^\lambda \ln^p(x)\left[\psi(x) - \sum_{i=0}^{r-1}\frac{\varphi^{(i)}(0)}{i!}x^i\right]dx + \sum_{i=0}^{r-1}\frac{\varphi^{(i)}(0)}{i!(\lambda+i+1)} \tag{3.0.9}$$

The following theorem from [23] are stated as below.

Theorem 3.0.1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the non commutative neutrix products fog and $fog'(or f'og)$ exists then the product $fog'(or f'og)$ exists and $(fog)' = f'og + fog'$.*

Theorem 3.0.2. *The non commutative neutrix products of $x_+^\lambda \ln^p x_+$ and $x_+^\mu \ln^q x_+$ and of $x_-^\lambda \ln^p x_-$ and $x_-^\mu \ln^q x_-$ exist and*

$$(x_+^\lambda \ln^p x_+)o(x_+^\mu \ln^q x_+) = x_+^{\lambda+\mu} \ln^{p+q} x_+ \tag{3.0.10}$$

$$(x_-^\lambda \ln^p x_-)o(x_-^\mu \ln^q x_-) = x_-^{\lambda+\mu} \ln^{p+q} x_- \tag{3.0.11}$$

for $\lambda+\mu < -1$ and $\lambda, \mu, \lambda+\mu \neq -1, -2, \ldots$ and $p, q = 0, 1, 2, \ldots$.

3.1 Product of distributions

In this section we provide the neutrix product of three distributions $x_+^\lambda \ln^p x_+$, $x_+^\mu \ln^q x_+$ and $x_+^\nu \ln^r x_+$. In the same way we can get the product of distributions $x_-^\lambda \ln^p x_-$, $x_-^\mu \ln^q x_-$ and $x_-^\nu \ln^r x_-$.

Theorem 3.1.1. *The non-commutative neutrix products of* $x_+^\lambda \ln^p x_+$, $x_+^\mu \ln^q x_+$, *and* $x_+^\nu \ln^r x_+$ *and of* $x_-^\lambda \ln^p x_-$, $x_-^\mu \ln^q x_-$ *and* $x_-^\nu \ln^r x_-$ *exist and*

$$(x_+^\lambda \ln^p x_+)o(x_+^\mu \ln^q x_+)o(x_+^\nu \ln^r x_+) = x_+^{\lambda+\mu+\nu} \ln^{p+q+r} x_+ \qquad (3.1.1)$$

$$(x_-^\lambda \ln^p x_-)o(x_-^\mu \ln^q x_-)o(x_-^\nu \ln^r x_-) = x_-^{\lambda+\mu+\nu} \ln^{p+q+r} x_- \qquad (3.1.2)$$

for $\lambda, \lambda+\mu, \lambda+\mu+\nu < -1$ *and* $\lambda, \mu, \nu, \lambda+\mu, \lambda+\mu+\nu \neq -1, -2, \ldots$ *and* $p, q, r = 0, 1, 2, \ldots$.

Proof. We will first prove the following results

$$(x_+^\lambda)o(x_+^\mu)o(x_+^\nu) = x_+^{\lambda+\mu+\nu} \qquad (3.1.3)$$

$$(x_-^\lambda)o(x_-^\mu)o(x_-^\nu) = x_-^{\lambda+\mu+\nu} \qquad (3.1.4)$$

Fisher has proved that

$$(x_+^\lambda)o(x_+^\mu) = x_+^{\lambda+\mu} \qquad (3.1.5)$$

$$(x_-^\lambda)o(x_-^\mu) = x_-^{\lambda+\mu} \qquad (3.1.6)$$

Using equations(3.1.5) and (3.1.6) in equations(3.1.3) and (3.1.4)

$$(x_+^{\lambda+\mu})o(x_+^\nu) = x_+^{\lambda+\mu+\nu} \qquad (3.1.7)$$

$$(x_-^{\lambda+\mu})o(x_-^\nu) = x_-^{\lambda+\mu+\nu} \qquad (3.1.8)$$

We will now first prove equations(3.1.7) and (3.1.8).

For proving equation(3.1.7) we are taking here $-s-1 < \nu < -s$, for some nonnegative integer s; $\lambda, \mu > -1, \lambda+\mu > -1$ and $\lambda+\mu+\nu \neq -1, -2, \ldots$. Let k be the smallest positive integer greater than $-\lambda-\mu-\nu$.

For this we have to use the result, which can be derived as

$$\begin{aligned} x_+^\nu * \delta_n(x) &= \int_{-1/n}^{1/n} (x-t)^\nu \delta_n(t)\, dt \\ &= \int_{-1/n}^{x} (x-t)^\nu \delta_n(t)\, dt + \int_{x}^{1/n} (x-t)^\nu \delta_n(t)\, dt \end{aligned} \tag{3.1.9}$$

Since

$$(x-t)_+^\nu = \begin{cases} (x-t)^\nu & for \quad x > t \\ 0 & for \quad x < t \end{cases} \tag{3.1.10}$$

Using equation(3.1.10) in equation(3.1.9) second integral vanishes and further using the property of $\delta_n(t)$ we get

$$\begin{aligned} x_+^\nu * \delta_n(x) &= \int_{-1/n}^{x} (x-t)^\nu \delta_n(t)\, dt \\ &= \frac{1}{\nu+1}\left[\delta_n(t)(x-t)^{\nu+1}\right]_{-1/n}^{x} + \frac{1}{\nu+1}\int_{-1/n}^{x} (x-t)^{\nu+1}\delta_n^{(1)}(t)\, dt \\ &= \frac{1}{(\nu+1)(\nu+2)}\int_{-1/n}^{x} (x-t)^{\nu+2}\delta_n^{(2)}(t)\, dt \\ &= \frac{1}{(\nu+1)(\nu+2)(\nu+3)}\int_{-1/n}^{x} (x-t)^{\nu+3}\delta_n^{(3)}(t)\, dt \\ &= \frac{1}{(\nu+1)(\nu+2)(\nu+3)\ldots(\nu+s)}\int_{-1/n}^{x} (x-t)^{\nu+s}\delta_n^{(s)}\, dt \\ &= \frac{\Gamma(\nu+1)}{\Gamma(\nu+1)}\left[\frac{1}{(\nu+1)(\nu+2)(\nu+3)\ldots(\nu+s)}\right] x_+^{\nu+s} * \delta_n^{(s)} \\ &= \frac{\Gamma(\nu+1)}{\Gamma(\nu+s+1)} x_+^{\nu+s} * \delta_n^{(s)} \end{aligned} \tag{3.1.11}$$

where Γ denotes the gamma function.

We have

$$\begin{aligned}\int_{-1}^{+1} x^i x_+^{\lambda+\mu}(x_+^\nu)_n dx &= \int_0^1 x^{i+\lambda+\mu}(x_+^\nu)_n dx \\ &= \int_0^{1/n} x^{i+\lambda+\mu}(x_+^\nu)_n dx + \int_{1/n}^1 x^{i+\lambda+\mu}(x_+^\nu)_n dx \\ &= \frac{\Gamma(\nu+1)}{\Gamma(\nu+s+1)} \int_0^{1/n} x^{\lambda+\mu+i} \int_{-1/n}^{x} (x-t)^{\nu+s}\delta_n^{(s)}(t)dt\, dx \\ &\quad + \int_{1/n}^1 x^{\lambda+\mu+i} \int_{-1/n}^{1/n} (x-t)^\nu \delta_n(t)dtdx \\ &= I_1 + I_2 \end{aligned} \tag{3.1.12}$$

On putting $nt = v$ and $nx = u$ in I_1, we have

$$\begin{aligned} I_1 &= \frac{\Gamma(\nu+1)}{\Gamma(\nu+s+1)} \int_0^1 \frac{u^{\lambda+\mu+i}}{n^{\lambda+\mu+i}} \int_{-1}^{u} \frac{(u-v)^{\nu+s}}{n^{\nu+s}} n^{s+1}\rho^{(s)}(v)\frac{dv}{n}\,\frac{du}{n} \\ &= \frac{\Gamma(\nu+1)}{\Gamma(\nu+s+1)} \int_0^1 \frac{u^{\lambda+\mu+i}}{n^{\lambda+\mu+i+\nu+1}} \int_{-1}^{u} (u-v)^{\nu+s}\rho^{(s)}(v)\, dv\, du \end{aligned}$$

$$I_1 = n^{-\lambda-\mu-\nu-i-1}\frac{\Gamma(\nu+1)}{\Gamma(\nu+s+1)} \int_0^1 u^{\lambda+\mu+i} \int_{-1}^{u} (u-v)^{\nu+s}\rho^{(s)}(v)dvdu \tag{3.1.13}$$

Thus,

$$N - \lim_{n\to\infty} I_1 = 0 \tag{3.1.14}$$

for $i = 0, 1, 2, \ldots k-1$

On changing the order of integration in I_2

$$\begin{aligned} I_2 &= \int_{-1/n}^{1/n} \delta_n(t) \int_{1/n}^{1} x^{\lambda+\mu+i}(x-t)^{\nu} dxdt \\ &\quad \text{for } nx = u \text{ and } nt = v \\ &= \int_{-1}^{+1} \rho(v) \int_{1}^{n} n^{-i-\lambda-\mu-\nu-1} u^{\lambda+\mu+i}[u-v]^{\nu} dudv \\ &= n^{-i-\lambda-\mu-\nu-1} \int_{-1}^{1} \rho(v) \int_{1}^{n} u^{\lambda+\mu+\nu+i} \left[1-\frac{v}{u}\right]^{\nu} dudv \\ &= n^{-i-\lambda-\mu-\nu-1} \int_{-1}^{1} \rho(v) \int_{1}^{n} u^{\lambda+\mu+\nu+i} \left[1-\frac{v}{u}\nu+\dots\right] dudv \end{aligned}$$

$$\begin{aligned} N-\lim_{n\to\infty} I_2 &= (i+\lambda+\mu+\nu+1)^{-1} \int_{-1}^{+1} \rho(v)dv \\ &= (i+\lambda+\mu+\nu+1)^{-1} \end{aligned} \tag{3.1.15}$$

for $i = 0, 1, 2, \dots, k-1$ and using property (iv) of the function $\rho(x)$

By equation(3.1.12), equation(3.1.14) and equation(3.1.15)

$$N-\lim_{n\to\infty} \int_{-1}^{+1} x^i x_+^{\lambda+\mu} (x_+^{\nu})_n dx = (i+\lambda+\mu+\nu+1)^{-1} \tag{3.1.16}$$

for $i = 0, 1, 2, \dots k-1$

On taking $i = k$ in equation(3.1.13), we get

$$\begin{aligned} I_1 &= n^{-k-\lambda-\mu-\nu} \frac{\Gamma(\nu+1)}{\Gamma(\nu+s+1)} \int_0^1 u^{k+\lambda+\mu} \int_{-1}^{u} (u-v)^{\nu+s} \rho^{(s)}(v) dvdu \\ &= \int_0^{1/n} x^{k+\lambda+\mu} (x_+^{\nu})_n dx \end{aligned} \tag{3.1.17}$$

If ψ is an arbitrary continuous function then

$$\lim_{n\to\infty} \int_0^{1/n} x^{k+\lambda+\mu} (x_+^{\nu})_n \psi(x) dx = 0 \tag{3.1.18}$$

since $k+\lambda+\mu+\nu>0$

Next if $x>1/n$, we have

$$\begin{aligned}(x_+^\nu)_n &= \int_{-1/n}^{1/n}(x-t)^\nu\delta_n(t)dt \\ &= \int_{-1}^{1}(x-u/n)^\nu\rho(u)du\end{aligned}$$

for $t=u/n$

$$\begin{aligned}(x_+^\nu)_n &= x^\nu\int_{-1}^{1}\left[1-\frac{u}{nx}\right]^\nu\rho(u)du \\ &= x^\nu\int_{-1}^{1}\left[1-\frac{\nu u}{nx}+\ldots\right]\rho(u)du \\ &= x^\nu+o(x^{\nu-1}n^{-1}) \qquad (3.1.19)\end{aligned}$$

$$\lim_{n\to\infty}\int_{1/n}^{1}x^{k+\lambda+\mu}(x_+^\nu)_n\psi(x)dx=\int_0^1 x^{k+\lambda+\mu+\nu}\psi(x)dx. \qquad (3.1.20)$$

Now let φ be an arbitrary function in $\mathcal{D}[-1,1]$. By the mean value theorem we have

$$\varphi(x)=\sum_{i=0}^{k-1}\frac{\varphi^i(0)}{i!}x^i+\frac{\varphi^k(\xi x)}{k!}x^k \qquad (3.1.21)$$

where $0<\xi<1$

then

$$\begin{aligned}\left\langle x_+^{\lambda+\mu}(x_+^\nu)_n,\varphi(x)\right\rangle &= \int_{-\infty}^{\infty}x_+^{\lambda+\mu}(x_+^\nu)_n\varphi(x)dx \\ &= \sum_{i=0}^{k-1}\frac{\varphi^i(0)}{i!}\int_{-1}^{+1}x^i x_+^{\lambda+\mu}(x_+^\nu)_n dx+\int_0^{1/n}\frac{x^{k+\lambda+\mu}(x_+^\nu)_n\varphi^k(\xi x)}{k!}dx \\ &\quad +\int_{1/n}^{1}\frac{x^{k+\lambda+\mu}(x_+^\nu)_n\varphi^k(\xi x)}{k!}dx.\end{aligned}$$

Using equations(3.1.15), (3.1.18) and (3.1.20)

$$\begin{aligned}
&N - \lim_{n\to\infty} \left\langle x_+^{\lambda+\mu}(x_+^{\nu})_n, \varphi(x) \right\rangle \\
&\quad = \sum_{i=0}^{k-1} \frac{\psi^i(0)}{(i)!} \int_0^1 x^{\lambda+\mu+\nu+i} dx + \int_0^1 x^{\lambda+\mu+\nu} \frac{x^k \varphi^k(\xi x)}{k!} dx \\
&\quad = \int_0^1 x^{\lambda+\mu+\nu} \left[\varphi(x) - \sum_{i=0}^{k-1} \frac{\varphi^i(0)}{i!} x^i \right] dx + \sum_{i=0}^{k-1} \frac{\varphi^{(i)}(0)}{i!(\lambda+\mu+\nu+i+1)} \\
&\quad = \left\langle x_+^{\lambda+\mu+\nu}, \varphi(x) \right\rangle \qquad (3.1.22)
\end{aligned}$$

This follows the result (3.1.7). Similarly we can prove (3.1.8). Thus equation holds on the interval $[-1, 1]$.

Now differentiating equation (3.1.7) partially with respect to λ, p times we get

$$(x_+^{\lambda} \ln^p x_+) \; o \; (x_+^{\mu}) \; o \; (x_+^{\nu}) = x_+^{\lambda+\mu+\nu} \ln^p x_+$$

which on differentiating partially with respect to μ, q times, yields

$$(x_+^{\lambda} \ln^p x_+) \; o \; (x_+^{\mu} \ln^q x_+) \; o \; (x_+)^{\nu} = x_+^{\lambda+\mu+\nu} \ln^{p+q} x_+$$

Again differentiating partially with respect to ν, r times, we have

$$\begin{aligned}
&(x_+^{\lambda} \ln^p x_+) \; o \; (x_+^{\mu} \ln^q x_+) \; o \; (x_+^{\nu} \ln^r x_+) \\
&\qquad\qquad = x_+^{\lambda+\mu+\nu} (\ln^p x_+) \; o \; (\ln^q x_+) \; o \; (\ln^r x_+)
\end{aligned}$$

By Fisher[36], we have

$$(\ln^p x_+) \; o \; (\ln^q x_+) = \ln^{p+q} x_+ \qquad (3.1.23)$$

which on using this gives (3.1.1) Similarly by the application of equation(3.1.8) we can prove (3.1.2). □

Theorem 3.1.2. *The neutrix products of* $x_+^\lambda \ln^p x_+$, $x_+^\mu \ln^q x_+$, $x_+^\nu \ln^r x_+$, *and of* $x_-^\lambda \ln^p x_-$, $x_-^\mu \ln^q x_-$, $x_-^\nu \ln^r x_-$ *exist and*

$$(x_+^\lambda \ln^p x_+)\ o\ (x_-^\mu \ln^q x_-)\ o\ (x_+^\nu \ln^r x_+) = 0 \tag{3.1.24}$$

$$(x_-^\lambda \ln^p x_-)\ o\ (x_+^\mu \ln^q x_+)\ o\ (x_-^\nu \ln^r x_-) = 0 \tag{3.1.25}$$

for $\lambda+\mu<-1,\quad \lambda+\mu+\nu<-1,\ \ \lambda,\mu,\nu,\lambda+\mu,\lambda+\mu+\nu\neq -1,-2,\ldots$ *and* $\ \ p,q,r=0,1,2,\ldots$

Proof. Fisher in [36] has given the neutrix product of $x_-^\lambda \ln^p x_-$ and $x_+^\mu \ln^q x_+$ and of $x_+^\lambda \ln^p x_+$ and $x_-^\mu \ln^q x_-$ in [36] as-

$$(x_-^\lambda \ln^p x_-) o (x_+^\mu \ln^q x_+) = 0 \tag{3.1.26}$$

$$(x_+^\lambda \ln^p x_+) o (x_-^\mu \ln^q x_-) = 0 \tag{3.1.27}$$

for $\lambda+\mu<-1, \lambda,\mu,\lambda+\mu\neq -1,-2,\ldots$ and $p,q=0,1,2,\ldots$
Composing equation (3.1.26)by $x_+^\nu \ln^r x_+$ and equation(3.1.27)by $x_-^\nu \ln^r x_-$ from the left, we get the required result. □

Theorem 3.1.3.

$$\Big(sgn\ x\,|x|^\lambda \ln^p |x|\Big)\ o\Big(\ |x|^\mu \ln^q |x|\Big) o\Big(sgn\ x\,|x|^\nu \ln^r |x|\ \Big) = |x|^{\lambda+\mu+\nu} \ln^{p+q+r} |x| \tag{3.1.28}$$

$$\Big(|x|^{\lambda}\ln^{p}|x|\Big)\, o\Big(sgn\ x\,|x|^{\mu}\ln^{q}|x|\,\Big) o\Big(\,|x|^{\nu}\ln^{r}|x|\,\Big)$$
$$= sgn\ x\,|x|^{\lambda+\mu+\nu}\ln^{p+q+r}|x| \qquad (3.1.29)$$

for $\lambda+\mu<-1, \quad \lambda+\mu+\nu<-1, \ \ \lambda,\mu,\nu,\lambda+\mu,\lambda+\mu+\nu \neq -1,-2,\ldots$ *and* $\ p,q,r=0,1,2,\ldots$

Proof. Fisher in [13] have shown that

$$sgn\ x\,|x|^{\lambda}\ln^{p}|x| = x_{+}^{\lambda}\ln^{p}x_{+} - x_{-}^{\lambda}\ln^{p}x_{-} \qquad (3.1.30)$$

$$|x|^{\lambda}\ln^{p}|x| = x_{+}^{\lambda}\ln^{p}x_{+} + x_{-}^{\lambda}\ln^{p}x_{-} \qquad (3.1.31)$$

In [36], it is proved that

$$\Big(sgn\ x\,|x|^{\lambda}\ln^{p}|x|\Big)\ o\ \Big(|x|^{\mu}\ln^{q}|x|\Big) = sgn\ x\,|x|^{\lambda+\mu}\ln^{p+q}|x| \quad (3.1.32)$$

$$\Big(|x|^{\lambda}\ln^{p}|x|\Big)\ o\ \Big(sgn\ x\,|x|^{\mu}\ln^{q}|x|\,\Big) = sgn\ x\,|x|^{\lambda+\mu}\ln^{p+q}|x| \quad (3.1.33)$$

Using equations (3.1.30), (3.1.31), (3.1.32) and (3.1.33)

$$\begin{aligned}
&\Big(sgn\ x\,|x|^{\lambda}\ln^{p}|x|\,\Big)\ o\ \Big(\,|x|^{\mu}\ln^{q}|x|\,\Big)\ o\ \Big(sgn\ x\,|x|^{\nu}\ln^{r}|x|\,\Big)\\
&= \Big(sgn\ x\,|x|^{\lambda+\mu}\ln^{p+q}|x|\Big)\ o\ \Big(sgn\ x\,|x|^{\nu}\ln^{r}|x|\,\Big)\\
&= \Big(x_{+}^{\lambda+\mu}\ln^{p+q}x_{+} - x_{-}^{\lambda+\mu}\ln^{p+q}x_{-}\Big)\ o\ \Big(x_{+}^{\nu}\ln^{r}x_{+} - x_{-}^{\nu}\ln^{r}x_{-}\Big)\\
&= x_{+}^{\lambda+\mu+\nu}\ln^{p+q+r}x_{+}\ +\ x_{-}^{\lambda+\mu+\nu}\ln^{p+q+r}x_{-}\\
&= |x|^{\lambda+\mu+\mu}\ln^{p+q+r}|x|
\end{aligned}$$

Similarly we can prove equation (3.1.29). □