Chapter-III

New generalized systems of nonlinear ordered variational inclusions involving \( \oplus \) operator in real ordered Hilbert spaces

The work of this Chapter has been communicated in the following journal:

- Journal of Inequalities and Applications (ISSN: 1029-242X, SCI).

Abstract

In this Chapter, we deal with two general systems of nonlinear ordered variational inclusions problem. We also construct some new iterative algorithms for finding approximation solutions to the general systems of nonlinear ordered variational inclusions and prove the convergence of the sequences obtained by the schemes. The results presented in the chapter are new and improve some known results in the literature.
New generalized systems of nonlinear ordered variational inclusions involving $\oplus$ operator in real ordered Hilbert spaces

“Mathematics is the science of what is clear by itself.”

Carl Gustav Jacob Jacobi

3.1 Introduction

A lot of work has been added into the theory of variational inequalities since 1964; when its seed was planted by Stampacchia. On account of its wide applications in physics and applied sciences etc., the classical variational inequalities have been very much contemplated and summed up in different ways [14, 26, 20, 22, 68].

Variational inclusions, examined by Hasounni and Moudafi [40], is very nice generalization of variational inequalities. They have proposed a perturbed iterative algorithm for solving the variational inclusions problem. From that point onward, this class has been summed up and stretched out by S. Adly [4], Ahmad and Ansari [6] and Ding [27] and some more researchers, see the references [1, 8, 16, 24].

A number of answers for nonlinear equations were presented and contemplated by Amann [7]. Theory related to the nonlinear mapping fixed point hypothesis and application has been
seriously examined in ordered Banach spaces [29, 75, 76]. In this manner, it is essential that summed up nonlinear ordered variational inclusions (ordered equation) are contemplated and talked about.

Fang et al. [31] introduced and studied $H$-monotone operator. With the help of this operator, a resolvent operator was designed and proved it’s Lipschitz continuity. They also introduce a class of variational inclusions in Hilbert space. In a current paper [34], Fang et al. additionally presented another class of generalized monotone operators called as $(H, \eta)$-monotone operator, which generalizes different classes of maximal monotone, maximal $\eta$-monotone and $H$-monotone operators, respectively. Recently, Lan et al. [43] presented another idea of $(A, \eta)$-accretive mappings, which generalized the current monotone or accretive operators, and concentrated on few properties of mappings. They examined a class of variational inclusions using the resolvent operator related with $(A, \eta)$-accretive mappings.

A plenty of research involving the ordered equations and ordered variational inequalities in ordered Banach spaces is done by Li and his co-authors, see [50, 52], and many references cited therein. Many problems concerning ordered variational inclusions are answered by the resolvent techniques linked with RME set-valued mapping [51], $(\alpha, \lambda)$-NODM set-valued mapping [49], $(\gamma_G, \lambda)$-weak RRD mapping [2] and $(\alpha, \lambda)$-weak ANODD set-valued map with strongly comparison mapping $A$ [52] and many more see, [5, 48, 70] and many citations therein.

In this Chapter, we make use of the resolvent operator approach for the approximation solvability of solutions of implicit system of generalized nonlinear ordered variational inclusions in real ordered Hilbert spaces.

### 3.2 Formulation of problems

Let $F_i : \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \to \mathcal{H}_i$, $A_i : \mathcal{H}_i \to \mathcal{H}_i$ and $g_i : \mathcal{H}_i \to \mathcal{H}_i$ be single-valued mappings, for $i, j = 1, 2, 3, \cdots, m$. Let $U_{ij} : \mathcal{H}_j \to CB(\mathcal{H}_j)$ be a set-valued map and $M_i : \mathcal{H}_i \to CB(\mathcal{H}_i)$ be set-valued weak-ARD mapping. Then, find $(x_1^*, x_2^*, \cdots, x_m^*) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m$ and $u_{ij}^* \in U_{ij}(x_j^*)$, for $i, j = 1, 2, 3, \cdots, m$ such that

$$0 \in \rho_i F_i(u_{i1}^*, u_{i2}^*, \cdots, u_{im}^*) \oplus \lambda_i M_i(g_i(x_i^*)), \quad (3.2.1)$$

where $\rho_i$ and $\lambda_i$ are given positive constants. The problem (3.2.1) is called generalized set-valued system of nonlinear ordered variational inclusions problem for weak-ARD mappings.
If \( U_{ij} = T_{ij} \) is a single-valued mapping, then problem (3.2.1) becomes:

Find \( x_j \in \mathcal{H}_j \), such that

\[
0 \in \rho_i F_i(T_{i1}x_1^*, T_{i2}x_2^*, \ldots, T_{im}x_m^*) \oplus \lambda_i M_i(g_i(x_i^*)).
\] (3.2.2)

This problem is known as generalized system of nonlinear ordered variational inclusions problem involving weak-ARD mappings.

**Remark 3.2.1.** Here, we discuss special cases for our problem (3.2.1), which was encountered by Li et al. [49, 50, 51].

**Case 1.** If \( i, j = 1 \), \( \rho_i = 1 \), \( \lambda_i = 1 \) and \( U_{ij} = g_i = I \), then the problem (3.2.1) is reduced to find \( x \in \mathcal{H}_1 \) such that

\[
0 \in F_1(x) \oplus M_1(x).
\] (3.2.3)

This problem was considered by Li et al. [50] and coined as general nonlinear mixed-order quasi-variational inclusions (GNMOQVI) involving \( \oplus \) operator in an ordered Banach space.

**Case 2.** If \( F = 0 \) (zero mapping), then problem (3.2.3) is reduced to find \( x \in \mathcal{H} \) such that

\[
0 \in M(x).
\] (3.2.4)

This problem was considered by H.G. Li for ordered RME set-valued mappings [51] and \( (\alpha, \lambda) \)-NODM set-valued mappings [49].

**Lemma 3.2.2** ([82]). Let \( \theta \in (0, 1) \) be a constant. Then the function, \( f(\lambda) = 1 - \lambda + \lambda \theta \) for \( \lambda \in [0, 1] \), is nonnegative and strictly decreasing and \( f(\lambda) \in [0, 1] \). Furthermore, \( f(\lambda) \in (0, 1) \) for \( \lambda \neq 0 \).

**Lemma 3.2.3** ([83]). Assume that \( \{a_n\} \) and \( \{b_n\} \) be two sequences of nonnegative real numbers such that

\[
a_{n+1} \leq \theta a_n + b_n,
\]

where \( \theta \in (0, 1) \) and \( \lim_{n \to \infty} b_n = 0 \). Then \( \lim_{n \to \infty} a_n = 0 \).
Lemma 3.2.4. Let \((x^*_1, x^*_2, \ldots, x^*_m) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m\) and \(u^*_{ij} \in U_{ij}(x^*_j)\) for \(i, j = 1, 2, 3, \ldots, m\). Then \((x^*_1, x^*_2, \ldots, x^*_m, u^*_{11}, u^*_{12}, \ldots, u^*_{1m}, u^*_{2m}, \ldots, u^*_{mm})\) is a solution of the problem (3.2.1) if and only if it satisfies

\[
g_i(x^*_i) = J^{I_i - A_i}_{\lambda_i, M_i} \left[ (I_i - A_i)(g_i(x^*_i)) + \rho_i F_i(u^*_{i1}, u^*_{i2}, \ldots, u^*_{im}) \right],
\]

where \(J^{I_i - A_i}_{\lambda_i, M_i}(x) = [(I_i - A_i) + \lambda_i M_i]^{-1}(x)\) and \(\rho_i, \lambda_i > 0\) for \(i = 1, 2, \ldots, m\).

Proof. The proof follows from the definition of the relaxed resolvent operator (1.2.29). □

3.3 Design of the algorithms

Algorithm 3.1
For \(i, j = 1, 2, \ldots, m\), choose \((x^n_i, x^n_j, \ldots, x^n_m) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m\) and \(u^n_{ij} \in U_{ij}(x^n_j)\). For \(n = 0, 1, 2, 3, \ldots\), set:

\[
x^{n+1}_i = (1 - \lambda) x^n_i + \lambda \left[ x^n_i - g_i(x^n_i) + J^{I_i - A_i}_{\lambda_i, M_i} \left[ (I_i - A_i)(g_i(x^n_i)) + \rho_i F_i(u^n_{i1}, u^n_{i2}, \ldots, u^n_{im}) \right] \right],
\]

From Nadler’s result, choose \(u^{n+1}_{ij} \in U_{ij}(x^{n+1}_j)\) such that

\[
\|u^{n+1}_{ij} + u^n_{ij}\| \leq \left( 1 + \frac{1}{(n + 1)} \right) D_j(U_{ij}(x^{n+1}_j), U_{ij}(x^n_j)).
\]

Remark 3.3.1. If we choose \(\lambda = 1\) and \(U_{ij} = T_{ij}\), for \(i, j = 1, 2, \ldots, m\), is single-valued operator, then Algorithm 3.1 reduces to the following algorithm for problem (3.2.2).

Algorithm 3.2
For \(n = 0, 1, 2, \ldots, i = 1, 2, \ldots, m\), choose \((x^n_1, x^n_2, \ldots, x^n_m) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m\), \(x^n_i\) is computed as follows:

\[
x^{n+1}_i = x^n_i - g_i(x^n_i) + J^{I_i - A_i}_{\lambda_i, M_i} \left[ (I_i - A_i)(g_i(x^n_i)) + \rho_i F_i(T_{i1}x^n_1, T_{i2}x^n_2, \ldots, T_{im}x^n_m) \right] + w^n_i,
\]

where \(w^n_i \in \mathcal{H}_i\) is the error, to take into account a possible inexact computation of the resolvent operator points, satisfying condition \(\lim_{n \to \infty} \|w^n_i\| = 0\).
3.4 Main Results

Theorem 3.4.1. Let \( A_i : \mathcal{H}_i \rightarrow \mathcal{H}_i, \) \( g_i : \mathcal{H}_i \rightarrow \mathcal{H}_i \) and \( F_i : \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \rightarrow \mathcal{H}_i \) be the single-valued mappings such that \( A_i \) be \( \lambda_{A_i} \)-ordered compression mapping, \( g_i \) be \( \lambda_{g_i} \)-ordered compression, \( (\alpha_1^i, \alpha_2^i) \)-ordered restricted accretive mapping and \( F_i \) be \( \lambda_{ij} \)-ordered compression mapping with respect to \( j^{th} \)-argument. Let \( U_{ij} : \mathcal{H}_j \rightarrow CB(\mathcal{H}_j) \) be a \( D_i \delta_{ij} \)-ordered Lipschitz continuous set-valued mapping. Let \( M_i : \mathcal{H}_i \rightarrow CB(\mathcal{H}_i) \) be \( (\gamma_{A_i}, \lambda_i) \)-weak rectangular different compression mapping with respect to \( A_i \) and if \( x_i \propto y_i \), \( J^{i}_{\lambda_i, M_i}(x_i) \propto J^{i}_{\lambda_i, M_i}(y_i) \) and for all \( \lambda_i, \rho_i > 0 \), then the following condition holds:

\[
\theta_j = \left\{ \alpha_1^j + \alpha_2^j \lambda_{g_j} + L_j(\lambda_{g_j} + \lambda_{A_j} \lambda_{g_j}) + \sum_{i \neq j, i=1}^{m} L_i \rho_i \lambda_{F_{ij}} \delta_{D_{ij}} \right\} < 1, \tag{3.4.1}
\]

for all \( j = 1, 2, 3, \cdots, m \), which in turn, imply that the problem (3.2.1) admits a solution \((x_1^*, x_2^*, \cdots, x_m^*, u_{11}^*, u_{12}^*, \cdots, u_{1m}^*, \cdots, u_{im1}^*, \cdots, u_{imm}^*)\), where \((x_1^*, x_2^*, \cdots, x_m^*) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \) and \( u_{ij}^* \in U_{ij}(x_j^*) \). Moreover, iterative sequences \( \{x_j^n\} \) and \( \{u_{ij}^n\} \) generated by Algorithm 3.1, converge strongly to \( x_j^* \) and \( u_{ij}^* \), for \( i, j = 1, 2, \cdots, m \), respectively.

Proof. Using Algorithm 3.1 and Lemma 1.2.13, for \( i = 1, 2, \cdots, m \), we have

\[
x_i^{n+1} + x_i^n = \left( (1 - \lambda)x_i^n + \lambda \left[ x_i^n - g_i(x_i^n) + J^{i}_{\lambda_i, M_i} (I_i - A_i) (g_i(x_i^n)) \right] \right) + \rho_i F_i (u_{i1}^n, u_{i2}^n, \cdots, u_{imm}^n) \]

\[
\leq (1 - \lambda)(x_i^n \oplus x_i^{n-1}) + \lambda \left[ x_i^n - g_i(x_i^n) \right] + \lambda \left( J^{i}_{\lambda_i, M_i} (I_i - A_i) (g_i(x_i^{n-1})) \right) \]

\[
\leq (1 - \lambda)(x_i^n \oplus x_i^{n-1}) + \lambda (\alpha_1^i x_i^n + \alpha_2^i g_i(x_i^n)) \tag{3.4.2}
\]
Using Definition 1.2.2, Proposition 1.2.32 and equation (3.4.2), we get

\[
\|x_i^{n+1} + x_i^n\| \leq \lambda C \| (1 - \lambda) + \lambda (\alpha_1 + \lambda_2) \| x_i^n + x_i^{n-1} \|
\]

\[
+ \lambda \lambda C \lambda L_i \| \left[ (I - A_i) (g_i(x_i^n)) + \rho_i F_i(u_{i1}, u_{i2}, \ldots, u_{im}) \right]
\]

\[
\oplus \left[ (I - A_i)(g_i(x_i^{n-1})) + \rho_i F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1}) \right] \|
\]

\[
\leq \lambda C \left[ 1 - \lambda (1 - (\alpha_1 + \lambda_2)) \right] \| x_i^n + x_i^{n-1} \|
\]

\[
+ \lambda \lambda C \lambda L_i \| \left( (I - A_i) (g_i(x_i^n)) \oplus (I - A_i)(g_i(x_i^{n-1})) \right) \|
\]

\[
+ \lambda \lambda C \lambda L_i \| F_i(u_{i1}, u_{i2}, \ldots, u_{im}) \oplus F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1}) \|
\]

\[
\leq \lambda C \left[ 1 - \lambda (1 - (\alpha_1 + \lambda_2)) \right] \| x_i^n + x_i^{n-1} \|
\]

\[
+ \lambda \lambda C \lambda L_i \| (\lambda_1 + \lambda_2, \ldots, \lambda_1 + \lambda_2) \| x_i^n + x_i^{n-1} \|
\]

\[
+ \lambda \lambda C \lambda L_i \| F_i(u_{i1}, u_{i2}, \ldots, u_{im}) \+
\]

\[
+ F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1}) \| . \tag{3.4.3}
\]

Now, from equation (3.4.3), we compute

\[
\| F_i(u_{i1}, u_{i2}, \ldots, u_{im}) \oplus F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1}) \|
\]

\[
\leq \| F_i(u_{i1}, u_{i2}, \ldots, u_{im}) \|\oplus
\]

\[
F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1}) \| , \ldots
\]

\[
+ \| F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1}) \| \| , \ldots
\]

\[
+ \| F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1}) \| \| , \ldots
\]

\[
F_i(u_{i1}, u_{i2}, \ldots, u_{im}) \oplus F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1}) \| \| , \ldots
\]

\[
\leq \lambda F_i \| u_i^{n} \oplus u_i^{n-1} \| + \lambda F_i \| u_i^{n} \oplus u_i^{n-1} \|. \tag{3.4.4}
\]

By the definition of $F_i$ as $\lambda F_i$-ordered compression map with respect to the $j^{th}$ argument, we have

\[
\| F_i(u_{i1}, u_{i2}, \ldots, u_{im}) \oplus F_i(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{im}^{n-1}) \|
\]

\[
\leq \lambda F_i \| u_i^{n} \oplus u_i^{n-1} \| + \lambda F_i \| u_i^{n} \oplus u_i^{n-1} \|)
\]

38
3.4. Main Results

\[ + \cdots + \lambda F_{im} \| u_{im}^n \oplus u_{im}^{n-1} \| \]
\[ = \sum_{i \neq j, j=1}^m \lambda F_{ij} \| u_{ij}^n \oplus u_{ij}^{n-1} \| \]
\[ \leq \sum_{i \neq j, j=1}^m \lambda F_{ij} \left( 1 + \frac{1}{(n+1)} \right) D_j(U_{ij}(x_j^n), U_{ij}(x_j^{n-1})) \]
\[ \leq \left( 1 + \frac{1}{(n+1)} \right) \sum_{i \neq j, j=1}^m \lambda F_{ij} \delta_{Dij} \| x_j^n \oplus x_j^{n-1} \|. \quad (3.4.5) \]

Using Proposition 1.2.32 and equation (3.4.5) in equation (3.4.3), we obtain

\[ \| x_{i+1}^n \oplus x_i^n \| = \| x_{i+1}^n - x_i^n \| \]
\[ \leq \lambda C \left[ 1 - \lambda \left( 1 - (\alpha_1^i + \alpha_2^j) \right) \right] \| x_i^n - x_i^{n-1} \| \]
\[ + \lambda \lambda C L_i \left( \lambda g_i + \lambda A_i \lambda g_i \right) \| x_i^n - x_i^{n-1} \| \]
\[ + \lambda \lambda C L_i \rho_i \left( 1 + \frac{1}{(n+1)} \right) \sum_{i \neq j, j=1}^m \lambda F_{ij} \delta_{Dij} \| x_j^n - x_j^{n-1} \| \]
\[ \leq \left\{ \lambda C \left[ 1 - \lambda \left( 1 - (\alpha_1^i + \alpha_2^j) \right) \right] + \lambda \lambda C L_i \left( \lambda g_i + \lambda A_i \lambda g_i \right) \right\} \| x_i^n - x_i^{n-1} \| \]
\[ + \lambda \lambda C L_i \rho_i \left( 1 + \frac{1}{(n+1)} \right) \sum_{i \neq j, j=1}^m \lambda F_{ij} \delta_{Dij} \| x_j^n - x_j^{n-1} \| , \]

which implies that

\[ \sum_{j=1}^m \| x_{j+1}^n - x_j^n \| = \sum_{i=1}^m \| x_{i+1}^n - x_i^n \| \]
\[ \leq \sum_{i=1}^m \left\{ \lambda C \left[ 1 - \lambda \left( 1 - (\alpha_1^i + \alpha_2^j) \right) + L_i(\lambda g_i + \lambda A_i \lambda g_i) \right] \| x_i^n - x_i^{n-1} \| \right\} \]
\[ + \lambda \lambda C \left( 1 + \frac{1}{(n+1)} \right) \sum_{i \neq j, j=1}^m L_j \rho_j \lambda F_{ij} \delta_{Dij} \| x_j^n - x_j^{n-1} \| \]
\[ = \sum_{i=1}^m \lambda C \left[ 1 - \lambda \left( 1 - (\alpha_1^i + \alpha_2^j + L_i(\lambda g_i + \lambda A_i \lambda g_i)) \right) \right] \| x_i^n - x_i^{n-1} \| \]
\[ + \lambda \lambda C \left( 1 + \frac{1}{(n+1)} \right) \sum_{j=1}^m \sum_{j \neq j, j=1}^m \sum_{i \neq j, j=1}^m L_j \rho_j \lambda F_{ij} \delta_{Dij} \| x_j^n - x_j^{n-1} \| \]
\[ = \sum_{j=1}^m \lambda C \left[ 1 - \lambda \left( 1 - \alpha_1^j + \alpha_2^j \lambda g_j + L_j(\lambda g_j + \lambda A_j \lambda g_j) \right) \right] \| x_j^n - x_j^{n-1} \| \]
\[ + \lambda \lambda C \left( 1 + \frac{1}{(n+1)} \right) \sum_{j=1}^m \sum_{j \neq j, j=1}^m \sum_{i \neq j, j=1}^m L_j \rho_j \lambda F_{ij} \delta_{Dij} \| x_j^n - x_j^{n-1} \| \]
\[ = \sum_{j=1}^m \lambda C \left[ 1 - \lambda + \lambda \left( \alpha_1^j + \alpha_2^j \lambda g_j + L_j(\lambda g_j + \lambda A_j \lambda g_j) \right) \right] \| x_j^n - x_j^{n-1} \| \]
where

\[
\theta^n_j = \left\{ \alpha^1_1 + \alpha^2_2 \lambda y_j + L_j (\lambda y_j + \lambda \lambda_j y_j) + \left( 1 + \frac{1}{(n+1)} \right) \sum_{i \neq j, i=1}^m L_i \rho_i \lambda F_{ij} \delta_{ij} \right\} < 1,
\]

and

\[
f_n(\lambda) = \max_{1 \leq j \leq m} \{ 1 - \lambda + \lambda \theta^n_j \}.
\]

From equation (3.4.6), we know that the sequence \( \{ \theta^n_j \} \) is monotonic decreasing and \( \theta^n_j \rightarrow \theta_j \) as \( n \rightarrow \infty \). Thus, \( f(\lambda) = \lim_{n \rightarrow \infty} f_n(\lambda) = \max_{1 \leq j \leq m} \{ 1 - \lambda + \lambda \theta_j \} \). Since \( 0 < \theta_j < 1 \) for \( j = 1, 2, \cdots, m \), we get \( \theta = \max_{1 \leq j \leq m} \{ \theta_j \} \in (0, 1) \). By Lemma 3.2.2, we have \( f(\lambda) = 1 - \lambda + \lambda \theta \in (0, 1) \), from equation (3.4.6), it follows that \( \{ \theta^n_j \} \) is a Cauchy sequence and there exist \( x^n_j \in \mathcal{H}_j \) such that \( x^n_j \rightarrow x^*_j \) as \( n \rightarrow \infty \) for \( j = 1, 2, \cdots, m \). Next, we show that \( u^n_{ij} \rightarrow u^*_{ij} \in U_{ij}(x^*_j) \) as \( n \rightarrow \infty \) for \( i, j = 1, 2, \cdots, m \). It follows from equation (3.4.5) that \( \{ u^n_{ij} \} \) are also Cauchy sequences. Hence, there exist \( u^*_{ij} \in \mathcal{H}_j \) such that \( u^n_{ij} \rightarrow u^*_{ij} \) as \( n \rightarrow \infty \) for \( i, j = 1, 2, \cdots, m \). Furthermore,

\[
d(u^*_{ij}, U_{ij}(x^*_j)) = \inf \{ \| u^*_{ij} + t \| : t \in U_{ij}(x^*_j) \}
\leq \| u^*_{ij} + u^n_{ij} \| + d(u^n_{ij}, U_{ij}(x^*_j))
\leq \| u^*_{ij} + u^n_{ij} \| + d(U_{ij}(x^n_{ij}), U_{ij}(x^*_j))
\leq \| u^*_{ij} + u^n_{ij} \| + \delta_{ij} \| x^*_j + x^n_{ij} \|
\leq \| u^*_{ij} - u^n_{ij} \| + \delta_{ij} \| x^*_j - x^n_{ij} \| \rightarrow 0 \ (n \rightarrow \infty).
\]

Since \( U_{ij}(x^*_j) \) are closed for \( i, j = 1, 2, \cdots, m \), we have \( u^*_{ij} \in U_{ij}(x^*_j) \) for \( i, j = 1, 2, \cdots, m \). By using continuity \( (x^n_1, x^n_2, \cdots, x^n_m) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \), \( u^*_{ij} \in U_{ij}(x^*_j) \) for \( i, j = 1, 2, \cdots, m \) satisfy equation (3.2.5) and so by Lemma 3.2.4, problem (3.2.1) has a solution \( (x^n_1, x^n_2, \cdots, x^n_m, u^n_{11}, u^n_{12}, \cdots, u^n_{1m}, \cdots, u^n_{m1}, u^n_{m2}, \cdots, u^n_{mm}) \), where \( u^n_{ij} \in U_{ij}(x^n_j) \) for \( i, j = 1, 2, \cdots, m \) and \( (x^n_1, x^n_2, \cdots, x^n_m) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \). This completes the proof. \]
Then the problem (3.2.2) has a unique solution

\[ \theta_j = \left[ \lambda_C \left( (\alpha_1 + \alpha_2 \lambda_j) + L_j(\lambda g_j + \lambda A_j \lambda g_j) \right) + \lambda_C \sum_{i \neq j, i=1}^{m} L_i \rho_1 \lambda F_{ij} \gamma_{ij} \right] < 1. \]

Then the problem (3.2.2) has a unique solution \( (x_1^+, x_2^+, \ldots, x_m^+) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \). Moreover, the iterative sequences \( \{x_j^n\} \) generated by Algorithm 3.2 converge strongly to \( x_j^+ \) for \( j = 1, 2, \ldots, m \).

**Proof.** Let us define a norm \( \| \cdot \|_* \) on the product space \( \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \) by

\[ \|(x_1, x_2, \ldots, x_m)\|_* = \sum_{i=1}^{m} \|x_i\|, \quad \forall (x_1, x_2, \ldots, x_m) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m. \]  

(3.4.7)

Then it can be easily seen that \( (\mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m, \| \cdot \|_*) \) is a Banach space.

Setting:

\[ y_i = x_i - g_i(x_i) + J_{L_i-A_i}^{I_i-A_i} \left[ (I_i - A_i)(g_i(x_i)) + \rho_i F_i(T_{i_1}x_1, \ldots, T_{i_{i-1}}x_{i-1}, T_{i_1}x_1, T_{i_{i+1}}x_{i+1}, \ldots, T_{i_m}x_m) \right]. \]

Define a mapping \( Q : \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \rightarrow \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \) as

\[ Q(x_1, x_2, \ldots, x_m) = (y_1, y_2, \ldots, y_m), \quad \forall (x_1, x_2, \ldots, x_m) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m. \]

For any \( (x_1^1, x_2^1, \ldots, x_m^1), (x_1^2, x_2^2, \ldots, x_m^2) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \), we have,

\[
\|Q(x_1^1, x_2^1, \ldots, x_m^1) + Q(x_1^2, x_2^2, \ldots, x_m^2)\|_* \leq \|Q(x_1^1, x_2^1, \ldots, x_m^1) - Q(x_1^2, x_2^2, \ldots, x_m^2)\|_* \\
\leq \|(y_1^1, y_2^1, \ldots, y_m^1) - (y_1^2, y_2^2, \ldots, y_m^2)\|_* \\
\leq \sum_{i=1}^{m} \|y_i^1 - y_i^2\|. 
\]  

(3.4.8)

First of all, we have to calculate \( (y_1^1 \oplus y_2^1) \) as follows

\[
(y_1^1 \oplus y_2^1) = (x_1^1 - g_i(x_1^1) + J_{L_i-A_i}^{I_i-A_i} \left[ (I_i - A_i)(g_i(x_1^1)) + \rho_i F_i(T_{i_1}x_1, \ldots, T_{i_{i-1}}x_{i-1}, T_{i_1}x_1, T_{i_{i+1}}x_{i+1}, \ldots, T_{i_m}x_m) \right]) \\
\oplus (x_2^1 - g_i(x_2^1) + J_{L_i-A_i}^{I_i-A_i} \left[ (I_i - A_i)(g_i(x_2^1)) + \rho_i F_i(T_{i_1}x_1, \ldots, T_{i_{i-1}}x_{i-1}, T_{i_1}x_1, T_{i_{i+1}}x_{i+1}, \ldots, T_{i_m}x_m) \right])
\]
Further, we calculate
\[
+ \rho_i F_i(T_{t_1} x_1^2, \cdots, T_{t_{i-1}} x_{i-1}^2, T_{t_i} x_i^2, T_{t_{i+1}} x_{i+1}^2, \cdots, T_{t_m} x_m^2)\Big)\]
\[
= \left( (x_i^1 - g_i(x_i^1)) \oplus (x_i^2 - g_i(x_i^2)) \right) + \left( J_{L_i, M_i}^{I_i - A_i} \left[(I_i - A_i)(g_i(x_i^1)) + \rho_i F_i(T_{t_1} x_1^1, \cdots, T_{t_{i-1}} x_{i-1}^1, T_{t_i} x_i^1, T_{t_{i+1}} x_{i+1}^1, \cdots, T_{t_m} x_m^1)\right] \right)
\]
\[
\oplus J_{L_i, M_i}^{I_i - A_i} \left[(I_i - A_i)(g_i(x_i^2)) + \rho_i F_i(T_{t_1} x_1^2, \cdots, T_{t_{i-1}} x_{i-1}^2, T_{t_i} x_i^2, T_{t_{i+1}} x_{i+1}^2, \cdots, T_{t_m} x_m^2)\right)\Big). 
\]
\[(3.4.9)\]

From the Definition 1.2.2 and Lemma 1.2.11, we have
\[
\|y_1^l \oplus y_1^r\| \leq \|y_1^l - y_1^r\|
\]
\[
\leq \lambda_C \left\{ \left\| (x_1^1 - g_i(x_1^1)) \oplus (x_i^2 - g_i(x_i^2)) \right\| + \left( J_{L_i, M_i}^{I_i - A_i} \left[(I_i - A_i)(g_i(x_i^1)) + \rho_i F_i(T_{t_1} x_1^1, \cdots, T_{t_{i-1}} x_{i-1}^1, T_{t_i} x_i^1, T_{t_{i+1}} x_{i+1}^1, \cdots, T_{t_m} x_m^1)\right] \right)
\]
\[
\oplus J_{L_i, M_i}^{I_i - A_i} \left[(I_i - A_i)(g_i(x_i^2)) + \rho_i F_i(T_{t_1} x_1^2, \cdots, T_{t_{i-1}} x_{i-1}^2, T_{t_i} x_i^2, T_{t_{i+1}} x_{i+1}^2, \cdots, T_{t_m} x_m^2)\right)\right\}\]
\[
\leq \lambda_C \left\{ \alpha_1^1\|x_1^1 - x_i^2\| + \alpha_2^2 \lambda_2^1\|x_1^1 - x_i^2\| \right\}
\]
\[
+ \lambda_C \left\{ \left( J_{L_i, M_i}^{I_i - A_i} \left[(I_i - A_i)(g_i(x_i^1)) + \rho_i F_i(T_{t_1} x_1^1, \cdots, T_{t_{i-1}} x_{i-1}^1, T_{t_i} x_i^1, T_{t_{i+1}} x_{i+1}^1, \cdots, T_{t_m} x_m^1)\right] \right)
\]
\[
\oplus J_{L_i, M_i}^{I_i - A_i} \left[(I_i - A_i)(g_i(x_i^2)) + \rho_i F_i(T_{t_1} x_1^2, \cdots, T_{t_{i-1}} x_{i-1}^2, T_{t_i} x_i^2, T_{t_{i+1}} x_{i+1}^2, \cdots, T_{t_m} x_m^2)\right)\right\}\]
\[
\leq \lambda_C (\alpha_1^1 + \alpha_2^2 \lambda_2^1)\|x_1^1 - x_i^2\| + \lambda_C \left\{ \left( J_{L_i, M_i}^{I_i - A_i} \left[(I_i - A_i)(g_i(x_i^1)) + \rho_i F_i(T_{t_1} x_1^1, \cdots, T_{t_{i-1}} x_{i-1}^1, T_{t_i} x_i^1, T_{t_{i+1}} x_{i+1}^1, \cdots, T_{t_m} x_m^1)\right] \right)
\]
\[
\oplus J_{L_i, M_i}^{I_i - A_i} \left[(I_i - A_i)(g_i(x_i^2)) + \rho_i F_i(T_{t_1} x_1^2, \cdots, T_{t_{i-1}} x_{i-1}^2, T_{t_i} x_i^2, T_{t_{i+1}} x_{i+1}^2, \cdots, T_{t_m} x_m^2)\right)\right\} \right\}. \quad (3.4.10)
\]

Further, we calculate
\[
\left\| J_{L_i, M_i}^{I_i - A_i} \left[(I_i - A_i)(g_i(x_i^1)) + \rho_i F_i(T_{t_1} x_1^1, \cdots, T_{t_{i-1}} x_{i-1}^1, T_{t_i} x_i^1, T_{t_{i+1}} x_{i+1}^1, \cdots, T_{t_m} x_m^1)\right] \right\|
\]
\[
\oplus J_{L_i, M_i}^{I_i - A_i} \left[(I_i - A_i)(g_i(x_i^2)) + \rho_i F_i(T_{t_1} x_1^2, \cdots, T_{t_{i-1}} x_{i-1}^2, T_{t_i} x_i^2, T_{t_{i+1}} x_{i+1}^2, \cdots, T_{t_m} x_m^2)\right] \right\}\]
\[
\leq L_i \left\{ \left\| (I_i - A_i)(g_i(x_i^1)) \oplus (I_i - A_i)(g_i(x_i^2)) \right\|\right\}
\]
\[
+ \rho_i \left\| F_i(T_{t_1} x_1^1, \cdots, T_{t_{i-1}} x_{i-1}^1, T_{t_i} x_i^1, T_{t_{i+1}} x_{i+1}^1, \cdots, T_{t_m} x_m^1) \right\|
\]
\[
\oplus F_i(T_{t_1} x_1^2, \cdots, T_{t_{i-1}} x_{i-1}^2, T_{t_i} x_i^2, T_{t_{i+1}} x_{i+1}^2, \cdots, T_{t_m} x_m^2) \right\} \right\]
3.4. Main Results

\[ \begin{align*}
\leq L_i & \left[ \left\| (g_i(x_i^1) \oplus g_i(x_i^2)) + (A_i(g_i(x_i^1)) \oplus A_i(g_i(x_i^2))) \right\| + \\
& + \rho_i \left\| F_i(T_i x_i^1, \ldots, T_i x_{i-1}^1, T_i x_i^1, T_i x_{i+1}^1, \ldots, T_i x_{i+m}^1) \oplus \\
& F_i(T_i x_i^2, \ldots, T_i x_{i-1}^2, T_i x_i^2, T_i x_{i+1}^2, \ldots, T_i x_{i+m}^2) \right\| \right] \\
& \leq L_i \left[ \left\{ \lambda_{g_i} \| x_i^1 \oplus x_i^2 \| + \lambda_A \lambda_{g_i} \| x_i^1 \oplus x_i^2 \| \right\} + \\
& \rho_i \left\| F_i(T_i x_i^1, \ldots, T_i x_{i-1}^1, T_i x_i^1, T_i x_{i+1}^1, \ldots, T_i x_{i+m}^1) \oplus \\
& F_i(T_i x_i^2, \ldots, T_i x_{i-1}^2, T_i x_i^2, T_i x_{i+1}^2, \ldots, T_i x_{i+m}^2) \right\| \right] \\
& \leq L_i \left[ (\lambda_{g_i} + \lambda_A \lambda_{g_i}) \| x_i^1 \oplus x_i^2 \| + \\
& \rho_i \left\| F_i(T_i x_i^1, \ldots, T_i x_{i-1}^1, T_i x_i^1, T_i x_{i+1}^1, \ldots, T_i x_{i+m}^1) \right\| \\
& F_i(T_i x_i^2, \ldots, T_i x_{i-1}^2, T_i x_i^2, T_i x_{i+1}^2, \ldots, T_i x_{i+m}^2) \right\] . \quad (3.4.11)
\end{align*} \]

Now, calculate the inner part estimate of the above expression with the help of the properties of the $F_i$-operator for $i = 1, 2, \ldots, m$. 

\[ \begin{align*}
\left\| F_i(T_i x_i^1, \ldots, T_i x_{i-1}^1, T_i x_i^1, T_i x_{i+1}^1, \ldots, T_i x_{i+m}^1) \oplus F_i(T_i x_i^2, \ldots, T_i x_{i-1}^2, T_i x_i^2, \\
& T_i x_{i+1}^2, \ldots, T_i x_{i+m}^2) \right\| \\
& = \left\| F_i(T_i x_i^1, \ldots, T_i x_{i-1}^1, T_i x_i^1, T_i x_{i+1}^1, \ldots, T_i x_{i+m}^1) \oplus F_i(T_i x_i^2, \ldots, T_i x_{i-1}^2, \\
& T_i x_i^2, T_i x_{i+1}^2, \ldots, T_i x_{i+m}^2) \right\| \\
& \leq \left\| F_i(T_i x_i^1, \ldots, T_i x_{i-1}^1, T_i x_i^1, T_i x_{i+1}^1, \ldots, T_i x_{i+m}^1) \right\| + \\
& \left\| F_i(T_i x_i^2, \ldots, T_i x_{i-1}^2, T_i x_i^2, T_i x_{i+1}^2, \ldots, T_i x_{i+m}^2) \right\| + \ldots
\end{align*} \]
Now, equation (3.4.8) can be rewritten as
\[
\bigoplus F_i(T_{i1}x_1^2, \ldots, T_{i-1}x_{i-1}^2, T_{ii}x_i^2, T_{i+1}x_{i+1}^2, \ldots, T_{im}x_m^2) \bigg| T_{ii}x_i^2 + \lambda F_{i2}\|T_{i2}x_2^2 + T_{i2}x_2^2\| + \cdots \\
\leq \lambda F_{i1}\|T_{i1}x_1^2 + T_{i1}x_1^2\| + \lambda F_{i2}\|T_{i2}x_2^2 + T_{i2}x_2^2\| + \cdots \\
+ \lambda F_{i-1}\|T_{ii-1}x_{i-1}^2 + T_{ii-1}x_{i-1}^2\| + \lambda F_{ii}\|T_{ii}x_i^2 + T_{ii}x_i^2\| \\
+ \lambda F_{i+1}\|T_{ii+1}x_{i+1}^2 + T_{ii+1}x_{i+1}^2\| + \cdots + \lambda F_{im}\|T_{im}x_m^2 + T_{im}x_m^2\|. 
\] (3.4.12)

By using the Lipschitz continuity of $T_{ij}$-operator, in equation (3.4.12), we have
\[
\|F_i(T_{i1}x_1^2, \ldots, T_{i-1}x_{i-1}^2, T_{ii}x_i^2, T_{i+1}x_{i+1}^2, \ldots, T_{im}x_m^2) + F_i(T_{i1}x_1^2, \ldots, T_{ii-1}x_{i-1}^2, T_{ii}x_i^2, T_{ii+1}x_{i+1}^2, \ldots, T_{im}x_m^2) \| \\
\leq \sum_{i \neq j, \lambda = 1}^m \lambda F_{ij}\|x_j^1 + x_j^2\|. 
\] (3.4.13)

Using equation (3.4.13) into equation (3.4.11) and then using it in equation (3.4.9), we have
\[
\|y_1^1 + y_2^1\| \leq \|y_1^1 - y_2^1\| \\
\leq \lambda C(\alpha_1^i + \lambda g, \alpha_2^i)\|x_1^e - x_2^e\| + \lambda C \|L_i(\lambda g, + \lambda A, \lambda g)\|x_1^e - x_2^e\| \\
+ L_i \rho_i \sum_{i \neq j, \lambda = 1}^m \lambda F_{ij}\|x_j^1 - x_j^2\|. 
\] (3.4.14)

Now, equation (3.4.8) can be rewritten as
\[
\|Q(x_1, x_2, \ldots, x_m) - Q(x_1^2, x_2^2, \ldots, x_m^2)\| \leq \sum_{i = 1}^m \|y_i^1 - y_i^2\| \\
\leq \sum_{i = 1}^m \left\{ \lambda C \left[ (\alpha_1^i + \lambda g, \alpha_2^i) + L_i(\lambda g, + \lambda A, \lambda g) \right] \|x_1^e - x_2^e\| \\
+ \lambda C L_i \rho_i \sum_{i \neq j, \lambda = 1}^m \lambda F_{ij}\|x_j^1 - x_j^2\| \right\} \\
\leq \sum_{i = 1}^m \left\{ \lambda C \left[ (\alpha_1^i + \lambda g, \alpha_2^i) + L_i(\lambda g, + \lambda A, \lambda g) \right] \|x_1^e - x_2^e\| \\
+ \sum_{i = 1}^m \lambda C L_i \rho_i \sum_{i \neq j, j = 1}^m \lambda F_{ij}\|x_j^1 - x_j^2\| \\
+ \sum_{j = 1}^m \lambda C \left[ (\alpha_1^j + \alpha_2^j \lambda g, \lambda ) + L_j(\lambda g, + \lambda A, \lambda g) \right] \\
+ \lambda C \sum_{i \neq j, j = 1}^m L_i \rho_i \lambda F_{ij} \gamma_j^1 \|x_j^1 - x_j^2\| \\
= \sum_{j = 1}^m \theta_j\|x_j^1 - x_j^2\|
\]
4. Main Results

\[ \leq \theta \sum_{j=1}^{m} \| x_j^1 - x_j^2 \| \]

\[ \leq \theta \| (x_1^1, x_2^1, \ldots, x_m^1) - (x_1^2, x_2^2, \ldots, x_m^2) \|_s, \quad (3.4.15) \]

where \( \theta = \max_{1 \leq j \leq m} \theta_j \). Finally, using equation (3.4.15), equation (3.4.8) can be written as

\[ \| Q(x_1^1, x_2^1, \ldots, x_m^1) - Q(x_1^2, x_2^2, \ldots, x_m^2) \|_s \leq \theta \sum_{j=1}^{m} \| x_j^1 - x_j^2 \| \]

\[ = \theta \| (x_1^1, x_2^1, \ldots, x_m^1) - (x_1^2, x_2^2, \ldots, x_m^2) \|_s. \quad (3.4.16) \]

It follows from the condition (3.4.1) that \( 0 < \theta < 1 \). This implies that \( Q : \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \rightarrow \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \) is a contraction which in turn, implies that there exists a unique \((x_1^*, x_2^*, \ldots, x_m^*) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \) such that \( Q(x_1^*, x_2^*, \ldots, x_m^*) = (x_1^*, x_2^*, \ldots, x_m^*) \). Thus, \((x_1^*, x_2^*, \ldots, x_m^*) \) is the unique solution of the problem (3.2.2). Now, we prove that \( x_i^n \rightarrow x_i^* \) as \( n \rightarrow \infty \) for \( i = 1, 2, \ldots, m \). In fact, it follows from equation (3.3.2) and the Lipschitz continuity of the relaxed resolvent operator that

\[ \| x_i^{n+1} + x_i^* \| = \left\| (x_i^n - g_i(x_i^n)) + J_{\lambda_i}^{I_i - A_i} \left[ (I_i - A_i)(g_i(x_i^n)) \right] + \rho_i F_i(T_{ij} x_1^n, T_{ij} x_2^n, \ldots, T_{ij} x_m^n) \right\| \]

\[ \leq \left\| (x_i^n - g_i(x_i^n)) \oplus (x_i^* - g_i(x_i^*)) \right\| \]

\[ + \left\| J_{\lambda_i}^{I_i - A_i} \left[ (I_i - A_i)(g_i(x_i^n)) \right] + \rho_i F_i(T_{ij} x_1^n, T_{ij} x_2^n, \ldots, T_{ij} x_m^n) \right\| \oplus \left\| J_{\lambda_i}^{I_i - A_i} \left[ (I_i - A_i)(g_i(x_i^*)) \right] + \rho_i F_i(T_{ij} x_1^*, T_{ij} x_2^*, \ldots, T_{ij} x_m^*) \right\| + \| w_1^n \oplus 0 \|, \]

From the previous calculations, we have

\[ \sum_{i=1}^{m} \| x_i^{n+1} + x_i^* \| = \sum_{i=1}^{m} \| x_i^{n+1} - x_i^* \| \]

\[ \leq \left\{ (\alpha_i^+ + \alpha_i^+ \lambda_{g_j}) + L_j (\lambda_{g_j} + \lambda_{A_i} \lambda_{g_j}) \right\} \]

\[ + \lambda_C \sum_{i 
eq j, i=1}^{m} L_i \rho_i \lambda_{F_j} \gamma_{ij} \sum_{j=1}^{m} \| x_j^n - x_j^* \| + \sum_{j=1}^{m} \| w_j^n \| \]

\[ = \sum_{j=1}^{m} \theta_j \| x_j^n - x_j^* \| + \sum_{j=1}^{m} \| w_j^n \|, \]

45
Chapter 3. New nonlinear ordered inclusion systems involving weak-ARD mappings

where \( a_n = \sum_{j=1}^{m} \|x_j^n - x_j^*\|, \quad b_n = \sum_{j=1}^{m} \|w_j^n\|. \) Algorithm 3.2 yields \( \lim_{n \to \infty} b_n = 0. \) Now by Lemma 3.2.3 implies that \( \lim_{n \to \infty} a_n = 0, \) and so \( x_j^n \to x_j^* \) as \( n \to \infty \) for \( j = 1, 2, \ldots, m. \) This completes the proof. \( \square \)