Chapter-IV

Sensitivity analysis for a new class of generalized parametric nonlinear ordered variational inequality problem in ordered Banach spaces

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Abstract

The aim of this chapter is to introduce and study a new class of generalized parametric nonlinear ordered variational inequality problem and discuss its existence result. We also discuss the sensitivity analysis of the solution for the said class by using the $B$-restricted-accretive methods in ordered Banach spaces. Several special cases are discussed, which can be obtained from our results.
Sensitivity analysis for a new class of generalized parametric NOVI

“One can imagine that the ultimate mathematician is one who can see analogies between analogies.”

Stefan Banach

4.1 Introduction

Generalized nonlinear ordered variational inequalities (ordered equations) have wide applications to many fields including, mechanics, physics, optimization and control, nonlinear programming, economics and engineering sciences.

Lately, much consideration has been given to the sensitivity analysis of variational inequalities. We think that sensitivity analysis is imperative for a few reasons, as follow:

Firstly, since assessing data information regularly presents estimation errors, sensitivity analysis helps in distinguishing touchy parameters that ought to be acquired with generally high exactness. Secondly, sensitivity analysis may anticipate the future changes of the equilibrium due to the changes in the governing systems. Finally, sensitivity analysis gives valuable data for outlining or arranging different equilibrium systems. Moreover, from scientific and engineering perspectives, sensitivity analysis can give new bits of knowledge with respect to issues being considered and can fortify new thoughts for critical thinking. In the course of
the most recent decade, there has been expanding enthusiasm for concentrate the sensitivity analysis of variational inequalities and variational inclusions.

We likewise contemplate the subjective conduct of the solution of the variational inequalities when the given operator and the feasible convex set change with a parameter. Such an investigation is known as sensitivity analysis, which is vital and significant. Sensitivity analysis provides us helpful data for outlining different equilibrium systems, and for anticipating the future changes of the equilibria because of the adjustments in the governing systems.

S. Dafermos in [25], utilized the fixed-point technique to consider the sensitivity analysis of the classical variational inequalities. This system has been changed and stretched out by numerous authors for considering the sensitivity analysis of different classes of variational inequalities and variational inclusions, for more details see [10, 17, 18, 19, 56, 67, 79, 85, 86].

H. Amman [7], in 1972, proposed the idea of finding the fixed points of the nonlinear mapping in ordered Banach spaces. Motivated by the idea of Amman, H.G. Li [53] introduced and studied the generalized nonlinear ordered variational inequalities and equations. Moreover, Li proposed and proved existence and convergence result for approximation solution for the generalized nonlinear ordered variational inequalities in ordered Banach space by using the $B$-restricted-accretive method of map $A$. H. G. Li [54] in 2009 introduced and studied a new class of general nonlinear variational inequalities and ordered equations involving $\oplus$ operator in ordered Banach space. The sensitivity analysis is also investigated by H. G. Li [55], in which an existence result was proposed for a new class of general nonlinear ordered parametric variational inequalities in ordered Banach spaces.

Motivated by the ongoing research in this direction, we present a new class of generalized parametric nonlinear ordered variational inequalities involving $\oplus$ operator in ordered Banach space and prove the existence and continuity of the solution of the said problem in this chapter.

### 4.2 New class of generalized parametric nonlinear ordered variational inequality

Let $\Sigma$ be non-void open subset of $\mathcal{E}$ in which the parameter $\rho$ takes values. Let $M, A, F, g, h : E \times \Sigma \to E$ be single-valued nonlinear ordered compression mappings such that
range $g(\cdot, \rho) \cap \text{dom} A(\cdot, \rho) \neq \emptyset$, range $h(\cdot, \rho) \cap \text{dom} F(\cdot, \rho) \neq \emptyset$ for any $\rho \in \Sigma$. Then, the following problem,

to find $x = x(\rho) : \Sigma \to \mathcal{E}$ such that

$$
\theta \leq M(x, \rho) + A(g(x, \rho), \rho) \oplus F(h(x, \rho), \rho),
$$

(4.2.1)
is called a new class of generalized parametric nonlinear ordered variational inequality problem with $\oplus$ operator in ordered Banach space.

### 4.2.1 Special Cases

**Case 1:** If $F(h(x, \rho), \rho) \equiv \theta$ (the zero map), then the problem (4.2.1) is converted to the problem (1.1) which was studied by H.G. Li in [55].

**Case 2:** If $F(h(x, \rho), \rho) \equiv \theta$ and $M(x, \rho) \equiv \theta$, then the problem (4.2.1) reduces to the problem (2.1) which was investigated by H.G. Li in [53].

**Lemma 4.2.1.** Let $\mathcal{E}$ be real ordered Banach space and $\mathcal{C}$ be a normal cone defined by the relation “$\leq$” having constant $\lambda_C$, and the mappings $M, A, F, (M + A \oplus F), g, h$ and $(M + A \oplus F) \wedge B : \mathcal{E} \times \Sigma \to \mathcal{E}$ be comparison mappings with each other. Let $x^*$ be a solution of the equation $\left[M(x, \rho) + A(g(x, \rho), \rho) \oplus F(h(x, \rho), \rho)\right] \wedge B(x, \rho) = \theta, \; (\theta \in \mathcal{E})$. Then $x^*$ is also a solution of generalized parametric nonlinear ordered variational inequality problem (4.2.1) with $\oplus$ operator in ordered Banach space.

**Proof.** The proof directly follows from the definition of $\wedge$ and the conditions that the mappings $M, A, F, (M + A \oplus F), g, h$ and $(M + A \oplus F) \wedge B : \mathcal{E} \times \Sigma \to \mathcal{E}$ are comparison mappings with each other. 

### 4.3 Existence result for generalized parametric NOVI with $\oplus$ operator

In this part of the chapter, we will prove the existence of solution of generalized parametric nonlinear ordered variational inequality problem (4.2.1).

**Theorem 4.3.1.** Let $\mathcal{E}$ be a real ordered Banach space and $\mathcal{C}$ be a normal cone having normal constant $\lambda_C$. Let $\Sigma \subset \mathcal{E}$ be an open subset in which the parameter $\rho$ takes values $M, A, F, B, g, h, (M + A \oplus F)$ and $(M + A \oplus F) \wedge B : \mathcal{E} \times \Sigma \to \mathcal{E}$ be comparison parametric
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mappings to each other. Let $M$ be $\lambda_M$-ordered compression, $A$ be $\lambda_A$-ordered compression, $F$ be $\lambda_F$-ordered compression, $B$ be $\lambda_B$-ordered compression, $g$ be $\lambda_g$-ordered compression and $h$ be $\lambda_h$-ordered compression mappings with respect to the second argument $\rho$, respectively. Further, let $(M + A \oplus F) : \mathcal{E} \times \Sigma \to \mathcal{E}$ be $B$-restricted accretive mapping with respect to the second argument $\rho$ for constants $\alpha_1$ and $\alpha_2$, and for any $\tau > 0$, the given condition

$$\tau \left[ \lambda_M + (\lambda_A \lambda_g + \lambda_F \lambda_h) \right] \vee \lambda_B < \frac{1 - \lambda_C \alpha_2}{\lambda_{C1}},$$

(4.3.1)

holds. Then the problem (4.2.1) admits a solution $x^*$.

Proof. Let $x_1 = x_1(\rho)$ and $x_2 = x_2(\rho)$ in $\mathcal{E}$, $\rho \in \Sigma$ such that $x_1(\rho) = x_2(\rho)$. We set

$$F(x_i(\rho), \rho) = \tau \left[ M(x_i(\rho), \rho) + A(g(x_i(\rho), \rho), \rho) \oplus F(h(x_i(\rho), \rho), \rho) \right]$$

$$\wedge B(x_i(\rho), \rho) + I(x_i(\rho), \rho),$$

(4.3.2)

where $i = 1, 2$. It follows from the conditions that $M, A, F, B, (M + A \oplus F), g, h$ and $(M + A \oplus F) \wedge B : \mathcal{E} \times \Sigma \to \mathcal{E}$ are comparison parametric mappings to each other with respect to the second argument $\rho$, and $x_1(\rho) = x_2(\rho)$ such that $F(x_1(\rho), \rho) \sim F(x_2(\rho), \rho)$. Using the conditions of restricted-accretive and the ordered-compression on suitable mappings with respect to the second argument $\rho$, and Proposition 1.2.7, we have

$$\theta \leq F(x_1, \rho) \oplus F(x_2, \rho)$$

$$\leq \left[ \tau \left( M(x_1(\rho), \rho) + A(g(x_1(\rho), \rho), \rho) \oplus F(h(x_1(\rho), \rho), \rho) \right) \right.$$

$$\wedge B(x_1(\rho), \rho) + I(x_1(\rho), \rho)$$

$$\oplus \left[ \tau \left( M(x_2(\rho), \rho) + A(g(x_2(\rho), \rho), \rho) \oplus F(h(x_2(\rho), \rho), \rho) \right) \right.$$

$$\wedge B(x_2(\rho), \rho) + I(x_2(\rho), \rho)$$

$$\leq \alpha_1 \tau \left[ \left( M(x_1(\rho), \rho) + A(g(x_1(\rho), \rho), \rho) \oplus F(h(x_1(\rho), \rho), \rho) \right) \right.$$

$$\wedge B(x_1(\rho), \rho)$$

$$\oplus \left( M(x_2(\rho), \rho) + A(g(x_2(\rho), \rho), \rho) \oplus F(h(x_2(\rho), \rho), \rho) \right) \right.$$ $$\wedge B(x_2(\rho), \rho) \right] + \alpha_2 (x_1(\rho) \oplus x_2(\rho))$$

$$\leq \tau \alpha_1 \left[ \left( M(x_1(\rho), \rho) + A(g(x_1(\rho), \rho), \rho) \oplus F(h(x_1(\rho), \rho), \rho) \right) \right.$$

$$\oplus \left( M(x_2(\rho), \rho) + A(g(x_2(\rho), \rho), \rho) \oplus F(h(x_2(\rho), \rho), \rho) \right) \right.$$

$$\wedge B(x_1(\rho), \rho) \right] + \alpha_2 (x_1(\rho) \oplus x_2(\rho))$$
Using the Definition 1.2.2 and Lemma 1.2.17, we obtain
\[
\forall \left( B(x_1(\rho), \rho) \oplus B(x_2(\rho), \rho) \right) + \alpha_2(x_1(\rho) \oplus x_2(\rho)) \\
\leq \tau \alpha_1 \left[ (M(x_1(\rho), \rho) \oplus M(x_2(\rho), \rho)) + (A(g(x_1(\rho), \rho) \oplus F(h(x_1(\rho), \rho)) \right) \\
\oplus (A(g(x_2(\rho), \rho) \oplus F(h(x_2(\rho), \rho)) \\
\oplus \lambda_B(x_1(\rho) \oplus x_2(\rho))) + \alpha_2(x_1(\rho) \oplus x_2(\rho)) \\
\leq \tau \alpha_1 \left[ (\lambda_M(x_1(\rho) \oplus x_2(\rho)) + (\lambda_A \lambda_g + \lambda_F \lambda_h)(x_1(\rho) \oplus x_2(\rho))) \right] \\
\vee \lambda_B(x_1(\rho) \oplus x_2(\rho)) + \alpha_2(x_1(\rho) \oplus x_2(\rho)) \\
\leq \tau \alpha_1 \left[ (\lambda_M + (\lambda_A \lambda_g + \lambda_F \lambda_h)) \vee \lambda_B \right] (x_1(\rho) \oplus x_2(\rho)) \\
+ \alpha_2(x_1(\rho) \oplus x_2(\rho)) \\
= \tau \alpha_1 \left[ (\lambda_M + (\lambda_A \lambda_g + \lambda_F \lambda_h)) \vee \lambda_B \right] + \alpha_2 (x_1(\rho) \oplus x_2(\rho)).
\]

Using the Definition 1.2.2 and Lemma 1.2.17, we obtain
\[
\|F(x_1(\rho), \rho) - F(x_2(\rho), \rho)\| \leq \lambda_C \Psi \|x_1(\rho) - x_2(\rho)\|,
\]
where \(\Psi = \tau \alpha_1 \left[ (\lambda_M + (\lambda_A \lambda_g + \lambda_F \lambda_h)) \vee \lambda_B \right] + \alpha_2\). It follows from the condition (4.3.1) that \(0 < \lambda_C \Psi < 1\), which in turn, implies that \(F(x(\rho), \rho)\) is a contraction mapping. Therefore, there exists \(x^* \in E\) which is a fixed point of \(F(x(\rho), \rho)\) and \(x^*\) is a solution of the problem (4.2.1), i.e., \(x^*\) is the solution of the generalized parametric nonlinear ordered variational equation with \(\oplus\) operator
\[
\left[ M(x, \rho) + A(g(x, \rho) \oplus F(h(x, \rho), \rho) \right] \wedge B(x, \rho) = \theta,
\]
for any parametric \(\rho \in \Sigma\). By the Lemma 4.2.1, there exists a solution \(x^*\) of the generalized parametric nonlinear ordered variational inequality (4.2.1).

\section*{4.4 Sensitivity analysis for generalized parametric NOVI with \(\oplus\) operator}

In this part of the chapter, we will discuss the sensitivity analysis for generalized parametric nonlinear ordered variational inequality problem (4.2.1).

\textbf{Theorem 4.4.1.} Let \(E\) be a real ordered Banach space and \(C\) be a normal cone having normal constant \(\lambda_C\), let \(\Sigma \subset E\) be an open subset in which the parameter \(\rho\) takes values,
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$M(\cdot, \rho), A(\cdot, \rho), F(\cdot, \rho), B(\cdot, \rho), g(\cdot, \rho)$ and $h(\cdot, \rho) : \mathcal{E} \times \Sigma :\to \mathcal{E}$ be the continuous parametric mappings with respect to the argument $\rho \in \Sigma$, $M, A, F, B, (M + A \oplus F), g, h$ and $(M + A \oplus F) \land B$ be comparison mappings to each other. Let $M$ be $\lambda_M$-ordered compression, $A$ be $\lambda_A$-ordered compression, $F$ be $\lambda_F$-ordered compression, $B$ be $\lambda_B$-ordered compression, $g$ be $\lambda_g$-ordered compression and $h$ be $\lambda_h$-ordered compression mappings with respect to the argument $\rho$. Further, if $(M + A \oplus F) : \mathcal{E} \times \Sigma \to \mathcal{E}$ is $B$-restricted accretive mapping with respect to the second argument $\rho$, and there exist constants $\alpha_1, \alpha_2$ and any $\tau > 0$ such that

$$
\tau \left[ \lambda_M + (\lambda_A \lambda_g \oplus \lambda_F \lambda_h) \right] \lor \lambda_B < \frac{1 - \alpha_2}{\alpha_1}
$$

(4.4.1)

holds. Then the solution $x(\rho)$ of the problem (4.2.1) is continuous in $\Sigma$.

**Proof.** Let $x(\rho)$ and $x(\bar{\rho})$ be two solutions of the problem (4.2.1) for any given $\rho, \bar{\rho} \in \Sigma$, then for any $\tau > 0$, we have

$$
x(\rho) = F(x(\rho), \rho)
$$

$$
= \tau \left[ M(x(\rho), \rho) + A(g(x(\rho), \rho), \rho) \oplus F(h(x(\rho), \rho), \rho) \right] \\
\land B(x(\rho), \rho) + I(x(\rho), \rho),
$$

$$
x(\bar{\rho}) = F(x(\bar{\rho}), \bar{\rho})
$$

$$
= \tau \left[ M(x(\bar{\rho}), \bar{\rho}) + A(g(x(\bar{\rho}), \bar{\rho}), \bar{\rho}) \oplus F(h(x(\bar{\rho}), \bar{\rho}), \bar{\rho}) \right] \\
\land B(x(\bar{\rho}), \bar{\rho}) + I(x(\bar{\rho}), \bar{\rho}).
$$

(4.4.2)

By the condition that $M, A, B, (M + A \oplus F), g, h$ and $(M + A \oplus F) \land B$ are comparison mappings to each other with respect to the second argument $\rho$ and Lemma 1.2.13, we have

$$
\theta = x(\rho) \oplus x(\bar{\rho})
$$

$$
= F(x(\rho), \rho) \oplus F(x(\bar{\rho}), \bar{\rho})
$$

$$
\leq F(x(\rho), \rho) \oplus \theta \oplus F(x(\bar{\rho}), \bar{\rho})
$$

$$
= [F(x(\rho), \rho) \oplus F(x(\bar{\rho}), \rho)] \oplus [F(x(\bar{\rho}), \rho) \oplus F(x(\bar{\rho}), \bar{\rho})].
$$

(4.4.3)

Furthermore, $(M + A \oplus F)$ is a $B$-restricted-accretive mapping with constants $\alpha_1, \alpha_2$, $M$ is $\lambda_M$-ordered compression, $A$ is $\lambda_A$-ordered compression, $F$ is $\lambda_F$-ordered compression, $B$ is $\lambda_B$-ordered compression, $g$ is $\lambda_g$-ordered compression, $h$ is $\lambda_h$-ordered compression, with respect to argument $\rho$, which by using Theorem 4.3.1, imply

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\[
F(x(\rho), \rho) \oplus F(x(\bar{\rho}), \rho) \leq \left[ \tau \left( M(x(\rho), \rho) + A(g(x(\rho), \rho), \rho) \right) \right.
\]

\[
\oplus F(h(x(\rho), \rho), \rho)
\]

\[
\wedge B(x(\rho), \rho) + I(x(\rho), \rho)
\]

\[
\oplus \left[ \tau \left( M(x(\bar{\rho}), \rho) + A(g(x(\bar{\rho}), \rho), \rho) \right) \right.
\]

\[
\oplus F(h(x(\bar{\rho}), \rho), \rho)
\]

\[
\wedge B(x(\bar{\rho}), \rho) + I(x(\bar{\rho}), \rho)
\]

\[
\leq \alpha_1 \left[ \left( \tau \left( M(x(\rho), \rho) + A(g(x(\rho), \rho), \rho) \right) \right) \right.
\]

\[
\oplus F(h(x(\rho), \rho), \rho) \right) \wedge B(x(\rho), \rho)
\]

\[
\left\{ \tau \left( M(x(\bar{\rho}), \rho) + A(g(x(\bar{\rho}), \rho), \rho) \right) \right.
\]

\[
\oplus F(h(x(\bar{\rho}), \rho), \rho) \right) \wedge B(x(\bar{\rho}), \rho) \right) \right) \] + \alpha_2(x(\rho) \oplus x(\bar{\rho})

\[
\leq \Psi(x(\rho) \oplus x(\bar{\rho})) \quad (4.4.4)
\]

where \(\Psi = \alpha_1 \left[ \tau(\lambda_M + (\lambda_A \lambda_G \oplus \lambda_F \lambda_h)) \right] \vee \lambda_B + \alpha_2 < 1\) for the condition (4.4.1), and

\[
F(x(\bar{\rho}), \rho) \oplus F(x(\bar{\rho}), \rho) \leq \left[ \tau \left( M(x(\bar{\rho}), \rho) + A(g(x(\bar{\rho}), \rho), \rho) \oplus F(h(x(\bar{\rho}), \rho), \rho) \right) \right.
\]

\[
\wedge B(x(\bar{\rho}), \rho) + I(x(\bar{\rho}), \rho)
\]

\[
\oplus \left[ \tau \left( M(x(\bar{\rho}), \rho) + A(g(x(\bar{\rho}), \rho), \rho) \oplus F(h(x(\bar{\rho}), \rho), \rho) \right) \right.
\]

\[
\wedge B(x(\bar{\rho}), \rho) + I(x(\bar{\rho}), \rho)
\]

\[
\leq \alpha_1 \left[ \left( \tau \left[ M(x(\bar{\rho}), \rho) + A(g(x(\bar{\rho}), \rho), \rho) \oplus F(h(x(\bar{\rho}), \rho), \rho) \right] \right)
\]

\[
\wedge B(x(\bar{\rho}), \rho)
\]

\[
\oplus \left( \tau \left[ M(x(\bar{\rho}), \rho) + A(g(x(\bar{\rho}), \rho), \rho) \oplus F(h(x(\bar{\rho}), \rho), \rho) \right] \right)
\]

\[
\wedge B(x(\bar{\rho}), \rho)
\]

\[
\leq \alpha_1 \left[ \tau \left[ M(x(\bar{\rho}), \rho) + A(g(x(\bar{\rho}), \rho), \rho) \oplus F(h(x(\bar{\rho}), \rho), \rho) \right] \right]
\]

\[
\oplus \left[ M(x(\bar{\rho}), \rho) + A(g(x(\bar{\rho}), \rho), \rho) \oplus F(h(x(\bar{\rho}), \rho), \rho) \right]
\]

\[
\vee [B(x(\bar{\rho}), \rho) \oplus B(x(\bar{\rho}), \rho)]
\]

\[
\leq \alpha_1 \left[ \tau \left[ M(x(\bar{\rho}), \rho) \oplus M(x(\bar{\rho}), \rho) \right] + A(g(x(\bar{\rho}), \rho), \rho) \right]
\]
Combining equations (4.4.3), (4.4.4), (4.4.5), and by making use of the Lemma 1.2.13, we obtain

\[
x(\rho) \oplus x(\bar{\rho}) \leq \left[\Psi(x(\rho) \oplus x(\bar{\rho}))\right] + \alpha_1 \left[\tau\left((M(x(\rho), \rho) \oplus M(x(\bar{\rho}), \bar{\rho})) \right.ight.
\]
\[
+ (A(g(x(\rho), \rho) \oplus A(g(x(\bar{\rho}), \bar{\rho}))
\]
\[
\oplus (F(h(x(\rho), \rho) \oplus F(h(x(\bar{\rho}), \bar{\rho})))
\]
\[
\vee (B(x(\rho), \rho) \oplus B(x(\bar{\rho}), \bar{\rho}))\right]\]
\[
(1 \oplus \Psi)(x(\rho) \oplus x(\bar{\rho})) \leq \alpha_1 \left[\tau\left((M(x(\rho), \rho) \oplus M(x(\bar{\rho}), \bar{\rho})) \right.ight.
\]
\[
+ (A(g(x(\rho), \rho) \oplus A(g(x(\bar{\rho}), \bar{\rho}))
\]
\[
\oplus (F(h(x(\rho), \rho) \oplus F(h(x(\bar{\rho}), \bar{\rho})))
\]
\[
\vee (B(x(\rho), \rho) \oplus B(x(\bar{\rho}), \bar{\rho}))\right]\]
\[
(x(\rho) \oplus x(\bar{\rho})) \leq \left(\frac{\alpha_1}{1 + \Psi}\right) \left[\tau\left((M(x(\rho), \rho) \oplus M(x(\bar{\rho}), \bar{\rho})) \right.ight.
\]
\[
+ (A(g(x(\rho), \rho) \oplus A(g(x(\bar{\rho}), \bar{\rho}))
\]
\[
\oplus (F(h(x(\rho), \rho) \oplus F(h(x(\bar{\rho}), \bar{\rho})))
\]
\[
\vee (B(x(\rho), \rho) \oplus B(x(\bar{\rho}), \bar{\rho}))\right]\]
\[
\leq \left(\frac{\alpha_1}{1 + \Psi}\right) \left[\tau\left((M(x(\rho), \rho) \oplus M(x(\bar{\rho}), \bar{\rho})) \right.ight.
\]
\[
+ (A(g(x(\rho), \rho) \oplus A(g(x(\bar{\rho}), \bar{\rho})) \oplus A(g(x(\bar{\rho}), \bar{\rho}), \rho))
\]
\[
\oplus A(g(x(\bar{\rho}), \bar{\rho})) \oplus (F(h(x(\rho), \rho) \oplus F(h(x(\bar{\rho}), \bar{\rho})))
\]
\[
\oplus F(h(x(\rho), \rho)) \oplus (h(x(\rho), \rho) \vee (B(x(\rho), \rho) \oplus B(x(\bar{\rho}), \bar{\rho}))\right]\].
\]

(4.4.6)

Using the continuity of the parametric mappings with respect to the second argument \(\rho \in \Sigma\), we have

\[
\lim_{\rho \to \bar{\rho}} \|g(x(\rho), \rho) - g(x(\bar{\rho}), \bar{\rho})\| = 0
\]

\[
\lim_{\rho \to \bar{\rho}} \|h(x(\rho), \rho) - h(x(\bar{\rho}), \bar{\rho})\| = 0
\]

\[
\lim_{\rho \to \bar{\rho}} \|B(x(\rho), \rho) - B(x(\bar{\rho}), \bar{\rho})\| = 0
\]
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\[
\lim_{\rho \to \bar{\rho}} \| A(g(x(\bar{\rho}), \rho), \rho) - A(g(x(\rho), \rho), \bar{\rho}) \| = 0 \\
\lim_{\rho \to \bar{\rho}} \| F(\cdot, \rho) - F(\cdot, \bar{\rho}) \| = 0 \\
\lim_{\rho \to \bar{\rho}} \| M(\cdot, \rho) - M(\cdot, \bar{\rho}) \| = 0.
\]

From the Lemma 1.2.11, we get

\[
\lim_{\rho \to \bar{\rho}} (x(\rho) \oplus x(\bar{\rho})) = \theta,
\]

and

\[
\lim_{\rho \to \bar{\rho}} \| x(\rho) - x(\bar{\rho}) \| = 0. \tag{4.4.7}
\]

This shows that the solution \( x(\rho) \) of the problem (4.2.1) is continuous at \( \rho = \bar{\rho} \). \[\square\]